Introduction to Stochastic Processes
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Lecture 33 - Stopping Time

So, in the last class, we started motivating the need for conditioning on a random time. When we defined a Markov chain, the initial one, we defined it condition on a deterministic time and when we condition at deterministic time, we said that future is independent of past given that, stated that deterministic time.

Then we also discussed, maybe sometimes it is not always we are going to condition a deterministic time, sometimes we will be faced with to condition on a random time, then we wanted to see if the Markov property holds when we have to condition on a random time.

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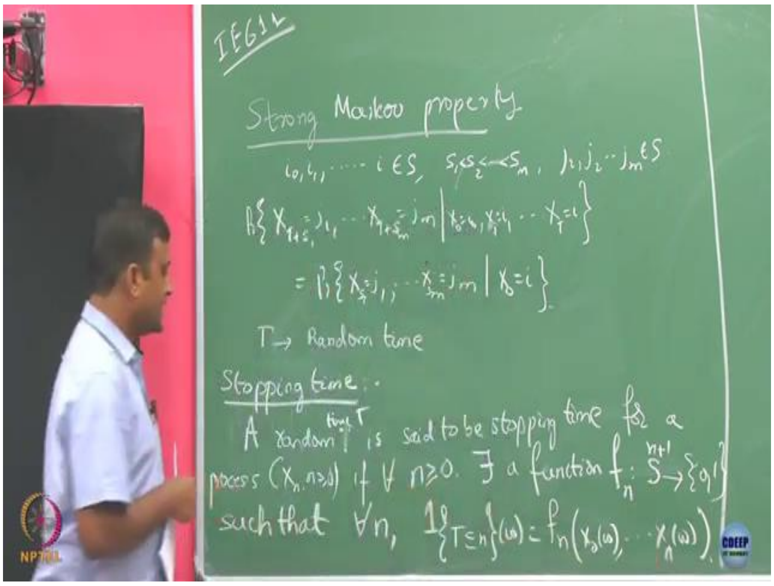

So, we saw an example where, when we are going to condition on a random time, the Markov property did not hold and then we said that if it holds then that Markov chain we are going to call it as and we are going to call that, Markov chain to satisfy strong Markov property which basically said that, X T plus S 1 to J 1, we get X T plus some S 1 is to J m given X naught equals to Y naught, X 1 equals to R 1 all the way up to X T equals to i.

What is this equal to? We said that is nothing but X of S 1 equals to J 1, all the way up to X of m equals to J m given X naught equals to i, as if from that time onwards as if the Markov chain begins afresh. Then the question is, here what is this? Here T is my random time.

So, I hope you guys solve the, remember the notations here, what I the way I have chosen this indices are this i 0, i 1, all the way up to i they are some states and also I have chosen S1, S2 all the way up to S m, they are all time indices, belong to time and also J1, J2, all the way up to J m they are all belongs to come from state space S, what S1, S2? Yeah, these are time indices, we have just said that S 1 is smaller than S 2 like this, they are not subsets, they are time, they are indices.

So, this is basically we are saying that whatever time you condition upon a random time from that S1 steps further and this S 2, sorry S 2 is further S 2 rounds from my time T. So, now if this T is a random time, our, now the question is whether this condition holds? For what random times? So, we have already seen a case where for arbitrary random times, this need not be the case.

So, if you recall our example, where we said that if I am T, my random time is one step before visit, one step before the time of my second visit to state J, then this was not the case, then the question is fine for arbitrary random time this property need not hold, but is there any special type of random time for which this property holds? It so happens that a random time if it satisfies the stopping time criteria, then this property holds.

Now, the question is, what is the stopping time? So, let me write this definition here, a random time T is said to be stopping time for a process, if they are all n greater 0 there exist a function. So, let us focus on the definition of, it is going to say that a random time T is said to be stopping time for a process, right now I am not saying this process is Markov process or anything, it is an arbitrary process here.

If for any n greater than 0, equal to 0, greater than or equals to 0, there exists a function F n, which takes n plus 1 states, remember S is what? S is the set of states, if you are going to take n plus n 1 elements into S, that maps it to either 0 or 1 that is F n is just like binary valued function takes 1, gives 0 or 1 based on n plus 1 inputs. Such that if you are going to ask this question on your random time T, what is a question, the question is, whether my T is less than or equals to n? If you ask this question, you need to say yes or no to this, to say yes or no I have made it indicator.

So, if this condition holds, this indicator is 1, if this condition is not correct or does not hold then this is 0. So, the left hand side is just yes or no kind of answer depending on whether this condition holds or not. If I can answer this question, using this function F n based on my n observations, I am going to make till the point n. So, what is this saying?

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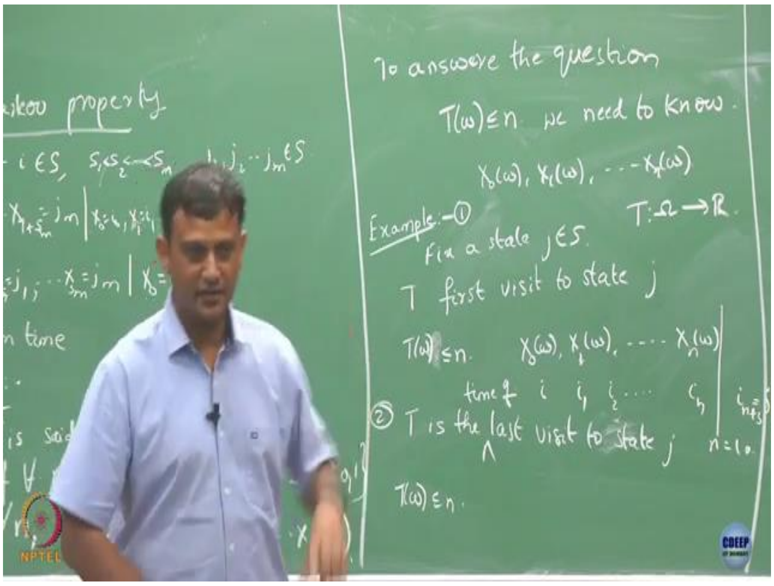

To answer question, so to answer this question, whether my random time T on the sample point omega is less than or equals to n, all I need to know is we need to know X 0 of omega, X 1 of omega all over to X n of omega. So, if this is just like expansion of this condition here, so if there exist some function F n, which tells you ok, if you want to know whether the T is less than or equal to n, at sample point omega, all you need to pass on is the n observations you have made about the state at the sample point omega. If you tell that then this function should answer this question.

Let us look at some examples also. Some trivial examples could be, before that let us see this. Suppose, let us say I fix a state J and I am going to now define a random time T to be first visit to state J. Now, if I define T to be like this and if I want to answer the question, whether T is less than or equals to n on a particular sample point let us say T of omega, is it enough, if I pass on you this much of state information till end?

So, let us take a sample, let us say this is a churn, this is the process evolving. Let say in round 1 I started with i, then I went to like another state let us call it i 1, then let us say I went to i 2 and all the way up to some i n state, and let say some of this that could be J here. So, the state J has appeared before n, so if I ask if I have shown you the sequence, can you answer this question? Yes or no. So, if you see that, okay, the J has occurred at the second slot itself, so then and if let us say n equals to 10, so whether T of omega is less than or equals to n, that is whether my first occurrence of state J has happened within ten slots, that is true.

By looking at the sequence is enough for you to answer this question. If whenever this happens then I am going to call this random variable, this random time to be what? A stopping time.

Student: Sir, (())(13:14).

Professor: Time T is a random variable, T is defined on what? T is a random variable from omega to R. If you are interested in a sample point and on that sample point you want to ask this question, whether on that sample point the random time is going to be less than n, what we are going to look is we are going to look at this sample path, you observed for that sample point.

So, this is a sample path for that sample point omega and then look at whether you can answer this question, you look at whether J has happened before, if that is the case then it is true. So, but this is a one case, where let us say J has happened, let us say none of the states contain J and J is let us say somewhere here i n after n plus 3, let us say J.

But when I ask this question, I am only going to tell you till this point, I am not going to tell what is the value of my sample point, sample path at X i n plus 1, i n plus 2, i n plus 3, they are not told to you, can you still answer this question, yes or no? Why? We are going to say, no, this value is not less than n, this has to be greater than n. So, either you say yes or no, then in that case no, this is not the case, the other is the case. So, your answer is no.

So, as long as I give you samples till time n or the first n samples, you are going to, you will be able to answer this question, either yes or no. Let us look into another example, let us say now T is the last visit to state J is the time of last visit to state J. This means after this J has happened J is not going to happen after that at all, is there meaning of this random time is clear to you? This random time T is the last visit of state J.

So, now if I ask the same question, can you answer this question based on your first n observations, why is that? Yeah, J might happened somewhere in between this, but it may still happen after that, that is not the end of happening of J, it may still happen subsequently. So, you cannot answer this question, either yes or no.

So, in this case this random time T is not a stopping time, we have seen another random time where we defined T to be one step before the second visit to state J, is that going to be a random time, sorry, is that going to be a stopping time? Check that, yes, no? Check that. Now, those who said no, can you tell me why that was not a stopping time?

Student: (())(17:20).

Professor: So, it depends to only when I have all the n plus 1 information I can affirmatively say yes or no, till that I cannot say anything clearly.

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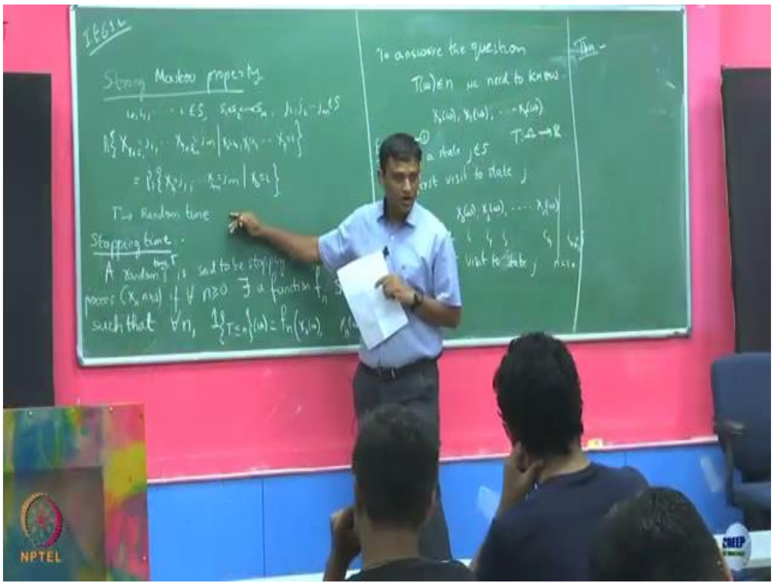

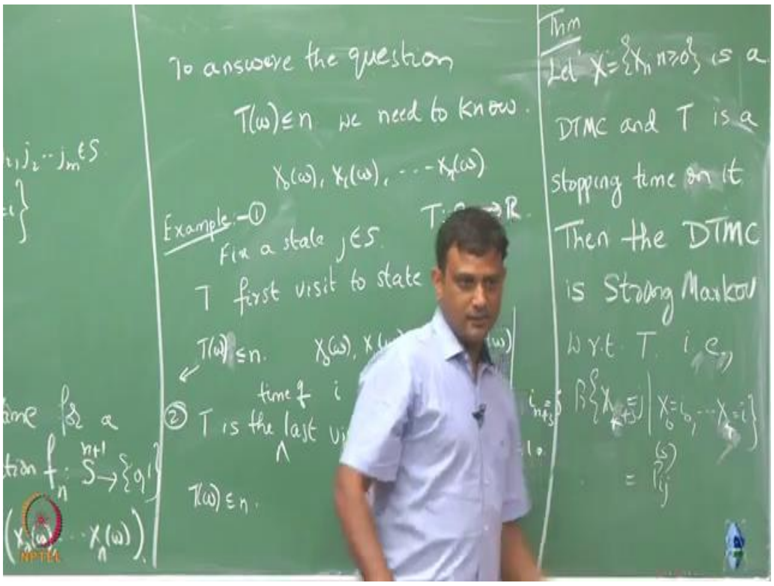

Now, the theorem says that, basically the result says that this Markov property holds if I tell you a priory that my T is a stopping time with respect to my process X n, with respect to my

Markov chain. So, let us say, here we are going to say that probability that X T plus S is equals to J given X naught equals to I naught, all that.

So, then I can write this probability here, even though this is a random time here, I could write this probability that I observed all the way up to T and I have noticed that I have taken state i at time T, then subsequently I am going to state J in the next S steps, that probability is simply P i J to the power S, what is P i J to the power S, this is the just a probability. So, now we are just saying that as long as this T satisfies this stopping time criteria, this property holds whether this T here is a deterministic one or a random one. Yeah?

That is a whole point of this, n was earlier deterministic. Now, that n is replaced by this random quantity T, T is the random time here, earlier used to condition on a n which was deterministic. Now, this is a random time here, you are a priory I am not telling you what is the small n, I am conditioning on any random T here.

Student: Can you explain that this T omega less than equal to n should be, information should be gettable from this n x n?

Professor: Yeah.

Student: This condition, how is it?

Professor: This is for like we have replaced this, this is what, this is a definition on T here, whether T is a random time, we are saying if this T is a random time, this property holds. That means to define this T to be less than or equals to n, I mean to verify this quantity T is less than or equals to n, all I need to see is the n observation, the first n observations. If that is the case, you can whatever this T it is, this is a random time, it can take different, different values.

If you are going to condition on that random time T and now look at what happens in the next S states, that transition is simply governed by your S step transition probability matrix, that is what we are saying.

So, earlier what? So just forget this, earlier our definition was this, right? If n is a fixed deterministic quantity, so the probability that i equal to state j and n plus S given all this n, first we said that it only depends on X n i. It does not matter on the previous one and we had called that P i j S. Now, we are saying instead of fixing this n, it could be arbitrary, it could be any T, a random quantity. We have already seen that if this T cannot be arbitrary T for this property to hold, you remember when we defined T to be 1 step before the second visit to state, particular state J, this was not the case, that did not equal to P i J S.

But now we are saying that this T random time has a special property which we called as stopping time then this is true, even though this T is also time index but this quantity is a random here. We are to apply this all, I want you to guarantee is first, tell me T is a stopping time. To verify that the T is the stopping time, you are going to do whether T of omega is less than or equals to n. Once you did this then you have satisfied the hypothesis of this theorem, then we are just going to say that ok that then this property holds.

And to apply this you need to have a stopping time that is where your question of whether we are going to say check T of omega is less than or, that is already we have, we have to have verified before we apply this.

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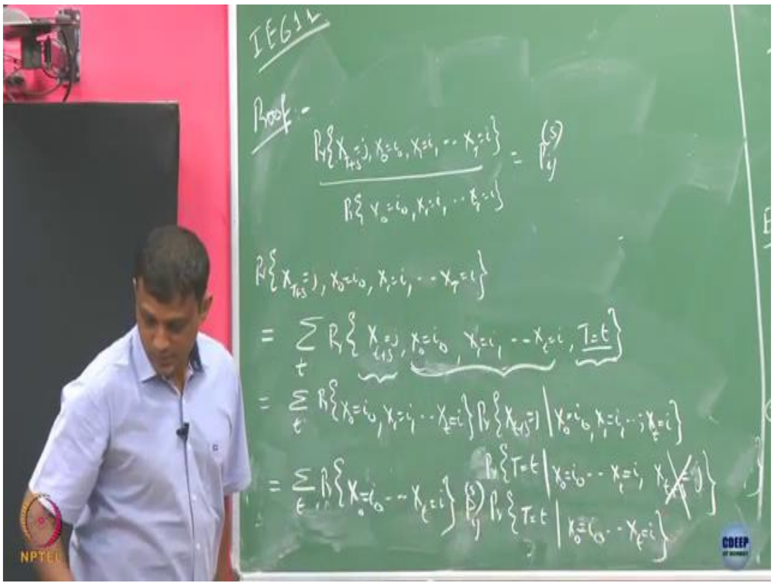

Now, how to prove this? Why this is true, so basically to prove this so what we need to show is, just reorganize this, this is a conditional distribution on my left hand side, this conditional distribution I will write as a joint distribution. So what I need to show is this is equals to P i J S. So, this left hand side I have just rewritten in this ratio format, this is just definition of my conditional probabilities.

So, now let us start looking into the numerator here, I want to show this and T here is a random quantity, random time and we have specifically further assumed that this is a random time. Now, let us try to exploit that property. So, what I will do is this T here is a random quantity but it takes values, integer valued, it takes integer values, it can take 1, 2, 3 like that.

So, what I will do is I will just try to add it for all possibilities. So notice that I brought in this T equals to T event there and I have just added it up all possible T values. So, these are still n equality here. Now, let us try to further unwind it. What I will do is, I am going to group it into, I am going to group in these parts, x naught, and then, probability given what?

Student: X t equals to j is not the (())(28:31).

Professor: So, first I took this and I wrote this as a probability, then this probability condition on this, then this probability condition on the previous two, all the way up to X 0 to X T and then X T plus S equals to J. So, first quantity, I am going to written like this, what is this quantity now? We are saying that now see that T is the deterministic quantity here, I have written this thing.

Student: (())(29:32).

Professor: Yeah. So, this is i go to state J at time slot T plus S given that at time T, I am already in state i and all the previous states are also given, but if I already assuming that this X 0, sorry this maximum sequence is X n sequence is DTMC. So, what is this quantity is going to be? Of S. Now, let us focus on this, now what I have given? I have given my DTMC all the way up to time T and further I have also given my DTMC value at T plus S.

Now, to answer my question whether T equals to t, now I have to basically compute probability that T equals to t, or this is same as knowing whether my T is equals to t if I know something about this. Now, if I am going to apply in my, stopping time property on this, to answer this question whether T is equals to t, what all the things I need to know here? I need to know only till t. Does it depend, this T equals to, whether T is equals to t does it depend on X T plus S? No.

So, I can get rid of this term here. To answer whether my random, my stopping time T is equals to t, all I need to know is only the sample still here. So, this guy vanishes, and now I can write it as simply, I am just rewriting this.

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Now, let us reorganize this a bit, I am going to pull this guy P i J outside because that is independent of T, summation t equals to 1 to infinity, this is basically for all values of T. Now, focus on this, what you have probability that X i equals to i naught, all the way up to X T i and then you have probability that T equals to t, conditioned and only these many terms.

So, now can I write it is a joint distribution, joint probability of these two quantities? So this is going to be probability that T equals to t, and now further this has been summed over all possible values of T, this probability. So, if I remove the summation then what is this quantity is going to be? So what is this quantity is going to be? See when I started this, from this step to this step I added the summation, because I have introduced this event. Now, I want to just reverse that process, so then what is this quantity?

Student: The quantity will become (())(33:24). That term will be equals, X T will become (())(33:30).

Professor: Summation will be not there but what is going to remain here? Till then X equals to i. So, are we done? Now, we have this quantity, take this quantity on the left hand side, this is exactly equals to P i J and that is what we wanted to show. So, what is that? Like the crucially we use the fact that my random time is a stopping time at this factor.

So, if I, because I could do this I could pull out my P i J term in this fashion. So, whenever we want to like whenever we are not certain that at from which point I want to see the future, then I definitely need to make use of this property like because if the point there at which I am going to look is a random quantity and then I want to use the standard whatever the Markov properties we have, then if I want to do that, I should first ensure that the random times that I am conditioning on is a stopping time.