Introduction to Stochastic Processes
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Lecture 34 - Hitting Time and Recurrence

Now, identifying the visits to states, like often when we have a Markov chain, like it is going to be taking different, different states, fine. I may be interested in when is the first time it is going to visit a particular state and to visit that state, possibly on an average how much time I am going to spend. So, such questions you may be incurring, for example on an average how many times my share price crosses a certain number and you want to have a, want to understand the distribution of such counts.

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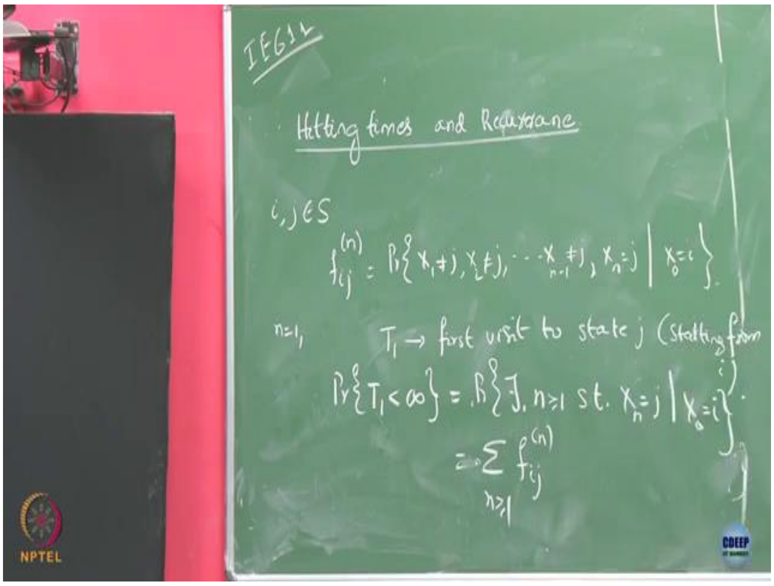

So, for that we have this notion of hitting times and recurrence. Now, let us take a state i and state J. I am interested, let say if I start from state i when is the first time I am going to hit state J. Now, I am going to define F i J n as probability that X 1 not equals to J, X 2 not equals to J all the way up to X n minus 1 not equals to J and exactly X n equals to J give a X naught equals to i.

So, what I have done here, starting from i I am interested in hitting state J only in the nth round, not before that. I am assuming that i and J are different here. Now, I could set n to be anything, I could just say n to be 1 that means what? I start from state i and I want to go to state J in one step, or I could state any n positive number and now this probability is giving me what is the, so this expression is giving what is the probability that I will go to that state in this exactly n many rounds.

Student: What is this P i J of n?

Professor: This one? Is it, this is it P i J of n? What is P i J of n? It is just like it is going, if you start from i and you want to be in state J after n steps, no, it does not say that tum bich me usme nahi ja sakte ho bolke, you can, this one will say that you cannot go there in between, this allows that option. Then, so now this is what in a way if I am interested in knowing my first visit to state J here starting from I, my first visit to state J starting from i in n number of rounds.

Now, suppose, let us say T n is first visit to state J and I am assuming that starting from i you can visit from state i to state J in either 1 steps, or after 2 steps, or after 3 steps or maybe like it can go on, you do not know how many times. Now, if I want to say what is the probability that T1 equals to finite, what is this, what is this probability? So, I could go from state i to state J in one step or two steps or three steps or whatever, can I express this in terms of F i Js, what will be that?

So, agar ek me bhi ja sakte he toh thik hai, dusre me ja sakte he to thik hai, three me bhi ja sakte he to thik hai, all these are fine. So, ultimately, I can say that like this can also I can say probability that there exist n such that X n equals to J starting from i, eventually I want to, I am going to hit state J starting from i.

So, this is nothing but F i J of n, and what is n? n is going to be greater than equals to 1. So, this is, I am going to hit n but right now this is not telling at what round and you could be hitting in 1, 2, 3, 4, 5. So, there exist some H such that I am going to hit state J starting from i. So, this is nothing going to be the sum of all these F i Js.

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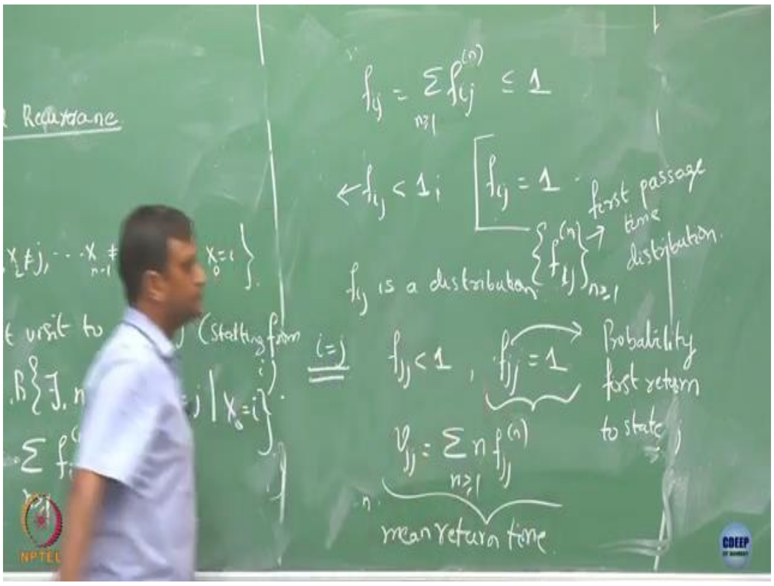

Now, I am going to denote this quantity as F i J equals to summation F i J of n, is this quantity is going to be less than or equals to 1, or equal to 1? So, I have just showed that this is nothing but some probability, probability or some event happening T1. So, this has to be necessarily it is going to be less than equals to 1.

Now, suppose let us consider the case when F i J is strictly less than 1, what does this mean? So, F i J less than, strictly less than 1 means, there is a possibility that when I start from state i, I will never come back to state J, that is why we are calling it as hitting time and recurrence, whether this is happening or not. Suppose, F i J is less than 1, we kind of understand that there is a positive probability that if I start from state i I will never come back to state J, I will never hit state J again.

And now if F i J is equals to 1, what does this mean? I will come back to State J starting from i in finite steps. Now, whenever let us consider this case, whenever F i J is equals to 1, that means what? This F i J does it make a probability distribution on n?

So all this so, we know that each F i J of n is going to be greater than or equals to 0, this quantity cannot be negative because this is a probability here and if we are going to say F i J equals to 1, that means this is a vector and this is a infinite dimensional vector and it sums to 1, so this is going to be a probability distribution, on what? On n whether it is going to take 1, 2, 3 or up to whatever value.

So, this is going to be a distribution, whenever that is the case let say whenever F i J is a distribution, by the way, when F i J is less than 1, we are going to say that state J is transient starting from i. What mean by transient? That means if it is possible that, it will never come back to that state and when this F i J is equals to 1, I know that starting from state i I will surely come back to state J, after sometime maybe whatever time it is going to take, I will surely come back to this.

Student: You said that F i J of n can be written equal to 1, is a random.

This is a distribution, let me write this, we call it as first passage time distribution, this is a distribution taking what values? Discrete values and what are these values, n equals to 1, 2, 3, 4. So, probability that it takes 1 is F i J 1, probability that it takes 2 is F i J 2, and probability that it takes value n is F i J n like this.

Now, let us say instead of starting from state i and looking at some others hitting others, let us take now, let us take i equals to J. Now, what I am interested in starting from state i starting from state J and again visiting that state. So, in that case if F J J is less than 1, that means if I start from state J there is a possibility that I will never come back to state J again and if F J J is equals to 1 that means if I start from state J, it is possible that I can come back to this.

So, for example let us say that your share price of a particular stocks hit some 100 rupees, then the probably, so let us say its base price started from some 100 rupees, probability that it come backs to 100 rupees again, if that state has F J J value equals to 1, that means it is going to come back with probability 1 again to state 1.

And then if this F J J has value 1, that means that it will never hit that 100 rupees value again, means with some positive probability it may never hit again, it may hit but there is also chances that it may not hit 100 rupees again. Now, I know when F J J equals to 1, this F J J of n forms a distribution, in that case further I am going to define another quantity called expected value, which is defined as F J J of n, now this F J J is…..

So, what is this F J J? This is a probability that the first time I am going to hit state J again in n rounds. So, what is this is going to give me? Expected time to hit state J again, expected time to return to state J. So, when I have this F J J, it is F J J I am going to call it as probability of first return to state J. So, I am calling it return because I starting from state J and again looking at coming back to this, or returning to that state J, and this is mean return time.

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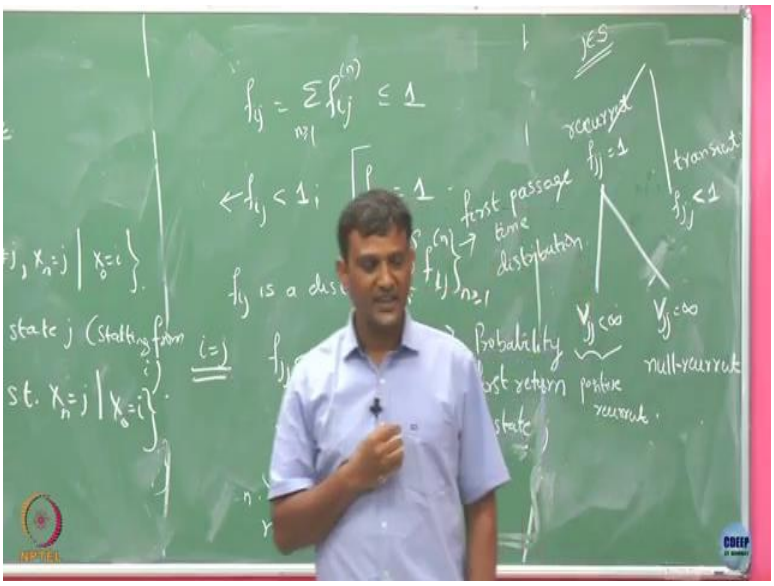

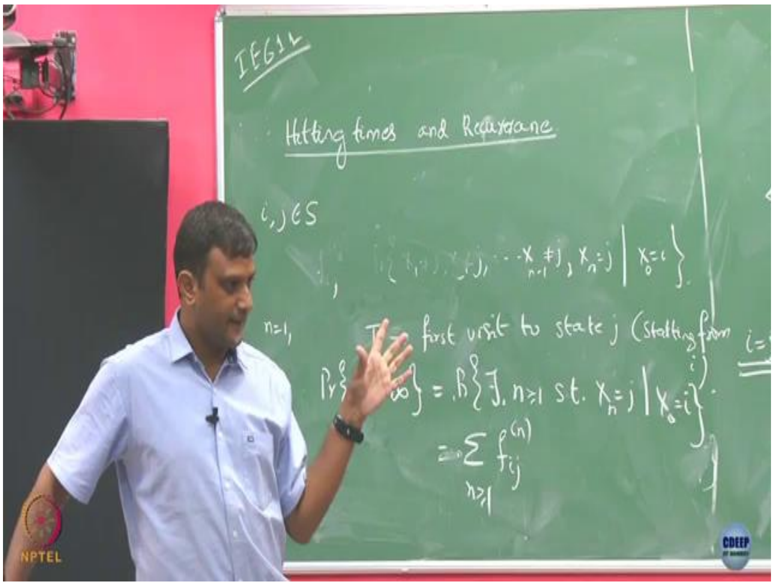

Now, as I said let us take a J S, now we can classify that state based on these observations. Suppose if my F J J is equals to 1 or my F J J can be less than 1, further when my F J J equals to 1, two things can happen, that is my V J J can be, so sorry, nu J J can be finite or my nu J J can be infinite.

So, as I said, when F J J happens to be 1, we are going to call that state J as recurrent, because we have a positive probability of coming back to this and when F J J is less than equals to 1, that means it is transient, matlab it may come back sometime and then die out, it may not come at all. So, in that case it is transient, now when it is recurrent, yes, we know that it is going to come back. Now the question is, whether it will come back soon with high probability or it will come back but after a long long time?

So, when you are, you know it is going to come back, now you may be interested in the frequency of coming back in a way, it is going to return quite often, or it will return but very rarely, you may be interested in these two scenarios. So, when it is going to come back frequently, what is the case we expect this mean return time to be? It will not be large in that case, it is coming back frequently.

So, that most of the times maybe n are smaller and those smaller n will have higher probabilities. And the larger n will have similar probabilities, but on the other hand if this guy is not returning so frequently, it will coming up very very long time or this is happening very very rarely, that means the n values which has high in positive value matlab mass, those are going to be very very large.

So, because of that this V J J may be large or may be unbounded also, so that is the case we are going to distinguish here. So, when it is recurrent, yes, we know it is going to come back but whether it is going to come back frequently or it is going to come back rarely after a long long time.

So, if it is going to come back after a long time, or it is going to come back rarely, we are going to call this as null recurrent and then we are going to call this case as positive recurrent. Let us assume that you are in insurance business, you start with some capital money, whatever you have, now how the insurance guys will work? They will all charge you some premium and whenever you get into trouble, you go back and ask for coverage and then they will pay you from this.

So, what insurance guys like? They like that you pay regularly the premium and do not come back and claim anything ever, just keep paying. So, but anyway they are supposed to pay you when you are in trouble, what if some tragedy happens and lot of claims come? If lot of claims come insurance company has to reimburse lot of money, and their money may, their balance may go negative also.

So, suppose let us say, so in that case what insurance like company want to do, it want to make sure that at any given time, its capital does not go below some threshold, you keep paying premium, its capital keep accumulating but at some point if large claims come and happen and small, small claims are fine like everybody is paying premium, some guys claim some coverage, the capital may be still growing but if sometimes large claims come, their capital may start going below.

So, what insurance companies want is they do not want to go bankrupt, they do not want their threshold to go below some level, maybe if it is goes once or twice it is fine but if it happens often, maybe they cannot survive in that business.

So, let us take such a, let us say insurance company is going to define some states, if my capital is this much, this is my state, if my capital hits this much, this is some states. Let us say some states and they will define one state which is they call it as let us say a danger level, if my capital goes this danger level, that means I am in danger.

Now, what that state the insurance company likes to be? They want it to be happened rarely, they do not want it to happen often. So, in that case, if you are going to model that case, you want if your capital whatever the state you are going to define, if it happens to be a null recurrent possibly, then you are going to enter into.

If you are initially a planner who want to get into this business and you have some capital with you and you have to decide, okay, whether I should enter this matter and let us say you are able to model all these things and come up with all these distributions and then you are able to also compute the expectation. Then the question is, should I enter? If you a priori feel that no, it looks like my state of getting into danger happens to be a positive recurrent. So, this is looks very very risky business, I do not want to get into this.

Student: (())(22:05).

Professor: So, see like calamities you cannot avoid, like so far whatever like let us say for some reason, some big tragedy happens and you cannot say that, that tragedy will not happen again and it can happen not in our control but…Yeah?

Student: Is it not saying FDD is 0, (())(22:35).

Professor: So, what? Okay, fine, this F J J….

Student: So what should we, we want that less than or why will you go for (())(22:50)?

Professor: So F J J is 1 that means if the risk you, some calamity has happened today, you cannot guarantee that the calamity is never going to happen in the future, it may happen in the future. So, you are going to visit again, that is what F J J 1 means, it is not saying that, so F J J is equals to 1 means, you cannot say that nothing bad happens in the future, it can happen but it may happen after long time or short time, that is what now matters to you.

Student: That can happen in F J J less than 1 also. Or F J J less than 1 there is a finite probability that nothing (())(23:35)

Student: It is not 0, it can be 0.7.

Professor: F J J equals to 1, it is just saying something bad will happen but it the rate at which it is going to happen what matters to us. If the bad things are again going to come again and again very frequently, then your, also here that is the opposite, here these bad things are actually good things here, because we are talking about this n to be large. So, when n is, so what is this? When n is equals to large that means bad things happen only very very later, that means actually null recurrent is good for you. If the bad things are happening very frequently, that is a bad thing for you, that is the other way around here.

Student: Sir, what I am saying that if this here get transient….

Professor: If you are in transient, yes.

Student: F J J for example is 0.7. That means bad things can happen with probability 0.7.

Professor: Yes, and actually we are in a good case like and also like I mean it depends on the value of F J J, it also say that you may never see a bad case again with 0.3 probability.

Student: (())(24:55).

Professor: Exactly. So, that as a modeler you do not want to assume that, it is basically assuming that okay, some positive probability nothing is going to, bad is going to happen in the future, you have to acquaint it, okay, something may happen but that may not be that frequent. So, this is all again, the design choice is like how risky or how risk hours you want to go for it. Based on that you are going to set these values.

One more thing before we compute the other distribution, so let us say this F i J n we know this is not P i J n but if, but can I compute this quantity only using my transition probability matrix t p m? Why is that, if yes? So, what I need to know okay, X naught equals to i. I am not coming back to state J till nth round. Can I compute this probability using transition probability matrix? Verify that and this is just like you just argue that, yes, we can compute this probability F i J J n using only transition probability matrix.

So, all this characterization of whether my state is transient or recurrent and further positive recurrent, null recurrent, this all can be explicitly stated just by knowing your transition probability matrix.

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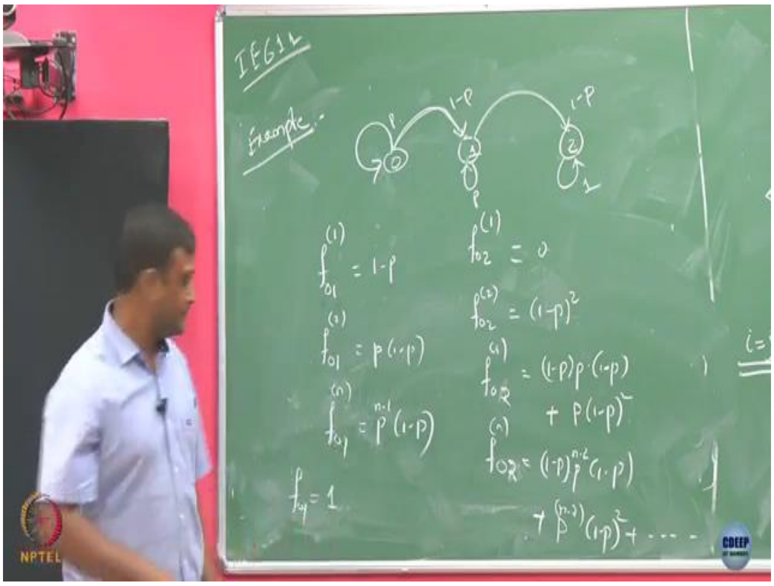

Okay, next, maybe before that let us look into this example, I know let us say I have been given some simple Markov chain with 3 states and I am going to express its transition probability in terms of the transition diagram like this. So, I have written in terms of transition diagram, you can also express in terms of the transition probability matrix, the same thing.

Now, can we compute F 01, 1? So, what is this say? That I start from 0, 1 and hit state 1 in 1 step, what is this probability? That is going to be 1 minus P. So, this is the only path first time I am going to do and now if I want to do it in 2 Steps, P into and if I want to do in n steps, P to the power n or P to the power n minus 1? And now can you give me what is F of 0 1, F 0 1, the summation of all will be what? So, this is going to be infinite.

So, it is going to be 1 minus P into summation of P to the power n, where n is going to start from 0 to infinity and that is, now just work out this and now what does this say? That if I start from state 0 that I go to state 1 is going to be 1, with probability 1 I am going to hit state 1 at some point and because of that my state 1 here starting from 0 it is going to be a recurrent one.

And now similarly, maybe you can just also, now let us do for 1, 2, 0, so what is this probability that I go to state 2 starting from 0 in one step? 0 and what is this is going to be? So the only way I can go from here to here is this route, in one step for the first time and then F of 0 3, is going to be what? So, I want to go from here to here in 3 steps.

So, one possibility is go from here and then stay here and then go here, so what is that probability? 1 minus P, into P into 1 minus P, is there any other possibility? So, you stay here in the first round on 0, then you go here. So, what is that value? P into minus P and is it possible to write generally this one for n? So, I want to do it in n steps, now in n steps.

Student: 0 to 1, then 2 and then…..

Professor: 0 to 1.

Student: (())(31:20)

Professor: No, we keep first time, it is not all possibilities we are looking at. So, I compute this then you have to just basically work out in how many possibilities. So, one possibility is just like go from here to here in one step, stay back here for n minus 1, n minus 2 steps and then go there, then other possibility is you spend n minus 2 steps there itself and then go here but there are other possibilities also. You can spend one step here, go here, then step, spend another step here like or whatever like.

So, there are many many other combinations which I can write down, so as you see that even for the simple things writing this can become too much of a combinatorial issue, you have to look at all possible paths that you can go from one state another to reach it for the first time.

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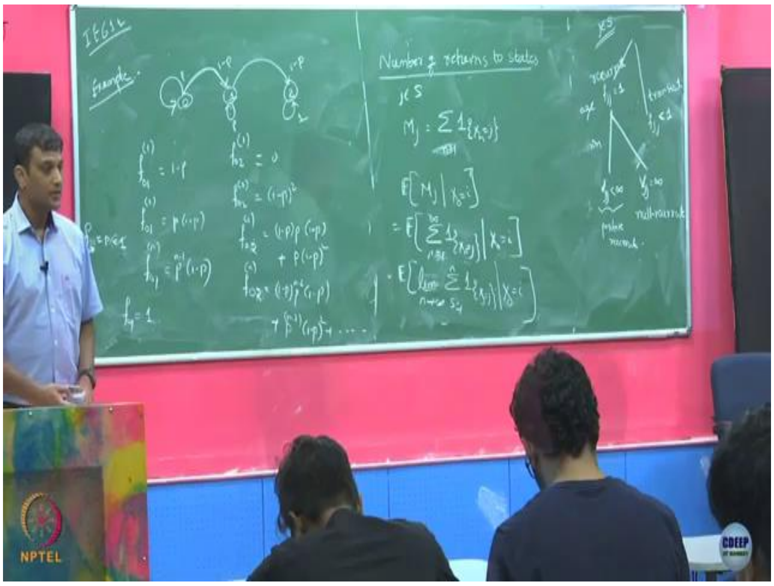

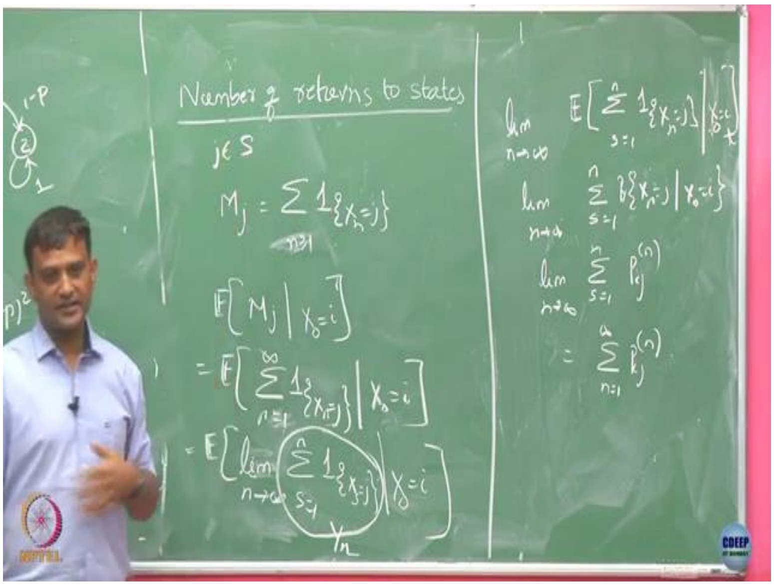

Now, let us compute this, so one thing is the number of rounds required to hit a particular states and other thing is when you are going to run it continuously, how many times you are going to revisit that state? So, revisit or return to that particular state, so suppose let us say I have a state J and I am interested in, yeah?

Student: For example, which are all the three states it is from?

Professor: In this…..

Student: I think only two terms will come.

Professor: Just to clarify here, to the way we have defined the recurrence, transient and all, we have stated like you start from that state and again see visit to that state but here what we have done is, we have taken some initial state which could be different from the state we are going.

So, if you want to, in this case you have said F 0 1 equals to 1 that means if you are going to start from state 0 then state 1 is recurrent in this. So, instead of this you can also compute what is F n 1, F 1 1 you start from state 1 and then again look at coming back to this in 1 round, 2 rounds or 3 rounds whatever. So, is it easy to check this? Let us say let us compute this F 0 0. Suppose if you start from 0, what is the probability that you come back to that state in one step? P, and then what is the probability that you come back to that in two states? 0. So, what is that in that case, this is going to be simply all other terms are 0.

Student: Y 0 P square (())(35:10).

Professor: You came up one times but once you left it, there is no return part to 0, you are not going to come back to that again. So, this is going to be simply P in this case, if suppose P is less than 1, what is state 0? Transient, that is obvious, ek bar tum idhar se hatke, state se then you may not come back to that state again and if P is equals to 1, in this case it is trivial like you will be always in that case.

So, like that you can compute for all the states and then based on that you can classify which one is transient and which one is recurrent and the recurrent one you can further classify whether it is null recurrent, or positive recurrent. So, to compute all this, all you need to tell you this transition diagrams or your transition probability matrix.

Now, look at this, what is this quantity guy is telling you? So, what is this? This is going to say 1 here if nth state is J, so what is this term is telling you? Number of times your hitting state J, okay in the entire process. So, that is why this is called number of returns to state J and now I may be interested in computing expected value of M of J starting from a particular state.

Now, what is this going to be expected value of, can anybody now see, see now this is a limit, so this nothing but what I can write it as expectation of, is this correct? So, this is nothing but n equals to 1 to infinity, let us assume this limit exist, limit as n goes to infinity, S 1 to S this one. Now, we are faced with, now I want to interchange this expectation and this limit, can I do so here, can I do so in this example?

So, let us say I call this as something as Y n, so I have the sequence of Y n's and I have limit as n goes to infinity of this Y n and now I want to interchange this, right now I do not know where this Y n the sequence Y n convergence in any of the senses, all I want to do is I want to check whether I can interchange this limit and expectation. When I can do this if at all I can do this, we can apply what research Monotone convergence theorem, you can we can, can we apply here in this case?

So, then so limit to expect, to interchange limit and expectation what all the results we have studied for a sequence of random variables? Yeah, bounded convergence theorem and dominated convergence theorem and?

Student: Monotone.

Professor: Monotone convergence theorem. So, to apply bounded and the dominated convergence theorem, what are the hypothesis we needed?

Student: (())(39:34).

Professor: That is one thing and we needed X n to be converging to some X n probability. So, we needed to know that sequence X n converges in some sense, but here and what is the last one? Monotone convergence theorem, to apply that we did not need to know what is the limit of my sequence, all to what is the hypothesis to apply my Monotone convergence theorem, non-negative?

Student: X 1 of omega should be less than equal to (())(40:10).

Professor: Yeah, for each sample point my process should be sample point. Is that happening here? Why is that? Summation of indicators. So, indicators are all non-negative value, so if you increase n for a given omega it is only going to increase and n increases. So, I can apply my Monotone convergence theorem and then interchange limit and expectation.

Then this one is a finite, there are only finite sum, I can further interchange this, given, and then I can interchange, this is going to be S equals to 1 into n, what is expectation of indicators at X n equals to J? Probability of X n equals to J given, what is this, is it X n equals to J given, X naught equals to i and what is this by our definition? P i J n and this is nothing but I can write it as n 1 to infinity.

So, my mean number of visits to state J starting from i can be expressed as simply as P i J of n's for n going from 1 infinity. So, let us stop here. Next class, so what we have just now done is, we are just written the expected value of this M j. In the next class, we will also try to see if we can derive by distribution of this M j itself, M j is the random variable here, which takes any value between 1, 2, 3 up to infinity, so how to find its distribution, we will see in the next class.