Introduction to Stochastic Processes

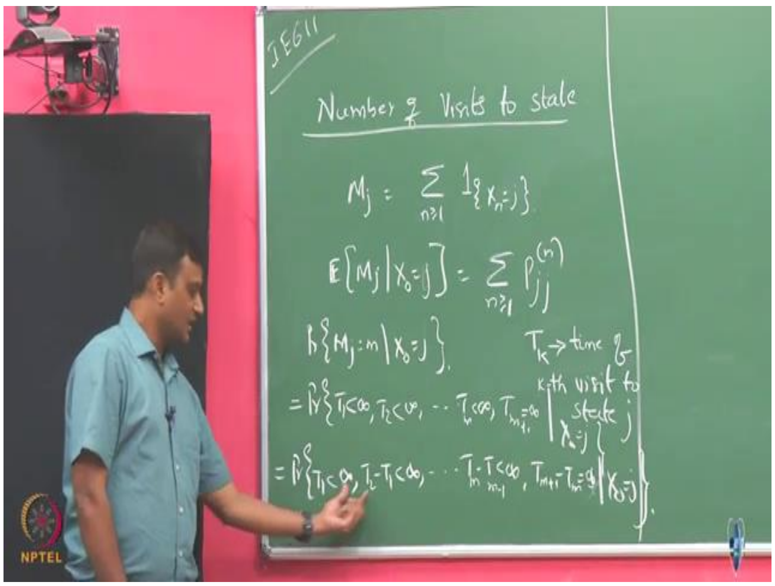
Professor. Manjesh Hanawal
Department of Industrial Engineering & Operations Research
Indian Institute of Technology, Bombay

Lecture 35

Mean Numbers of Returns to a State

So, in the last class we started talking about hitting times and recurrence time. So, we defined what we mean by different states like what is transient, what is recurrent state further current state we further classified as positive recurrent and null recurrent and then we started defining what we mean by mean number of visits.

(Refer Slide Time: 00:51)



So, we defined the number of visits to state. So, we said that if I have a state j then this quantity Mj is defined as, number of visits of Markov chain to the state and then I was interested in the question, what is the expected number of visits given that I start from a particular state or I could just start with j and we said that is nothing but Pjj of n, n equals to 1.

We show we argued that this follows from after applying my monotone converges theorem. So, finding expected value of Mj is straightforward here. But suppose, let us say I am interested in finding probability that Mj is equals to m. Let us say if I start from some j. So, what is Mj is telling number of visits to state j. So, the Mj here is a discrete valued random variable here. It can take value 1, 2, 3 all the way up to infinity. Now, this is the expected value that is expressed in terms of n-step transition probability.

Now instead of this expected value. I am interested in more refined description of this Mj and I want it to know exactly what the distribution of Mj itself. Now how to go about this? So, what is this is basically saying that. If I start the state j what is the probability that I am going to hit this state exactly m times again in the future. Now how to go about this? Or how to find this distribution? Now, let us try to compute this.

Student: (())(3:24)

Professor: Yeah, X naught we want to Mj is number of counts of particular state j here. So, here I want is Mj to be exactly m. That means this (())(3:41) X n has taken this state j exactly m times. Now what does this mean? The probability that Mj equals to m means what? I hit my state once after that I hit it second time and I hit it nth time after that? I never hit it again.

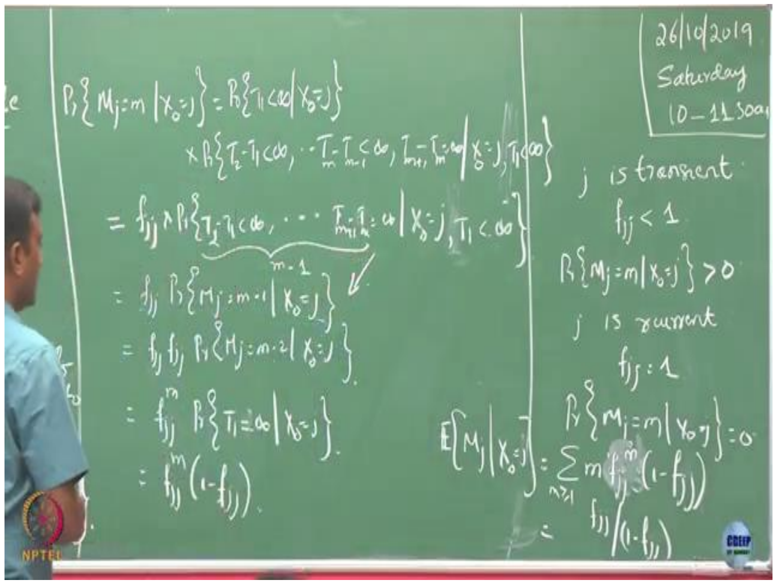
Suppose I define Tk. Could be the time of Kth visit to my state j. So, when this K equals to 1, T1 denotes what time of first visit, when K equals to 2. That means time of my second visit like that. So, now if Mj has to be m, what is that should happen? T1 should be some finite number. I should be hitting my state j in finite quantity, finite time and what about T2.

T2 should also be finite. I am hitting it for the second time and like that T2, T3 all the way up to Tm plus 1 and what should be Tm plus 1, it should be infinity, then only it is the case that Mj is exactly equals to m. So, let us write that. So, if this is the case then it must be a case that T1 should be finite, T2 it should be finite all the way up to Tm should be finite and Tm plus 1 should be finite given whatever X naught equals to j.

So, Mjis equals to m all it is saying is I should have hit state j m times in the future. It is not telling it at what time you may hit it after some time for the first time, subsequently after some more time, you might have hit another time. So, if Mj equals m the first m visits must have happened in finite time, and after that the mth 1, m plus 1 hit might have not happen that means Tm plus 1 equals to infinity. I could then write this as.

So, is this T2 is going to be greater than T1. That is the second is it so, I could write this as T1 less than infinity like this and write it like all the way up to Tm minus finite and the Tm plus 1 all the way up to infinity, given it is not the same thing I have expressed in this format. So, now instead of looking at the time, I am looking at the interval. So, T1 is the first time I visit. This is the time duration more required to hit for the second time like this.

(Refer Slide Time: 07:18)



Now, let us write this. I am going to now apply my chain rule here. I just applied a chain rule now what is this quantity probability that T1 is less than infinity, given that X naught equals to j. So, what is the meaning of probability that T1 less than infinity. That means I have probability of error hitting state j starting from j, first with it means I can just think it of ever, whenever I am going to hit state j.

Probability of error hitting state j starting from X naught equals to j what is this? So, did we define something in terms of the fjj, what was fjj? So, there are something called fjj n also, what was fjj of superscript N? Starting from state j probability that you are going to hit state j for the first time in n times and we define fjj to be the sum fjj of ns. What was that interpretation we said what does the fjj then told us?

Yeah probability of ever hitting state j. So, is it not exactly that probability that, T1 less than infinity means probability of error hitting state j. So, this is fjj not audition and then same thing, same thing X naught is equals to j. There should be condition here, which I missed (())(09:48) T1 less than infinity.

Well, let us focus on this. What it is saying? You started from state j initially you hit state j in some finite time that is what meaning of T1 less than infinity and then after that you are asking probability that you are going to hit it for the next time that happens in finite time and all the way up to. Now can I think of this as. Now what I am saying you are going to hit state j at some finite time and after that conditioning on this does not going to matter because this is a Markov Chain and then I am asking, you are going to hit state j again, again. Now I will be looking at it. For how many times here

Student: m minus 1

Professor: m minus 1 times. Now, I have already condition that you are hit it first time, at some point after that, you are again going to hit it m minus 1 time. So, now is it not just case that this probability is nothing but, I can think of, now I can then interpret that whenever you hit T1 less than infinity. From that point again you are visiting your state j again for the m minus m rounds. So, check that. So, check that this is indeed correct like you can just write like this.

Student: (())(12:01)

Professor: So, the T1 has been absorbed into this, like you have been the T1 less than infinity now I am whenever that hit happens for the first time. Now, I can think of that is the starting point and from there, subsequently visiting m minus 1, steps. So, yes X naught equals to j in some time. I have it state j again. Now I can think of Markov chain from that point and then look at it hitting again m minus 1 times subsequently.

So, that is what this I can, this point is actually the time when I have hit for the first time, but I can think that as my original point, origin and write this. So, you say this. This has nice recursion in terms of fjj's. So, I can further write it as fjj again fjj and what is this going to be? m minus 2 like that. So, if you keep on writing this. What will eventually end up with is fjj just write that here, we will end up with this and then at the end, we will end up with probability that T1 less than infinity, x naught equals to j. This is the last one it is going to be. You will make T1 equals to infinity.

See here T2 minus T1 become here infinity in the last one. This will be just T1 equals to like basically this is T1 minus 0 infinity. So, that is basically T1 equals to infinity. Now what is this probability you started with state j and then asking the question that your first time hit, is going to happen at infinity. That means basically you are not hitting it. What is this probability? 1 minus jjj

Now, if you are going to look at this Mj as a random variable, which takes integer valued numbers and it has this kind of distribution can you associate it with what kind of geometric distribution that what parameter?

Student: (())(14:39)

Professor: With parameter fjj, that is a parameter here. So, now we have fjj here. Now we know something about fjj depending on either it is transient or recurrent. Suppose my state j is transient. What is fjj means in that case transient case less than 1. So, in that case this probability will be non-zero probability, it will be strictly positive quantity.

So, in that case we will have is going to be, now suppose j is recurrent. What is this quantity fjj is going to be what 1 and what is this quantity is going to be? That case fjj is equals to 1 and my probability is going to be 0. So, does this makes sense? So, do you expect this probability to be 0 for a recurrent case? Why?

Student: (())(16:18)

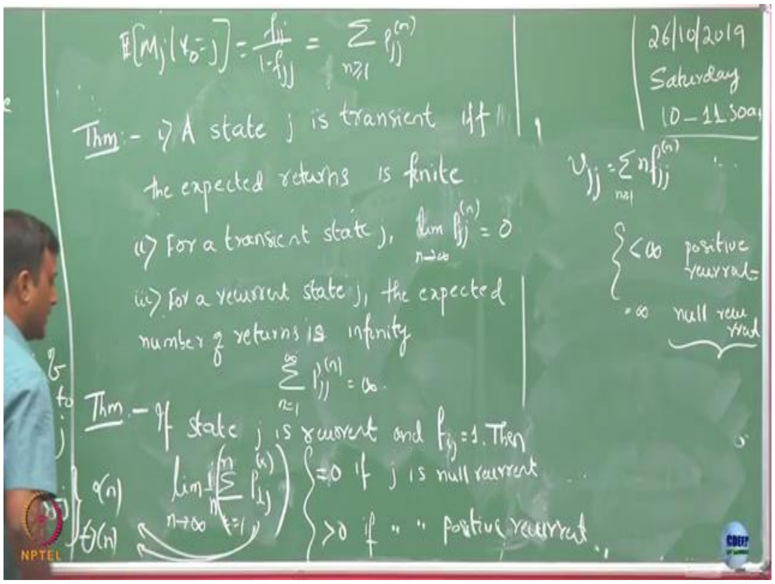
Professor: So what is recurrent means? I keep on hitting the state at right. So, that means what? What is the value mjis going to take it is going to take infinity right, because in this term here I am Xn is going to take j many, many times. So, it this value cannot be finite in that case. So, that is what this term is going to be 0. This probability is going to 0 and this is going to be 1 only at m equals to infinity, possibly. That is only place where it is have full marks.

Now we have this. Now, let us see based on this probability, we can find out the expected value of Mj, Mj expression we already have here. Let us see, what is the expression I get from this probability? Now what is the expected value of Mj given x naught equals to j, how can I write this? I am going to write it as m into a probability and what is the probability? Probability is exactly this quantity here.

Can somebody quickly compute this and tell me what is this value is going to be expected value. If this expected value correct here, I take value m with this probability. So, that is what this is expression for my probability. Now what is its value? You can check that this value turn out to be fjj minus 1 minus fjj.

So, again if fjj is going to be 1, this quantity is unbounded. This quantity is basically infinity. That means expected number of visits is going to be infinity. But if suppose this fjj strictly less than 1 that is just j is a transient quantity, then this expected quantity is going to be what? It is going to be finite. So, based on this we can make the following, we can claim the following result.

(Refer Slide Time: 19:14)



So, let me write this, we have come up with two formulas for this, one is fjj divided by 1 minus fjj and another one is what? This we also said that this is also same as a state j is transient. If and only if, expected returns is finite. So, this statement here directly followed from this. If this if j is transient. I know that fjj is less than 1 and this guy is going to be finite, and if this guy is finite, I know that fjj has to be less than 1. So, that is why a state j is transient. If and only if it is expected returns are finite.

Now, is this clear also, I am saying that if j is transient then the limit as n goes to infinity. The sequence Pjj of n equals to 0. Why is this true? Because j is transient, I know that this sum is finite, that means this series is converges. If that is the case, it must be the case that, this sequence of Pjj n should go to 0 and then for the current state and we know that, if state j is recurrent, this guy is going to be infinity.

So, that is and you also have that n equals to 1 to infinity, this Pjj of n is going to be infinity. So, basically we have tried to characterize what kind of state it is. Based on its mean return to mean return times, I turn the mean return time we have expressed in terms of my first passage times, fjj is basically my first passage time or the what we called as a recurrence time, we have already given its definition.

So, from this theorem it is clear that. How I am going to check whether a state is going to be transient. All you need to do is, see if expected number of visits is 0 or see if a transient j state is

j. Then we know that also it is sequence simply converges to 0. Now what about the recurrence state?

For the recurrence state, we know that the current state is further classified into null recurrent and positive recurrent. From the here we have not given any condition. We just know that when the state is going to be recurrent, whenever the sum is going to be infinite that it is going to be that it is going to be remember. But what about when it is going to be null recurrent or positive recurrent? For that I am just going to state the result, which we will take it as granted we will proof that results a bit later.

I know my state is recurrent to further classify that it is going to be positive recurrent or null recurrent, what I will do is, suppose I look into this sequence. There should be 1 by, now I am looking at the average of the first n terms here in the sequence of p, i, j, K ‘s. Now, I would look at this average if in the limit n goes to infinity, if this average goes to 0 starting from an i, for which f, i, j, K equals to 1, then I know that this j is null recurrent. If this K takes some positive value other than 0 then it is going to be, the first case is null recurrent, then if it is going to take a positive value then we are going to call it as positive.

So, what is that how to use this result. Suppose let say you know j is recurrent that you would easily find out if you know that fjj is equals to 1. You have, let us say you have already found it out. Now, to know that my state j is positive recurrent or null recurrent. One option is go with the definition we already have. Look at n time’s fjj of n. What is that the new jj we have given new j is what.

Look at this and see that if this is going to be less than infinity then it is going to be positive recurrent, and if this is going to be infinity null recurrent either go and use this definition for this you need to have these quantities fjj that it is a first passage, times or first passes distribution you need to have or If you do not want to go and but you have this sequence, Pijs that you have this in-step transition probabilities, then you can look into this average quantities and see that if this goes to 0. Then you know that this j is null recurrent. Otherwise it is going to be a positive recurrent. So, any intuition why if this goes to 0, this is going to be null recurrent, and when this is greater than 0, it is positive, recurrent.

Student: (())(30:03)

Professor: So what would you say when it is going to be positive recurrent.

Student: (())(30:13)

Professor: So, fine see if j is recurrent we already know that fjj equals to 1. You can just think that it to be that particular j and then apply this. But we are saying that what is f i j equals to 1 means. That means if you start from state i with probability 1 you are going to hit state j. If you are going to start with such a state from where, you will surely that you are going to hit state j. Then look starting from that j you compute this.

You can as well take i to be j. So, let us go back to this. So, when this is finite. What would you say and when this is infinity. So from that at least, you can correct why this could be possibly make sense. This result possibly makes sense, let us focus on this particular case. So, this is going to be null recurrent? When we said that null recurrent then we said that it is going to come back but going to take a long time possibly to come back.

So, in that case, so, in that case, we are saying that this in this when you are going to look into this n basically dominates this summation and basically keeps the summation. What is the meaning of 1 by 1 this going to 0. So, basically this means that this quantity over here, this is small o of n. You understand the meaning of this o of n. So, in this quantity if I just divided by n that if this n dominates this and eventually kills it, that means.

Student: ((32:36))

Professor: Then you divide this quantity by small n and let n go to infinity. Then this quantity goes to 0 and in a similar way when this happens this means the second case. This means this is like some theta n, that means this quantity is eventually going to constant. That means this quantity. So, it is n here some constant times this quantity. So, that is why as n goes to infinity, you are going to get it to some constant value.

Student: (())(33:18)

Professor: So, recurrent means it is one of them positive recurrent or null recurrent.

Student: (())(33:43)

Professor: What is not satisfied? No it is like 1 by 1. It is not just this. You are not just taking this summation infinity. You are also dividing by n and then taking this to infinity. That is a difference between this and this. See recurrent means it is both here positive and null recurrent. So, this quantity is going to be infinitely for both positive and null recurrent.

Now, I want to I want to further classify whether it is positive recurrent or not. For that instead of looking at this, I start looking at running averages. So, this is the running average version of this, as n going to be infinity.

Student: (())(34:29)

Professor: is finite j is recurrent then weather it is going to a positive recurrent or null recurrent I am going to decide based on this criteria here. So, just think about this like why this will kind of thing is going to translate at least. Just think about this intuitively.