Introduction to Stochastic Processes

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Lecture 37
Class Properties Continued

So, we started talking about this communicating classes in a Markov chains. Then in the last class we said this so the relation that i communicates to j and i and j communicates with each other. That relations is an equivalence relation and that partitions my state space into classes which are equivalence classes. So, then we started looking into some properties of this communicating classes. What are the properties we studied about this communicating classes?

Student: Class property theorem.

Professor: We said one class property theorem which says that, if we have states in a particular class they must be all are of the same type whether transient, positive recurrent or null recurrent. We look into its proof today building on that we will see some more properties related to these classes and then we will move on to something called invariant distribution.

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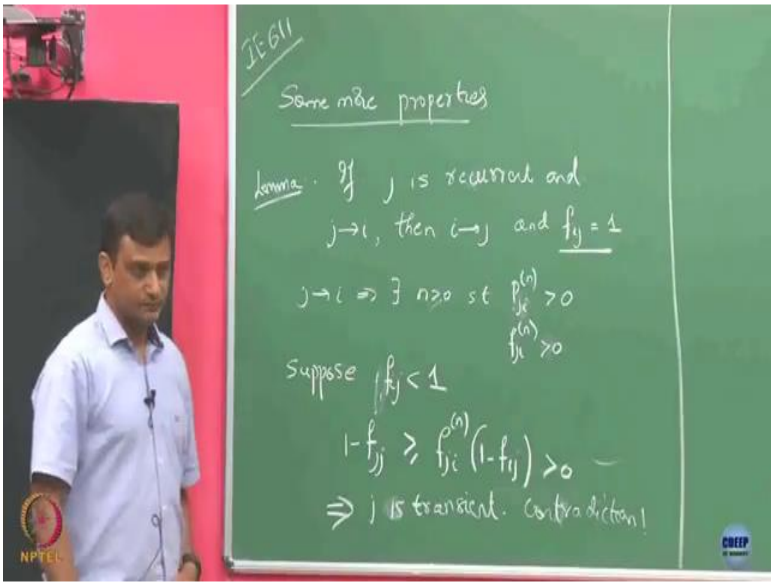

So, these are some more properties. Suppose let say I have so what we are saying is suppose you know that j is a state which is recurrent and if you say that j communicates with i then is it obvious that ji some state i, I have to come back to state j then i communicates with j. So, we are, so there are saying j is recurrent by our notion of recurrence that means I should be hitting this state j again and again.

I should be coming back to this state with probability one. But, suppose I have from the state j I have reached to some other state j in some finite time. Then if I have to again come back to their state then it must be a case from that state j I must return to state j. This also implies that, Fij equals to 1. So, how you show this? So since j goes to i, I know that in some finite time there exist some n such that Pij of n is going to be greater than 0.

Since, j such that there exist n positive such that, Pij is greater than 0. So, the same condition also implies that I should be reaching, so this is telling that at some point n, I am going to reach state j with positive. I can also translate to the first time visit to state. So, this is ji first time visit to state i starting from j then that should be also happening to with some positive probability. So, I can also take that to be, may be this need not be the same as this n, it could be some different n. when it is going to hit that particular state i for the first time.

Now, what I have so now let us try to prove this. Suppose let say I am taking this to be suppose let say Fij less than 1. Let us start with this negation of the statement given to me. Now, let look at this quantity Fjj and this not returning to state j, I am looking at one minus less than this. So, now it may happen that, so I am asking about not returning to state j starting from j. So, I am now can look at I hit some special state in between and from that I do not come back to j again and which state I hit in between could be many possible ways.

Now, I am going to take one particular possibility. So, that is why I am going to look at a lower bound. So, let say first I am going to hit this particular state i and after that, I do not come back i from there I do not return to state j. So, this is 1 minus Fjj is not returning to state j.

So, what I am asking now looking at a one particular possibility of that. Where first I go to state i and from there i again this is 1 minus Fij tells that, if I start from i, i never go back to j again. So, this gives me one possibility of not going back to state j again. So, that is why this is going to be lower bound.

Now, we know that this guy Fjj of n is strictly positive and if this Fij is strictly less than 1, this guy is also positive. So, this product is positive and then what we are saying is, in this case if this is if we start with this assumption we are just proving that, this Fjj is less than 1. If Fjj is less than 1 what does that mean?

Student: (())(07:12)

Professor: It transient, but we have initially assumed that this j is recurrent so it contradicts. So, if j is recurrent this cannot happen. So, hence it must be the case that j is recurrent and if Fij is equals to 1 it must be the case that, starting from i, i should be communicating to j at some point. So, i is going to communicate with j. So, what we are saying is, if my state j is recurrent, if i leave state j and go to some other state at some point. Then it must be the case that, from that state i should comeback to state j again. Then only I am going to visit my state j again and again, otherwise I am not going to visit again.

Student: (())(08:27)

This is the whole contradiction if I am going to take this Fij. Suppose let say Fij is not equals to one. But it is Fij is less than 1. Now, what we are getting by using this assumption, we are getting that Fjj is?

Student: (())(08:42)

Professor: Strictly less than 1. That means j is?

Student: Transient

Professor: Transient that (())(08:45) hypothesis is what, j is recurrent. So, that is what is getting violated here.

Student (())(09:04)

Professor: This one. So, what is this? This is a probability of this you nor coming back to state j. This can happen in many ways, so I am looking at a particular possibility where from you go to state i and from there you do not comeback and this is just one possibility that is why this probability is a lower bound on this event. You are not coming back, in it happen in many ways. So, that is why this lower bound and I get greater than 0 of here. Now, we have already said that all my states get clustered into different communicating classes.

And now if I am going to take and this communicating classes can be either open or closed. Now, suppose if I know this communicating class is open or this communicating class is closed. Can I further say that what kind of it has? So, let say I have a communicating class which is open then what kind of states possibly that communicating class will have. If my state j is recurrent I have to be hitting again and again so there should be a reversed path also. But, in the open case that may not be the case. So that is why if you have a.

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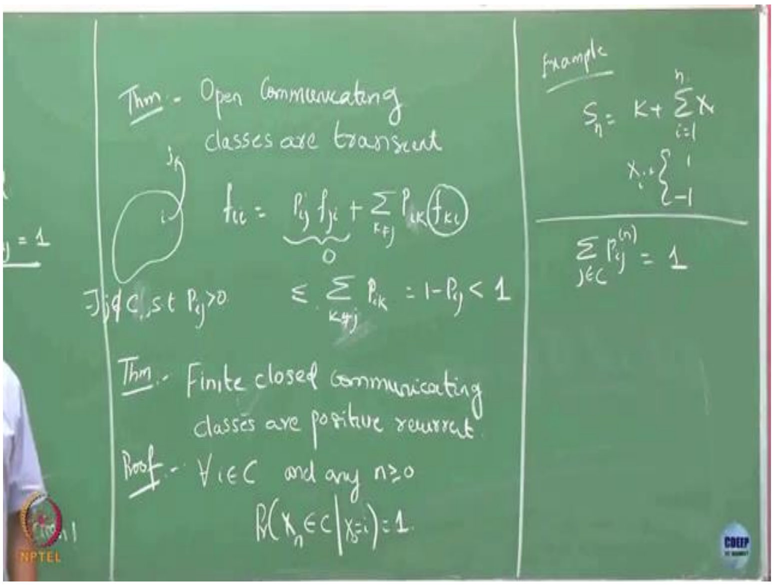

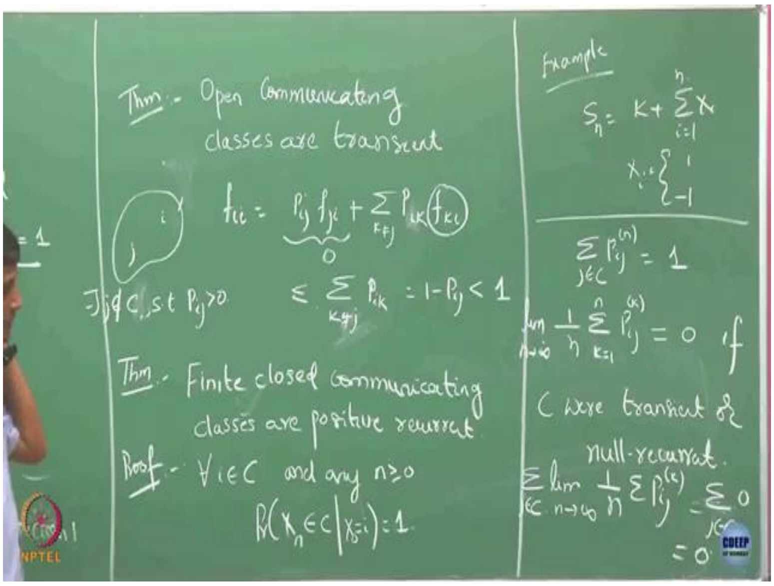

So, let me also state it as result, so this is (())(11:06) clear to us now how to show this. Suppose let say have a class here which has one particular state let say i here and because this is an open communicating class I should have a state j outside this class. So, that the probability of reaching from this state to this should be positive, that is the definition of our open communicating class. So, if that happens how can I show that, this state i is transient. Fii equals to…

Student: (())(11:58)

Professor: Let us write what is Fii? Fii is the probability of ever hit again hitting state i again starting from i. So, I am going to now look at one step thing like I leave from that state i and then again I come back to that. So, I am now looking it in one state whereas when I did this, I looked at going to something in n steps and then coming back to that state.

So, what is the possibility in the first i? I could go from with probability Pij, I can go from state i to j and then I can look at from there I can look at coming back to i. So, this is one possibility but other possibilities could be instead of i I could be going to some other states and then if you could be FKi. So, there are many possibilities like or maybe all these K naught equals to j could be maybe contained within the same place.

What I am asking is, in one step from the state a, you go to some other state and from there you comeback to state i. So, this is like this particular state which I know which is outside my class. So I first go there and from there I will come back to state i again at some point and this is not just i not just j.

I will go back to some other state K and from there I will come back at some point i. Now, what I know, so I know that Pji is what there exist a j, if this my class is open communicating class. There exist j outside my class so there exist j which is outside my class such that Pij is what greater than 0 and then in this case this j is such that, what should be Fji? Why is that?

Student: (())(14:27)

Professor: So nor comeback if you are if Fji is not 0, then there is a positive possibility that you will come back. In that case means that actually i and j communicate they should be in the, would have been in the same class, not outside. So, what is this term is going to be with this term and now what I will do is. I am going to I know that this FKi is are all upper bounded by greater than or equals to 1. So, I use this bound and get a bound like this. So, why this is our upper bound? Because FKis I have put one for them.

I know they are probability terms and they are at most 1. Now look at this, so now I am adding PiKs, what is PiK? Probability of going from state i to K and now I am looking at all possibilities of states from where I can jump from state i except for j.

Student: (())(15:43)

Professor: So this is what 1 minus. Now, I have said that this Pij is what strictly positive. So, what this should be?

Student: (())(16:01)

Professor: One, so this is what I wanted to show that state i that is inside this class is transient. That is Fii is going to be strictly less than 1. So, we already know that if I have a communicating class which is open that means every state in that should be transient. Because if I have one state which is transient in this communicating class, so it must be the case that all these states in this communicating class must be transient. So, this going to some other.

Student: (())(16:52)

Professor: So, but that is not like say the whole point is, if at all you are going to from j you are going to comeback i, through some going to other class and other class. That means with some positive probability and some finites n we are going to come back to this state. That means j is communicating with i.

Student: (())(17:16)

Professor: So, just what is our definition of Fii, Fij. Fji in this case this is the sum of Fji superscript n. So, if any term there is positive this term would have been positive. But, there if this is 0 it must all the term should be 0 in that case. So, now what about close communicating classes. So fine, open communicating I just write away said that all the states there are transient. What about close communication class?

If I am going to say this is going to be a closed communicating class everybody should be communicating with other that is fine. Within some finite time and because of that is it required that they have to be either only positive recurrent or null recurrent anyway. What we have is it is not necessary that a close communication class if I look into a generality like arbitrary close communication class which could potentially have infinite many states in that.

Then that could be any of the things transient, positive recurrent or null recurrent. I cannot in general, where we said that Sn equals to K plus summation xi, i equals to 1 to n and what was this i's? i is such that it could take 1 and minus 1.So, what would be say about this Markov chain, so how many states it had?

Student: Countably infinite.

Professor: Countably infinite and we said that is an irreducible. So, irreducible means? We already kind of said that this is a communicating class. One communicating class so one big communicating class. So, if it is one big communicating class and is that going to be closed or open? Closed like just everybody is there in the same class. Now, what did we conclude each state there as, what did we conclude. We said each state there is transient, so even though we could have a communicating class, which is irreducible.

It could potentially have it could still be transient like we had an example. But, it so happens that, if we have a communicating class a closed communicating class which has only finitely many states in it. Then it is going to be positive recurrent.

So finite, see what we are saying is, a set of states based on our communicating relation can be partitioned into many classes. One class may have finite number of states in it and it is a closed communicating class. That finite closed communicating class is such that each state in that class is going to be positive recurrent, positive recurrent.

The how the proof of this? Let us see we can decide for this proof. So, is this clear to you I am just saying that take any state in my communicating class and look at any step n. if you start from i the probability that you in the nth state remain in class C itself is going to be probability 1. That is that should happen because this is a close communicating class. You are not going to escape from this, in any round it must be placed that if you whatever you are going to start in that every n step you have to remain in the same C.

So, this is same as saying that you going from state i to j, so I am looking at nth step. This sum of this probability should be such that it should be equals to 1. It is just like rewriting this probability in this fashion. So, now look into this case now I am going to do is, I started with some state i. Now, what I am doing is now I am looking at another state in j only this is another state. Now, I am going to look at this quantity here. So, what is this quantity?

It is just like I am looking at transition from i to j in different rounds and I am just taking average of this. So, I know that what I know about its limit as n goes to infinity, if my state j is recurrent. What I know about this? Where does this limit go? So we had said, if it is I have said suppose my state j is recurrent, we had a criteria to further distinguish whether it is going to be a positive recurrent or null recurrent. So, what is this limit is going to when it is a null recurrent.

It goes to 0 and what happens to this sum as n goes to 0 when my state j is transient. So, we said that summation P if my state j is transient and if I start from initial state i let say, we had said that summation Pij script n what is that is going to be, we said that, that is finite as n goes to infinity.

So, that is finite what happens if I take 1 by n? If i divide it by n, what happens to this, this is going to 0. So, this is equals to 0, if C were transient or null recurrent. So, if C is my communicating class is such that if the elements in that are as a transient or null recurrent. It must be the case that this limit happens, that is the property we have already shown.

Now, this is true for any state in j. So, if you state j you take any j, if it is a transient or null recurrent this should happen. So, now what I will do is I will just add this over C this limit I am just adding both left hand limit both over j now and this is still going to remain 0. The right side I am also adding over all summation but the summation contains only finitely many elements. So, this is basically and this is going to be still 0 because this is a finite summation here that is just going to be 0.

Now, what we are going to show that, we are going to actually show by contradiction here. If my elements the states in C are transient or null this has this quantity is supposed to be 0. But, it is not actually the case the right hand side has turned out to be 0. But let us look into the quantity what this quantity is more about.

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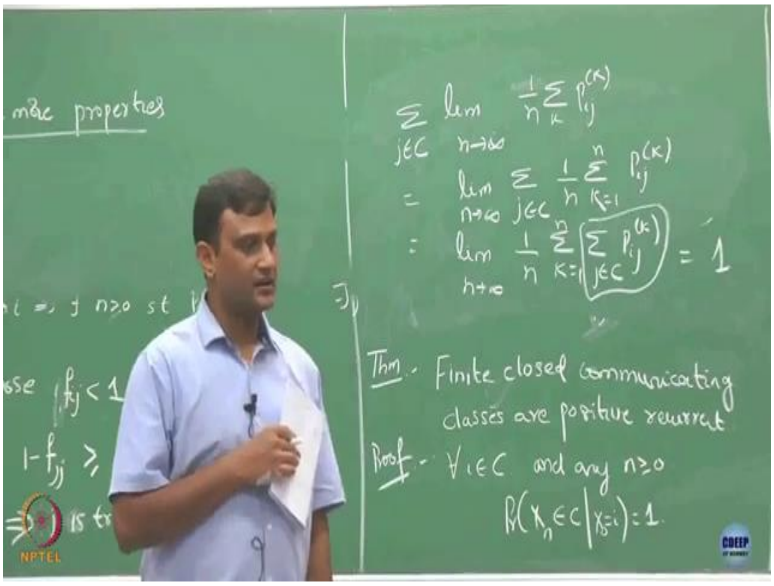

So, let us look into this quantity. At this point for each j, I have a limit here. Can I interchange this limit and this finite summation here? Since this is a finite summation, I can interchange. Had it been C had been an infinitely many state in this, I may not so easily would have been able to interchange this.

But, now that I know that this C is finite, I can blindly interchange this limit. Now, further these are two summations over a finite number of elements. This is C is finite and this is going from K to n, I could easily again interchange these limits. Now, what is this quantity now? So what are these quantities? This is going from state i to state j, in K steps.

And now I am summing it over all possible states. So, this is basically jumping from state i to another state within j in the Kth step. This is going to be one because this is a communicating class close communicating class. So, these quantities going to be 1 which we have already shown here. So, this is one and this is what is this summation further is going to do.

Student: (())(31:13)

Professor: This is going to be n and this is going to divide by n. What is this quantity is going to be one and as limit this is going to be a constant in this case and what is this going to lead to you is?

Student: (())(31:28)

Professor: So, had it been the states here are either transient or null recurrent what we are ending with is a contradiction here because 1 cannot be equal to 0. See this is what like you had to be careful in interchanging the limit and the summation here.

So, in this case we could interchange the limit because C is a finite. But, had we blindly ignored it and even when C was infinitely many terms in this we have blindly ignored this. That would have given us a result which was not at all consistent. So, what we are saying is as if just tell me my communicating class is closed.

I can tell you that it is going to be positive recurrent as long as this is finite. If you tell me that I do not know it is finite, it could be countably infinite by communicating class is countably infinite. Then I any of the cases could be possible, you need to prove me in that case what is that possibility. For example, we said in this case you had a case where we had a communicating class with countably infinite things. But, the states were all transient there.