Introduction to Stochastic Processes
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Lecture 26
Law of Large Numbers

So now let us look at some of the important results in this probability called Convergence in Weak Convergence and Strong Convergence. So, did we already define what we mean by iid sequence or we have not yet formally defined it, we have said it. What do we mean by iid sequence? Independent and identically distributed. When we have a sequence of random variables. We actually did this when we are trying to define joint distributions of sequence of random variables.

So, what we mean by independent that, so if I have a sequence of I have a random process, I am going to say this is to be iid, if each of my random variable is independent of others and all the random variable in the sequence have a same distribution. This is as simple as that.

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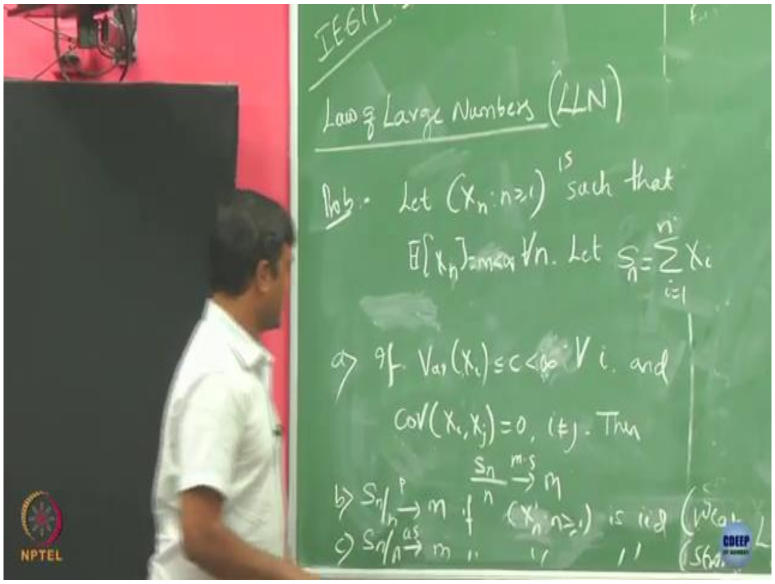

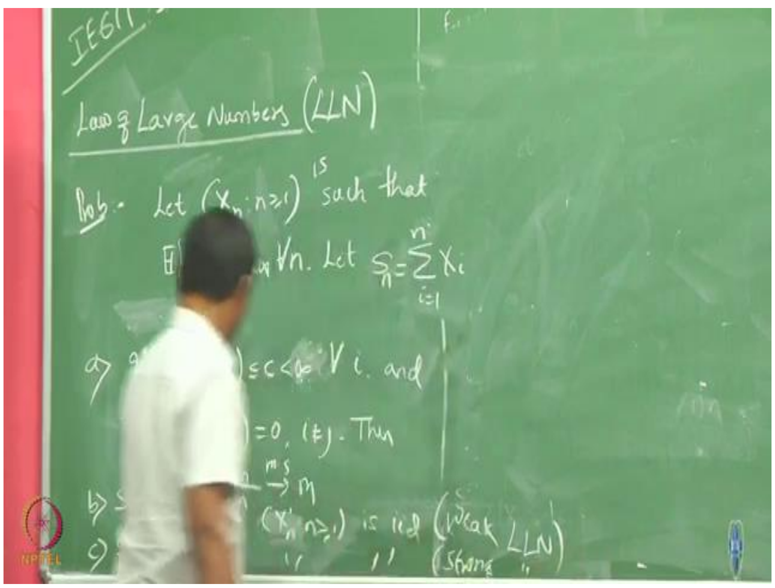

Now let us talk about, that, we have the following results. So, let Xn greater than n greater than or equals to 1 is such that expectation of Xn is finite for all n and then we are going to define Sn to be summation of Xi, i running from 1 to n divided by n. So, this is the average of the first sum. This is the average of first n term. Then if, so let us say now I have this given sequence of random variables where all of them have the same mean that is the expectation of Xn is going to be m for all n and m is assumed to be finite. That is mean of all these random variable is finite.

And I have defined this Xn which is the average of the first n random variable. Then we are going to say that if further each of my random variance of my random variables is bounded and their covariance of any pair of random variables is 0, then this Sn by n. So, I have to be careful here. I have to I am going to define Sn to be simply Xi, i to 1 to n.

So, this is just as sum of the first n random variable and if I am going to look at their average Sn by n that converges to this m in the mean squared sense there upper bond they are bounded by the same constant. They are uniformly bonded but need not be same and they are saying there pairwise coherence is 0. What does this mean?

Student: They are in (())(6:29)

Professor: So, this is uncorrelation. They are saying they are uncorrelated. That means they are pairwise independent. It's not they are Independent. This sequence in this bullet a, we are not saying this sequence Xi are independent. We are only saying that they have the same mean and their variance is are bounded. Then the second point says that if this sequence of our iid. In that case what? They are all independent in that case, they have to behave the same necessarily a mean and also they have to necessarily have to same variance.

If that is the case then they convergence in probability to the same value mean. And one can show a stronger result under the same iid assumption that this sequence convergence in almost sure sense also. We know that almost sure is a stronger notion than convergence in probability. So, if you can show c that already implies b.

So, this part the b part often called as weak laws large numbers and the last part is called strong law of large numbers. Let us try to understand the first part. Why that is true.

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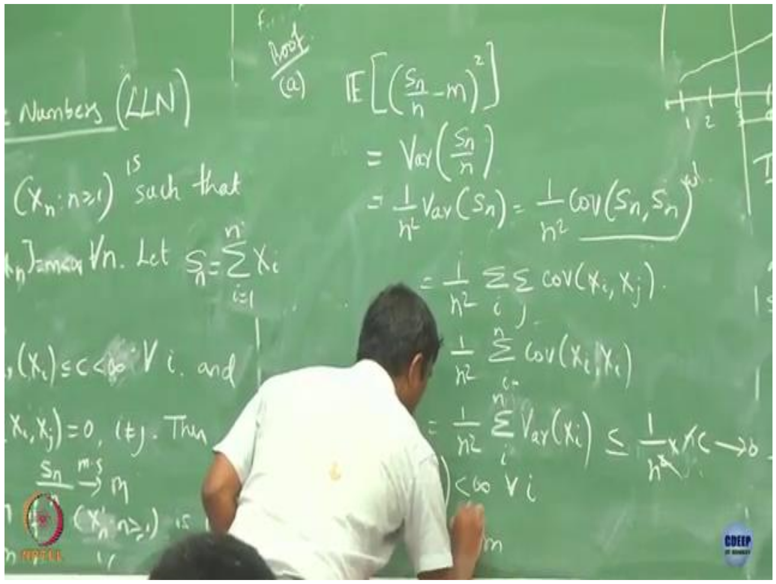

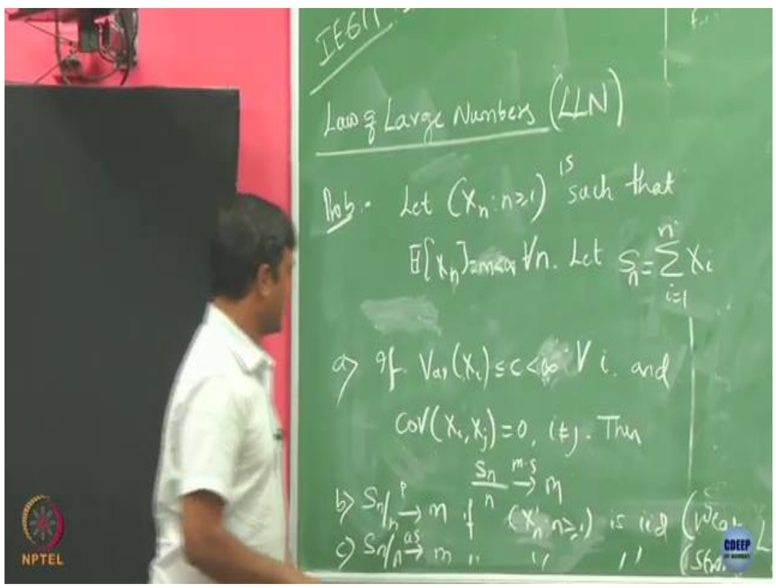

So, if I want to show that Sn by n goes to m in the mean square sense, what I need to show I need to show that off Sn by n by m. This goes to 0. If this goes to 0, then I can argue that Sn by n converges to m in the mean squared sense. So, what does the mean of Sn by n? Before that what is the mean of S of n, Sn it is going to be n times m. Because I can just, this is nothing but the, expect some of the n expectations where each one is value m. So the expected value of Sn minus n is already m.

So, if you say nothing but then in this case this is nothing but variance of Sn by n. So if this is variance of n I know that this is nothing but 1 by n square of variance of Sn. And we already know that this we can write it as 1 by n square. This is coherence of Sn with itself, correct. Did not we say that variance is nothing of a random variable nothing but covariance with itself. Now, if you expand this so Sn is nothing but for sum of n terms here, this is also some of n term and we know how to find the expected, covariance of such terms.

So, if you are going to expand this what you are going to get is, we did this exercise I think, sometime back the class. That value turns out to be, simply covariance of Xi minus Xj and we know that when i and j are different, we are going to use this uncorrelated ness property. And because of that if you simplify this this is nothing but covariance of Xi, i 1 to n and now let us coherence of Xi with itself this is nothing but its variance and what we say variance if Xi is upper bounded by?

Student: c

Professor: c, we are going to make use of this assumption that we are making. So, if you do this, what is this going to be so c by n and this c we are assume to be finite. So, if I let n go to infinity this go to 0. So now is this true that this guy Sn by n convergence in means squared sense. So, that is what we have exactly proved that we have showed that as n equals to infinity this goes to 0. So, fine As long as my variances are bounded we have the same means and pairwise my random variables are independent. The convergence in means squared sense. What about this part. How to prof b? Can we say can we say something about b using a part a? So, when does convergence in mean squared sense implies convergence in probability?

Student: Always implies.

Professor: Always implies? But now I am assuming I am trying to state this b for the case when my sequence is iid. So if my sequence is iid I know that this already holds. This already holds because all of them have the same distribution. So there means are necessarily same. What is not guaranteed is the whether the variance is going to be uniformly there are all going to be same. But we do not know where they are going to be finite. They could be infinity as well, variance.

So, if in this case for this case b. If I make an extra assumption that in my iid, they are iid but further I assume that the variance are finite. Then we are done then part a already automatically implies. So part b if variance finite for all i then, Sn by n converges to m in probability. We just argued that. In this case yes if my sequences is iid in addition if I assume that my variances are finite all this all these things in my part a are already proven all these conditions in all the assumptions in part a already holds.

Then under this assumption, we have shown that it converts in mean squared sense. And we already know that if it converts mean square sense that automatically implies convergence in probability and then comes a case what about if this variance are not necessarily bounded. They could be infinity. So, that makes a bit more analysis. So, we will skip it. Actually that can be done. We will do a proof later which will have a similar flavour that you can use to prove this part also. Then comes the last part. We want to know (())((15:11)) theorem that this sequence Sn by n converges in mean squared sense.

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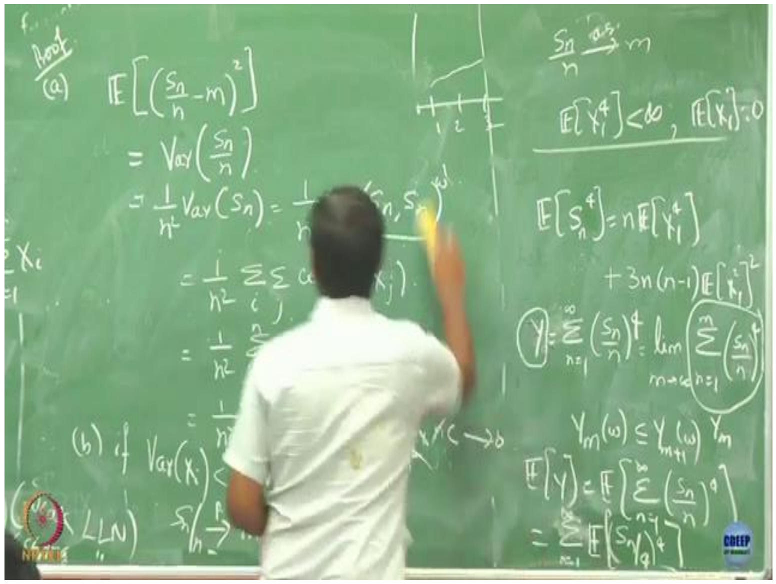

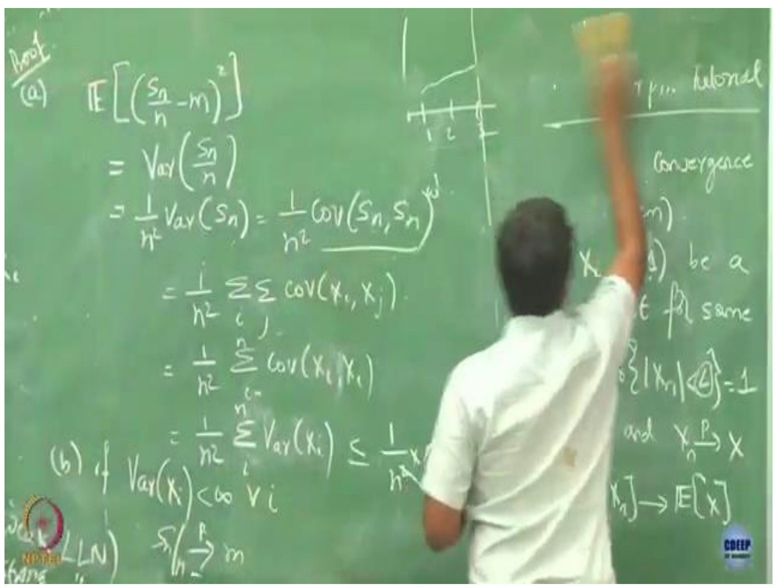

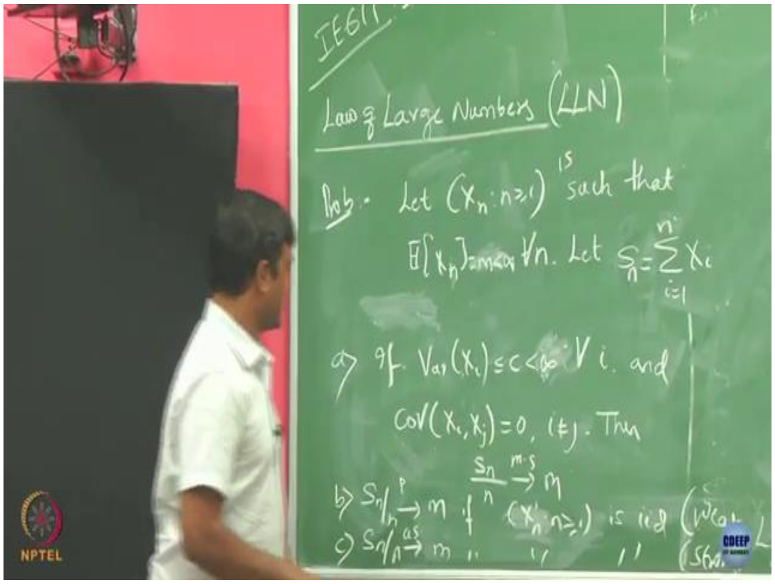

So, again in this case I mean it's, to prove it in gentle in very generality. It's going to be difficult. We are going to argue it like the way we did it for part b under a restricted case to prove this. Now we are going to assume that. My fourth moments are finite. So, I am only writing the fourth movement of x1 is finite. That means the fourth moment of all the random variables is going to finite. Because I am assuming iid sequence. So, we already I think sometimes discussed. We know that the fourth movement is finite. Then all the moments lower than this are also finite, fine.

So, if we can show this under that assumption that already implies part b. Because the part b only required variance to be finite which is a second order statistic. But whereas I am assuming a bound on the fourth order moment. So, if I assume this the second order moment already finite. So, that is already that is what I have used to sure convergence in probability. Now let us see how if I assume this. Why is that convergence in almost sure senses true, again.

So, this is where the proof monotone convergence comes into picture. How it comes, first thing to note is Sn which is the sum of all the random first n random variables. If I am going to take the fourth movement and if I am going if I apply further my iid assumption on that. It simplifies to. So and also I am going to assume that expectation of x1 is fine 0. I am going to like to simplify the argument we are going to make this assumption is this fourth moment is finite. Second is the second, the mean is going to be 0.

Student: (())(18:29)

Professor: If I say x1 is 0. That means mean of all random variables is 0. . Because this is an iid sequence. So, we are proving it under this iid assumption. But x1 of x1 is not equals to 0. This expression becomes too long. So, to just look get rid of, see we have this x4 term x2 term then comes a join term x1, x2 all pairs. And also we have to write too many terms. If expectation of x1 is 0 that simplifies and we get this.

Now I am going to define y to the sum of Sn by n rest to the power 4. So, what is this? So, this is how to interpret this, this is how to interpret as limit m goes to infinity summation n1 to m. You understand what is the limit of the series. It is not a sequence here it is a series here. That is like as I am going to infinity n. We have summing n and we are getting. Now why this y is well defined by this I can define like this. This is a limit, I can write a limit when this exist. Now notice that Sn by n. I am raising to the even power. So, all these terms are going to be non-negative.

So, because of that if you are going to look this. So, let us let us take this as yn. So, let us say this is now a sequence ym. ym is defined like this is this sequence ym is monotonically increasing? Yes, because it is adding more and more positive or not negative terms. And that is true for any omega. You take omega for that point and if you do this so for each omega I have a monotonically increasing sequence. So it will converge maybe to infinity but it will converge.

So, that is what that limit I am going to that limiting value I take it as a y. Now by defining my y as the limit of this ym and the way you have defined it is true that ym of omega. If he is going to be better than y n plus 1 omega monotonically increasing sequence. If that is the case. Can I invoke my monotonic converges theorem and can apply expectations on both sides and it will change the limits. So what I want to do is I want to do is expectation of y is nothing but expectation of this quantity. But now this is a limit because I have argued that this is a monotonically increasing sequence. I have the liberty to interchange my limit and expectation.

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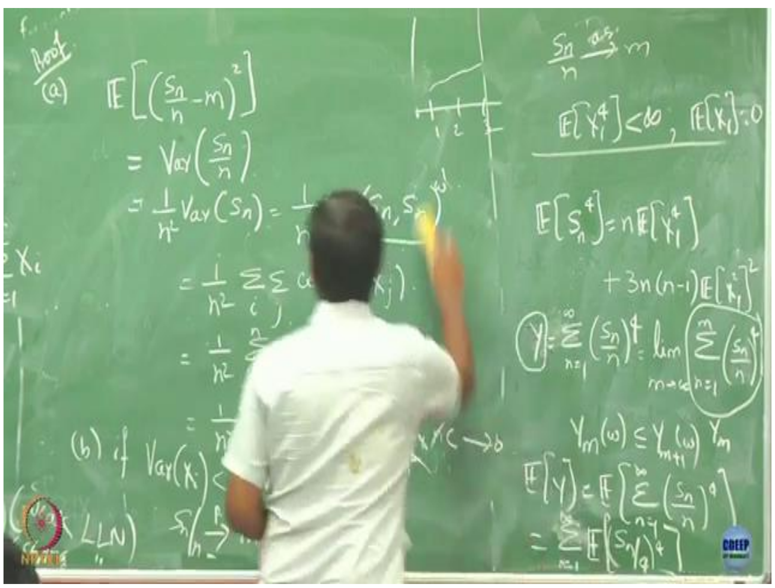

So, what we just said is expectation of y is nothing but fine I mean I am just I am doing nothing. I am just playing with definitions that have used. And that is fine if you guys have lost already, what are the definitions we are using. So but just see that what are the things that we are manipulating here? Now I am going to use my Sn relation from this. So if I am going to do that the expression I get is. Now here what I am going to invoke the condition that this fine this fourth moment is going to be finite.

So, I have invoke this condition that expectation of x1 is 0. When I wrote this expression here. So, had this expectation of x1 is not equal to 0 that would have been many more terms. Thanks to this assumption I could only write it simply like this and now I am going to assume. So what I have assumed is this expectation of the fourth moment is so this fourth moment is finite. That also meant that this second moment is finite. This guy is finite here, this guy is finite here and this guy here is at most n square and this guys is n. but on the denominator I have n to the power 4.

You can verify that this series is going to, so just let us say I also do not know what tell me but let us write it. So, this is going to be expectation of x1 4 divided by n cube plus… So, this is going to be what 3, 1 by n squared minus 1 by n cube and expectation of x1 square 2 and I do not think this is this. This is this is not a sequence, like this is the sum. So, this is going to be this series we have. So, this sum is going to converge to some finite value. You can check that because I know that 1 by n square summation 1 by n square converges.

So, summation 1 by n cube should definitely converge then and similarly this 1 by n Square converges and this 1 by in cube also converges. Their difference is also going to be some finite. So whole of this quantity here this series is going to be finite. I do not know what is that quantity. But what we are trying to argue is that this expectation is going to be finite. So if this expectation is finite. We know what probability that y is finite is going to be happening. Probability 1. Yes what are the conditions? We have we have establish expectation of y is finite. If expectation of y is finite, it must be the case that that y takes finite value with probability 1.

So, this is always true. If expectation of y is finite then this is true. But other is not always true. So, let us quickly complete this. Now the last point we need to argue is. How the way we have defined y. We have defined it like a like this right series. If summation of an converges. So, let us if an converges. Let us say an converges to some quantity what will an look like.

Student: (())(28:10)

Professor: Yeah, what when the an will converge to, this is going to converge to 0. So, by this we know that my Sn by n to the power 4 it converge to 0. Then I can take the fourth root of that and can also argue that Sn by n converges to 0 for all omega. Yeah.

Student: (())(28:52)

Professor: I have not yet said that, like so if y is going to be finite with probability 1, then Sn by n. That sequence is going to converge to 0. So, now we have to argue that this convergence implies that almost sure converges y is that because y being finite happened with probability with 1 and that is what we are using this implies this. If this is all very preliminary to 1, what can be the probability of this? It should be one? Then what does. What is that? Then what in what sense it converges.

Student: Almost sure

Professor: It converts in almost sure sense. So, is that clear if this is y is less than infinity it must be the case that this sequence should converge this event should happened and this should event should happen with probability y. So, this is the standard result in analysis y not. So, just to you take the contrapositive case suppose an does not converges to anything it diverges. That means an's are going to infinity then do you expect this to converges all the series no way.

Student: (())(30:41)

Professor: yeah, it can converge to infinity but let say we are talking about, we have a case where this is finite.

Student: (())(30:53)

Professor: Yeah, so it could converge to infinity but we have specifically shown that this is a finite case

Student: (())(31:00)

Professor: No it's not necessarily always the case. For example take this case an is to be 1 by n summation 1 by n goes to infinity whereas 1 by n goes to 0. So, let us stop here.