Introduction to Stochastic Processes
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Lecture 32 - Strong Markov Property

So, when we basically went from this standard Markov chains, where we are conditioned upon a particular index and asked whether the future from this time onwards is going to be independent of past. Now, when we just instead of asking for a deterministic time when you move to a random time, that slight technicality, but that technicality is important to handle because most of the times we will be facing such cases.

For example, in the example I gave here you are not always going to like if you go to a stockbroker and if you are interested in stock trading, you will not just say that you sell my shares on hundredth day, you will tell him to sell your shares when its value is above certain number. So, then only you feel that it is profitable for you. So, that is why we have to worry about conditioning on a random time. And also like in the aircraft example I give you, so, you want to always keep track of what is your fuel level or whatever some other parameter and you want to say that okay, if this parameter exceed something, you will see how, what things happen in the future. So, that is why it is important to handle this technicality.

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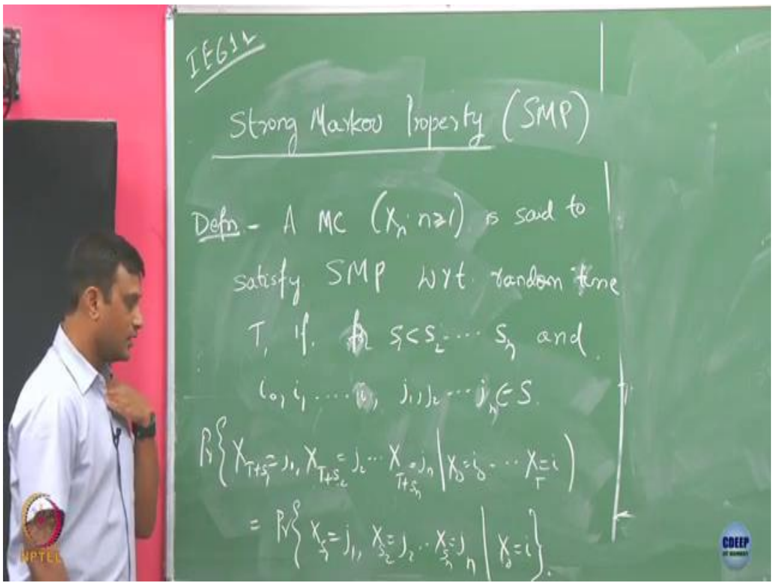

And to understand this, we have to kind of slightly introduce some notions called, we want to bring in something called strong Markov property, this S n, sorry j n. So now what we want is we are going to define a Markov chain to satisfy a strong Markov property with respect to random time T. So this strong Markov property is defined with respect to some random time if this condition holds, what is this condition? Till random time T you give all the state of my Markov chain then asked the question, from that time onwards if you go S1 steps ahead S2 steps ahead and S3 Sn steps ahead, that distribution is going to be equal to this probability where this probability says that….

Student: (())(5:48)

Professor: No. Okay. Now that is still correct. So this says that you can, as if now you can think of your Markov chain starting from this time and then going to in the next S1 states to step j 1 and further in state in time S2 to j2 like that.

Student: There should be X T equal to i T.

Professor: X T, No, so, that has now become your starting point.

Student: 0.

Professor: That will become your starting point, yeah. Whatever that XT, the time where you took i that you will now take it as your origin and from there you start going in the next S1 step you take j2 and in S2 you take j2. So, what it basically Markov (prop) this strong Markov property is telling that if you go into condition till a random time, then the future you can think of as if your origin is that point and from there you are looking at the same amount of future ahead.

So, what you did, here till X T you looked into, then the future you looked into further S1 rounds in the future and then another S2 rounds in the future. So, here just saying that then you can imagine that the process has started right from there, then look jumping to j 1 in S1 steps ahead in the future. So, like that basically what it is telling is if strong Markov property holds the random time, whenever condition holds whenever observation is giving till that random point, the future I can think of as it I am starting a fresh Markov chain is starting a fresh from that state. Yeah. This part is deterministic, this part is deterministic there is no randomness involved in this case.

Student: Sir, that is the property of homogeneous Markov Chain.

Professor: That is the property of homogeneous Markov chain. So, what we did say, it just does not depend on n, but how does it is connecting here this is with respect to random time. And we are now talking about, you look into the distribution starting from that point.

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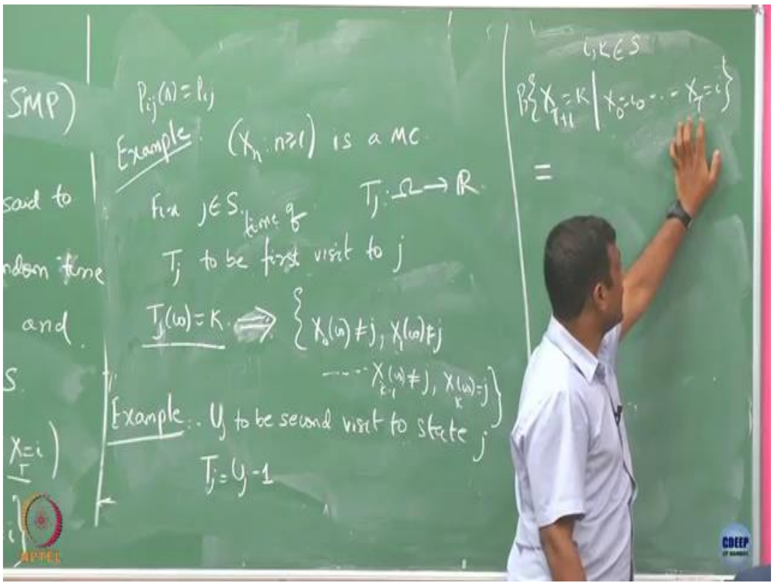

So, what did Markov property told you? Sorry, homogeneous such that P i j n this P i j for all n and that is not this definition, we are going from step state i to state j from step n is the same irrespective of which step you are looking at. So, here it is about if you are given me all observations still time T, I can now think of my process starting at that point into the future. This is the definition, so this is what we are going to call it as strong Markov property. If my process at all satisfies this, my Markov chain I am going to call it as a strong Markov property or a strong Markov chain.

So now let us see, is it true that at any random variable will, if I have a Markov chain, is it like if I take any random variable, T which is integer value, this property will be true, true? So, let us look some examples. Let us look at some examples of random times, okay T. Let us take my Markov chain X n, now let us fix state j.

I am focused on one particular j, and now I define P to be first visit to j, we are going to just take one particular state and now you will be defining your T to be first visit to j. Your Markov chain starting maybe at some point, let us say hits state j. Whenever it hits, that time slot is given by this T. So when it is going to hit j, that depends on your transition probability matrix and also on your initial distribution, or maybe like from which point you are going to start.

So, T here is a random time. Now, I want to ask this question what is, what is So, T is now what? T is a map from omega to natural numbers. Now, I want to ask this question for some particular sample what is I want to ask this question, what is this when this is going to happen, when this event is going to occur?

Student: If we use (())(11:52)) not depending on previous..

Professor: T is simply a random number. It has nothing to do as of now with the Markov chain. Markov chain is there, now I am just defining T on this Markov chain as follows, T is the to be the time of first visit to j. Now, I want to ask, so maybe like let me make its index by j so that this j was more explicit.

So if this is the case, on some sample point, that my random variable takes value k, what does that mean? It must be the case that X of omega is not equals to j, X1 is not equals to j all the way up to X k minus 1 of omega is not equals to j and then X k of omega is j. So what is this mean? I am looking at this event, when I write to pehla, this is not first state, 0 state is not j, first state is not j. All the way, like your second is not j, even k minus 1 state is not j, but kth state is j. So this is the time, T is the time of first visit, time.

Student: T is uniformly distributed.

Professor: T, I am not talking about its distribution, I am just defining a random variable. This distribution right now, I am not worried about, so is this random variable definition clear to you? This random variable is just telling it is focusing on a particular state and just state it is looking it to it and seeing when I am going to first hit that state.

If it is the first hit, it must be the case that previously I should not have hit that state, right? So the previous time indices, so previous time indices, k minus 1, k minus 2 all the way up to 1 and 0 these are not state j. They could be anything else but on the kth state it is going to be j.

Student: Previously you said that at T j it is equals to k means T j equals to k, then (())(14:33)

Professor: I am not selecting anything here. I am just defining what is the meaning of this. So, T j is a random variable, right? So T j is what? T j is from omega to R. Now on a particular point omega, what is the meaning of T j omega, value is k, what does this mean? So, our definition is the first time it visits state j. The time of first visit so, then in that case if this it has taken this T j has taken k, value k on sample point omega, this must have happened. Only in the kth round it would have hit j not before that. Yeah?

Student: If this is happening then?

Professor: If this happened then that is exactly T j equals to k on that means.

Student: (())(15:53)

Professor: This is, you do not need to write so, this means yeah, fine. I mean this is the basically definition, right? Definitions means they are both way, okay? Implies, yeah this. So now let us see. Okay, I can always define some different, different random variables like this which are random times. I could say instead of first visit to state j, I could say T j is the time of second visit to j. That means it is going to look, keeps on looking when first time happens and then looks, okay then second time happens, that time slot, it is going to give it as value of T j.

Like that I can define many many random times here. But the question is, is it true that for any random time this kind, this property is going to be satisfied? If so happens that not really it is the strong Markov property is not necessarily need to be satisfied by any random points. So to see that, let us look at another example. So I am going to define U j.

Student: U j is the time of second visit to state j.

Professor: Yeah.

Student: So will that hold?

Professor: Yeah, this condition? I do not know. I mean, I do not know because right now, if this is if it holds, then I am going to call it as strong Markov property. If it holds, whether it holds, that is a different question, when it holds or for what random points it holds? I am just telling you that I am giving you examples of random times, I am not saying anything about whether this random time satisfies this property.

If it satisfies at all then we are going to call Markov chain to be strong Markov property with respect to that random time. Okay, so this is just an example of a random time. So now let us take another example where my random time is second visit to my state j. And now U j I am going to define my random time to be just one step before this. Let us say it is going to hit second time my state j, and the one step before is my random time. I will be interested in one step just before it hits the second time the state j.

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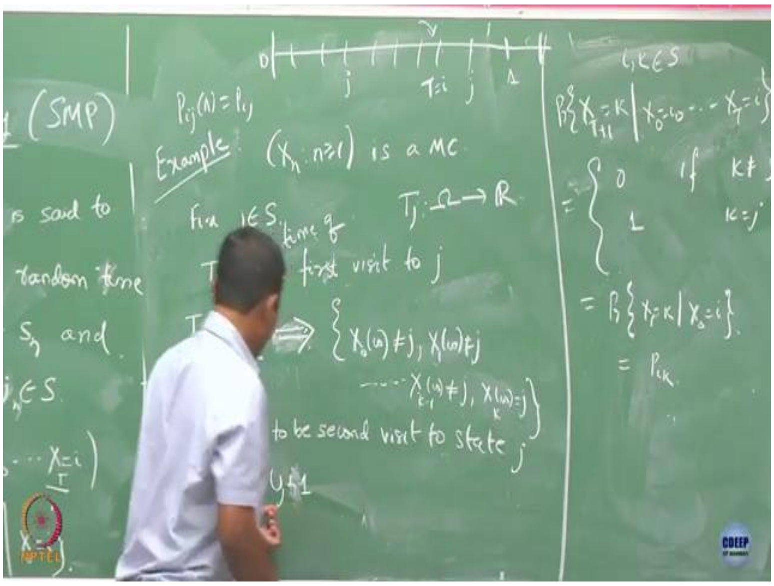

Okay, now let us see, I want to compute this probability. Okay, let us take some i and j and now let us try to understand what this probability is. So what we are saying, so T is what? T is specific to particular state and we are saying that one step before, so what is the T giving you? Let us say you have this time slot. Let say j has happened here and j has happened here on these time slots and the T is basically giving you this time, just one step before my second j and now when I gave this right till time T, I have given you and let say so, this T is going to always be this in this realization just one step before this right and we have told that time, this state is what? i, by this definition what is X T plus 1 is going to be? j.

So, if this k is not j, what is this probability is going to be?

Student: k not equal to 0.

Professor: And if k is equals to j, this is going to be 1. If k is equals to j, by definition this state has to be j, right? So by probability 1 and if k is not equals to j, then it cannot happen, this event cannot happen. So then that is why it is going to be 0.

So, this probability is 0 and 1. But is it going to be same as what we want according to this definition? So according to this definition, we want it to be probability that x 1 equals to k given x naught equals to i or basically this is same as saying P i k, that is my definition. But P i k can be anything instead of being 0, 1, right? I can construct a Markov chain where this P i k can be anything.

But if I am going to use such a random time, what is happening is this probability is not equal to this probability. So at least with this respect to this random time, my strong Markov property, whatever I am defining it here, this is not satisfied. So let us see, what is making this strong Markov property failed in this example.

So what is this, this random variable T j here in a way already anticipates what is going to happen in the next round. So, T j is one step below my second visit. So, that means I already know the next step is going to be j. So, it is kind of anticipatory here. So when such a thing is going to happen, my strong Markov property is not satisfying. So we can anticipate in that case whenever my stopping time is such that it is not anticipatory in nature, maybe my strong Markov property is satisfied. So you see that already any random time is not going to satisfy my strong Markov property.

Student: Even if we take U j plus 1.

Professor: Okay, let us take it U j plus 1 and then so this is one step after j. So, you are already looking into the future in this case like, so, I mean, this is not a good definition to apply here. So, suppose you have been told that my state is i of in the next state after I visited my state 2 for the second time. So, in that case it may be I do not know, like this could be satisfying or not satisfying, I do not know.

But when it was like this, it was clear that this should happen when you have in this case. So, when you have plus here, yes, you have been told that after visiting this, here, you have hit state 1. And then, after this what is going to happen? I do not know. That is the question you are asking here, right?

Student: In this case, first he addresses (())(25:10)…

Professor: But here you are just take immediately the next slot so, U j is what? U j is exactly this where the, you have hit j for a second time and then you are telling T j to be immediately in the next slot to either say direct beech mein kuch nahi hora hey. So then you have to see if you want to apply your definition, what have been told is, at this point your state is i. Now what you want to ask what is x T plus 1 is going to be k, that I cannot say anything with this. So that is why I cannot, I do not know what is happening here. But if you have defined like this the previous case, it is not going to happen. Okay, let us stop here.