Introduction to Stochastic Processes

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Lecture 25

Convergence in Expectation

Let us continue our discussion on Convergence of Sequence of Random Variables. We have talked about our different notions of convergence. In the last class, we looked at Cauchy criteria. What is the other thing we looked in the last class, other than Cauchy criteria? So, we looked at some other means of other conditions or other properties that one can verify, to check whether a sequence converges to a sequence to us to a limiting random variable.

One was this correlation property of the random variables. So, we move on now, like we want to now study, fine. We have a sequence of random variable whether they converge to some limiting random variable. So, often instead of looking at the random variable itself, we may be interested in looking whether the means of this random variable converges to some value. So, when I look at the means of each of these random variables, I will end up with a deterministic sequence.

After taking expectation of each of this random variable, I will have a deterministic sequence and it converges. So, and if it converges, now, I want to understand when the mean will converge? Or are there any properties that will help me to infer from the properties of convergence of the random variables in other, what you have already studied like convergence and probability.

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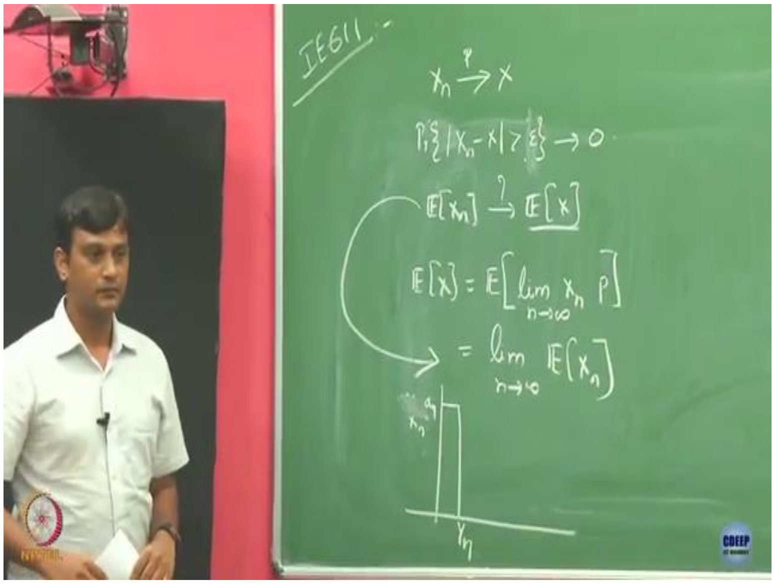

So, let us say that I have Xn converging to X in probability. So, what does this mean? This means basically limit has, sorry. So, this basically we said that probability that is equals to 0. Now, suppose if this happens, can I say that expectation of Xn goes to expectation of X? What I am basically asking is? So, what is X here? Expectation of X is nothing but expectation of limit as Xn and this limit is what is in the probability sense, whether this happens or what I am been saying to ask whether this is same as limit as n tends to infinity of expectation of Xn.

When I said this, this is what exactly I meant. Whether expectation of Xn converges to expectation of X is basically saying whether expectation of n, as n goes to infinity is equal to expectation of Xn. Yes, we have said but we are doing this we are asking, we are asking this question? Whether this holds? We are not right now saying that this is true, we want to see whether this holds, if at all it holds when does this hold?

So, when you studied some examples, when you are studying convergence of random variable in, in probability sense and the means squared sense. We came across an example where it said that it converges in probability, but it did not convergence means squared sense. We, we had an example like that, because convergence in probability does not always imply convergence in mean squared sense.

That happened in that example, because it so happened that, even though the example I had their Xn's they are putting values on a smaller and smaller intervals as I put n goes to infinity, but their amplitude was also exploding. So, you remember like, we had this example where, so, it was like this and this was some sequence what was we called it An as n increases, this was putting on a smaller interval but it was amplitude was also increasing, and we had argued that unless An grows much smaller than n, this guy will diverge, I mean, the mean squared error will diverge and this will not converge in the mean squared sense.

So, similar behavior happens here, like that is what we want to capture. If we are looking at the expectation of the random variable, even though it may be putting that mass on a smaller and smaller interval, but, it may take a very large value, right, even though because of that, it may converge in probability sense in this fashion that limit and this may be almost same everywhere except for a small interval differing by epsilon or like they may be, they may be different at only some small interval, but on that interval, my Xn may be taking a very large value because of that expectation could be very large there and in that way, we cannot, we cannot say that always, this kind of behavior happens.

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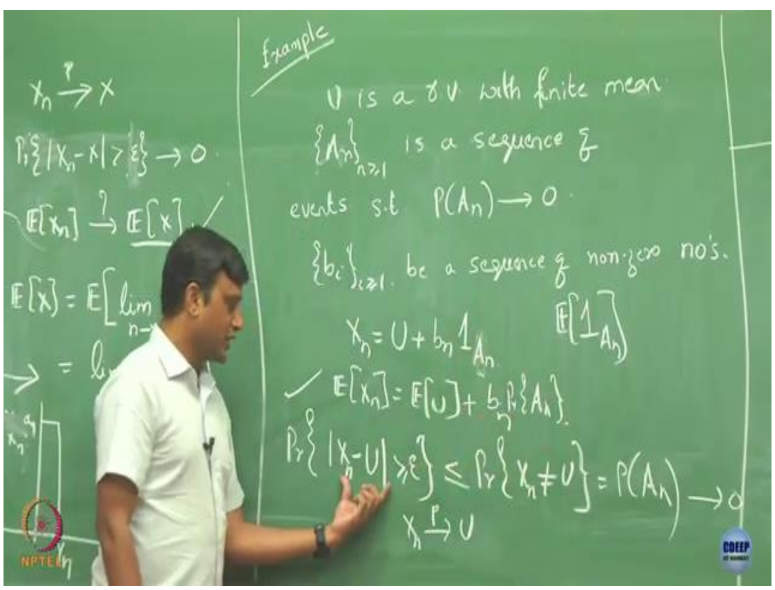

To make this more concrete let us look at an example. So, let us say U is a random variable, now, I am going to come. So, let us say I have random variable U, I have a sequence of Ai's, which are events, and these events are such that their probability goes to 0 as n goes to infinity. And I am also going to look at this sequence bi. These are another, set of sequence of numbers, some numbers that are given to me.

Let me see I, so and I am going to assume that they are non-zero. So, this is I am starting with and I am going to define a sequence of random variables based on this in the following fashion. So, this U does not depend on any n it is just like one random variable U. I am going to define for each n, a new random variable, which is the sum of these two random variable U and, this bn is thus, deterministic quantity which is coming from this given sequence and it is multiplied by this Indicator function which depends on An and An is the event that I am, I am, I am assuming it will be such that it probably goes to 0.

Let us say I have such a setup. Do you understand what I mean by indicator of An, fine. Now, if I am going to look at his expectation what his expectation is? I will assume that with finite means. Plus, what is this? What will be if I take expectation? This will be bn into; it is going to bn expectation of indicator function. What is then? It will be probability of An. So, it looks like its expectation for any n is going to look like this.

So, now let me ask this question. I have a sequence of random variables like this, does this converge in probability? So, to verify that what I need to do, probability that what you guess it should be converging to if at all it converges to U. So, let us say our guess is. Let us assume I am going to take to be U. What is this quantity? If I can show that that as n go to infinity, this probability goes to 0, then I know that Xn converges to U in probability sense.

Now, shut that or see that this probability is same as Xn is not equals to U. See, when I say Xn is not equal to U that means the difference is going to be some non 0 quantity. So, it could be either greater than epsilon or it could be much greater than that, that is what this is going to be an upper bound, fine? So far? Now, come back to this. Now what is the probability that Xn is not equal to U?

It is going to be, so, probability that Xn are not going to be equals to you, that is enough that if this quantity happens to be positive then this going to be different and when this count is going to be positive when this is true, and what is that probability? That probability is nothing but the probability of event An. So, the way I have constructed this, Xn naught is equals to U is nothing but probability that, of An.

So, the, the problem is that they are not differing is that this quantity is going to be positive, this quantity is positive is nothing but this An has to be this, this indicator function has, this condition or this indicator function has to satisfy and that is going to be satisfied probability An and what is our assumption? We have said that I will come up my assumption said that this An is such that this guy goes to 0.

So, then we, what? So, what we have shown that; so what we just have shown is that Xn converges to U in probability sense. Now, let us come back to this. So, if Xn converges here to U in probability, fine. Now, I wanted to ask the question, can I also do this and can I also say that expectation of Xn converges to expectation of X, X in this case is going to be U. Because that is the limiting distribution.

But here if you focus on this even though this probability of An's are going to 0 as n goes to infinity, I could choose bn's such that this product does not go to 0. So, in that case, this expectation of Xn need not be the same as expectation of U as n goes to infinity. They, they can be different. So, as you say that even though Xn can converge to some distribution in probability, but that does not automatically allow us to take expectation of the sequence to be the expectation, the limit of expectation of the sequence to be the expectation of the limiting random variable.

So, then the question now is: When does this happen? We want to now, look at the conditions when does this happen? Yeah, yeah.

Student: (())(15:10)

Professor: So, indicator of an event, expectation of indicator of event is what? Is the probability that event itself? How can you write this? Just write it into 0 and just expand it, so expectation of this. Only the one term remains the other term is 0, the one time that remains is exactly this 1 into Pr of An. This one? Let us look at this event Xn not equals to U. That means the difference between them is not 0.

It is other than 0, it could be positive or negative, then you are going to look at the absolute value that is going to be the difference is going to be greater than 0. What you are saying, all you are asking is, this difference is going to be greater than 0. That could be either greater than 0, greater than epsilon or it could be much more than that. Exactly. So, this event is going to be subset of this event that is why I am going to get an upper bound, fine.

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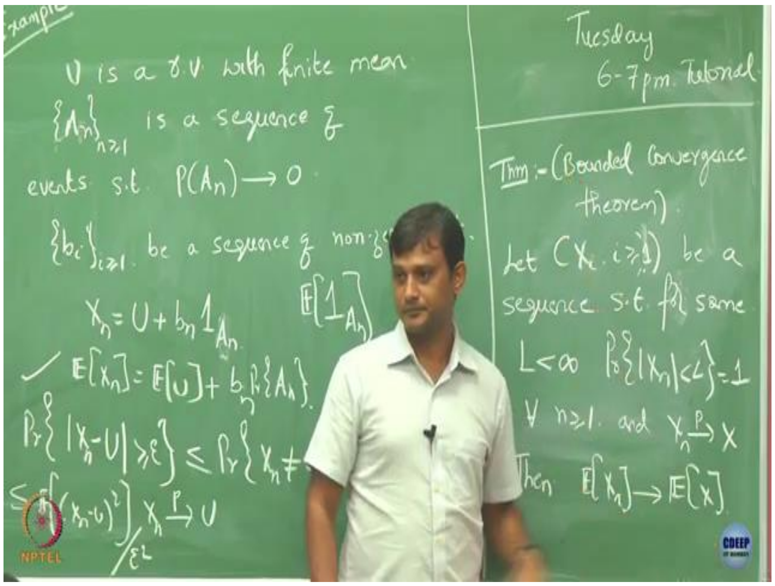

So, then I am going to now state couple of results, which actually helps to do this kind of interchange. What I mean by interchange is exactly this. The first one is called bounded convergence theorem. See, some of these results are you may not be able to directly relate it to where they come in practically, but some of these results are useful to state some results, some, some of that you are going to see later in the class today, that gives us very practical, some of uses some intuition about some of the practical things.

So, these are like some intermediary building steps that we need to understand, to understand some larger result, for example, later in the class today I am going to talk about Central Limit Theorem, that is one of the important result and which has many practical implications. So, to understand that we have to understand these results. So, at every result, do not try to connect it with practical things, then you may be lost but just try to follow what we are trying to define and then eventually where we are kind of going to connect it.

Let, this be a sequence such that there exist some L. So, this, this theorem tells gives us the first condition; it says that, suppose you have a sequence and such that you have some L which is finite and all my random variables, their absolute values is bounded by this random variable with probability 1, that may then convergence in probability to a random variable X implies that their expectations, expectation of the sequence of random variable converges to the expectation of the limiting random variable.

So, this is kind of this is a very intuitive statement. So, that is why it is called bounded convergence. What you are saying is your random variables are always bounded. That naturally means that even when you are limiting random variable cannot be unbounded. So, if you have everything bounded then there is nowhere that in the sequence you will incur a case where something explodes.

So, because of that there is nothing observed behavior when I have this boundedness and everything goes well. Let us look to proof of this. So, then I make this kind of assumption that all the random variables are bounded like this, where I made such as such case again. Did I use such an assumption earlier also? Where?

Student: In random variable 1 (())(21:14)

Professor: Yeah. So, in which guy of convergence I used it?

Student: If Xn converges to Xn (())(21:22)

Professor: In means quite sense, when I had a mean squared set I wanted all the second moments of all the random variables to be bounded there. So, if I have a random variable like this and if L is finite and if I take the second moment, will it be still bounded and that limit is going to be what? Instead of L it is going to be L square. So, this condition and I had, when I said convergence in probability implies convergence in distribution, but convergence in probability does not always imply convergence in mean squared sense.

But I said that convergence a probability implies convergence means squared sense under some condition, what was that? Yeah, so, if my sequence are bounded by some random variable, so here is it; can I say that, here convergence in P already implies convergence in mean squared sense? Yes, because that y is nothing but L in this case and we have already said that…

Student: (())(22:48)

Professor: Just L square, L square should be finite, fine. So, and we have already said that so under this condition, we know that if Xn converges to X in probability it already converges to mean squared sense and in the mean squared sense, when I discussed it, I already said that the limiting random variable, the second moment is already finite.

That was a consequence which I asked you to verify, I do not know you guys did. So that came from Triangular inequality. So, by the same logic, we can also say that here, this x is going to be bounded, its expectation is going to be bounded. So, this is also going to be bounded. Now, let us understand.

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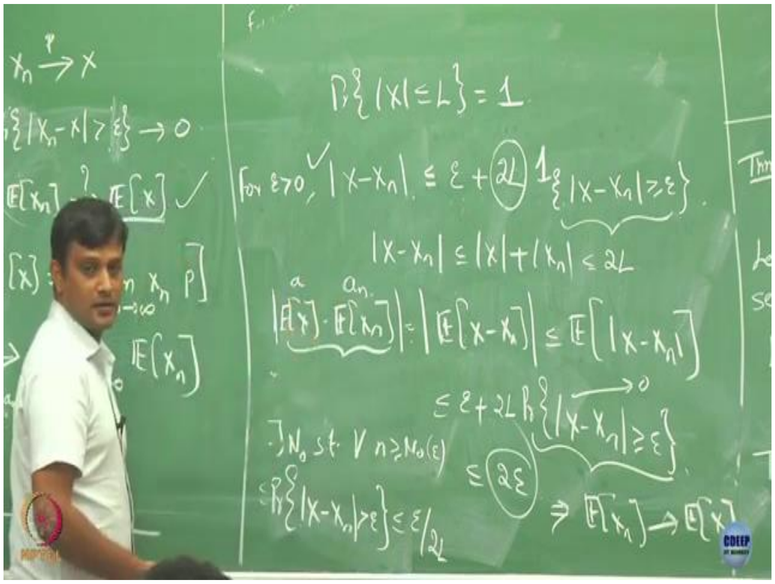

So, you can verify that I am just leaving you this, that X is going to be also going to be L, probability 1; whatever that limit is. After this, let us focus on this inequality. Take any epsilon greater than 0. Now, my claim is this X minus Xn I can write upper bounded in this fashion. So, this is the main inequality I need to make this final claim.

So, let us see, we understand this inequality. So, taken epsilon greater than 0, if it is such that this X minus Xn, the absolute value is greater than epsilon, greater than equals to epsilon, if this is true or like let us say this is not true that is this difference is upper bounded by epsilon, this term is already 0. And the then this condition already true like x minus Xn is upper bounded by epsilon that is what I have already assumed.

In the other case, where let us say this holds. Well X minus Xn is greater than epsilon, it is going to be at least anyway epsilon. It is going to be plus something else and what is that something else? We can always write X minus Xn as upper bounded by X plus X of n. And both of them are upper bounded by one with probability 1.

So, I could read them as upper bounded by the probability 1. So, just verify that is why I could write this will be greater than 2L at most. I mean, see like this X minus Xn, I could have already written. This is upper bounded by 2L, but I am expressing in terms of this epsilon because this gives me a tighter bound. That is if this condition holds, yes, 2L is there 2L is there; if this condition does not violate, then epsilon is the bound that is why tighter bound.

Anyway, so, I have this inequality here. So, now let us see what I want? I want to, now I want to find expectation. So, what is the median growth expectation of Xn goes to expect, expectation of X that means, if I am going to look at the absolute difference of this expectations this should go to 0 as n tends to infinity or alternatively what I can show as if this is upper bounded by epsilon for some n, which is sufficiently large, right, for all n after some point. So, now fine this is there this I could write it as expectation of X minus Xn, why I could do this?

Student: Linearity.

Professor: Because of the linearity of the expectation. And now, I could do that. Now, I have taken this absolute term inside the expectation, is this true? Because the differences I have made absolute so my expected value is going to be larger, fine. Now, I am going to bring in this bound which I applied here. And what is this is going to be? So, if I use this bound, this is going to be epsilon plus 2L and expectation of this indicator.

So, that is going to be… Now, I know that this guy goes to 0, this probability goes to 0. And I know that because of that, then I can say that there exists some and n naught such that for all n greater than or equals to n naught I can replace this guy by maybe there exists for all n greater than n I can write it as this there is going to be less than or equals to, so I can always write it as.

So, you give me epsilon, I will take epsilon by 2L and I will look after which point this guy is going to fall below epsilon by 2n. I can always, there always exists such an n. So, now you should do that for sufficiently large n, this guy is going to be upper bound plus 2 epsilon. This I know this guy goes to 0, right? As n goes to infinity, ok. Whatever epsilon it is or whatever on you again it goes to 0.

Now, what I will do is whatever epsilon you give me I will be looking at epsilon by 2L and I know that after some n naught, this guy is going to be below epsilon by 2L. It is because this guy is going to 0. So, I just planted this quantity here for all n which are greater than this n, then it becomes 2 epsilon, it is just like… What?

Student: Why have we taken (())(30:38), we can take anything.

Professor: Yeah, you can take anything I just want to bring it simplifying all this to epsilon. Now it is clear right like now I am more familiar territory. Now, I can say that, I have shown that this difference is upper bounded by 2 epsilon for all n greater than n not and this and not is a function of epsilon. And now this, if this holds, if I can do this for every epsilon, then what is that? What does this imply? This epsilon is arbitrary. So, then this is exactly the definition that this Xn converges to X.

So, this implies, so, I am saying that this implies this, are you convinced or not? So, what I could show? What can now think of this as some this was An sequence, and this is the limit, what I have shown is An minus A is going to be upper bounded by 2 epsilon for n sufficiently large and this this I am doing for any epsilon that you are given to me. So, that means that is what the definition of the limit, an converging to A and I could do that, because I am exploiting the fact that I already know that Xn converges to X in probability.

I am just using that convergence to claim that this guy converges. So Fine. So, this is one simple thing, like if I know that my bound random variables are bounded, and if that may and my random variable convincing probability, then I can do this interchange of the expectations. Yeah, so what is the other things?

Student: (())(32:53)

Professor: Not necessarily, it is not only if and only if condition. If this holds then we say we can do this but if we can do this at all, I do not know that is it is that always implies that such a thing implies.

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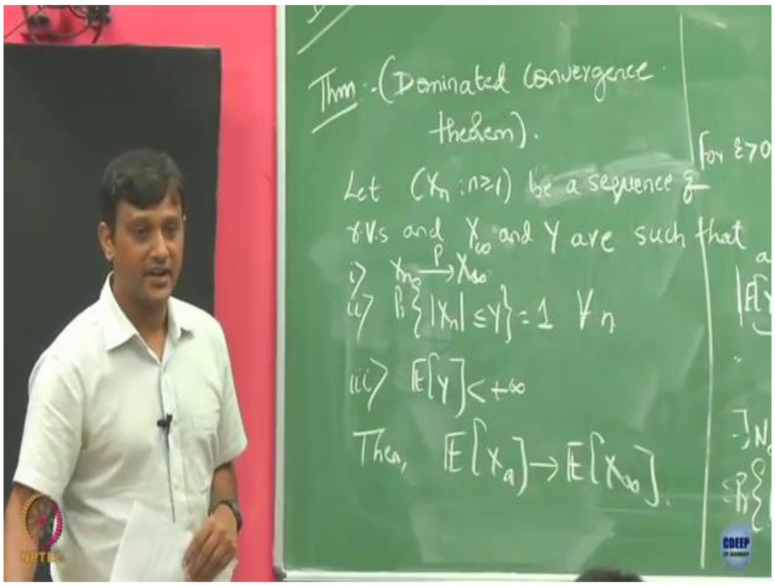

So, let us say this other things which tries to generalize this bit. See when I try to state this, I wanted this Xns to be bounded, right and this bound was a deterministic bound. But you can relax this and say that instead of the deterministic bound, I can have this bound with some porb, using a prob, using a random variable. So L I can replace by random variable y, if such a y exists which always dominates Xn, then also it is possible.

So, let me make that. So this is called, so this is just a slight generalization of this bounded convergence instead of boundedness now you look domination. We are going to say that these sequences such that they are all dominated with probability 1 by random variable y and is dominating random variables such that its mean is bounded and then if I have the sequence converging to some X infinity in probability then, I can do this interchange.

So, it is same as this except for the fact that I am replacing this L by this y which dominates the sequence, but that y are further one that it has a finite expectation. So, if I can find such a random variable which has a finite mean dominates my sequence and then I can interchange this limit if my sequence converges n probability. So, we are not going to prove this, but just take this like you should understand how to apply this results. So, before you apply, you should better check all these conditions.

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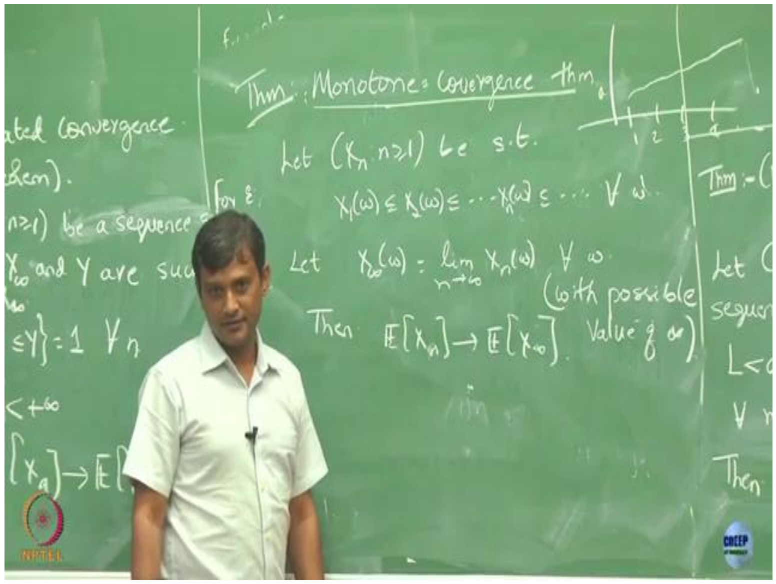

So, the last one called monotone convergence theorem. So, this is as the name indicates, this is an another version of monotone convergence we have already come across deterministic sequence. So, what do you know about a deterministic sequence when it is monotonically increasing? So, let us say I have a sequence, deterministic sequence, let us talk about deterministic case.

If it is always increasing like A of n plus 1 is going to be greater than equals to A of n, what happens to limit An? The limit exists to convergence, what? Depends on n, why? So, any sequence which is monotonically increasing necessarily convergence, right? Either finite one or it could be like even the limit could be unbounded. It at least it does not diverge, right?

So, if I have a monotone let us say I have let us say these are A1, A2, 3, 4, like that, right, my and this is my sequence, if my sequence is like this increasing it is increasing, increasing that either it blows up and goes to infinity or like it saturates at some point, right? So, it always converges. So, similar log.

Student: If you do not know, how will you unbound it?

Professor: So, this is the convergence in the extended domain. So, we allow the limit to be infinity as well. So, it is not necessary that when we talk about convergence, the limit has to be always finite even the limit could be infinity, right? It is just like that it is just that it does not happen that while it is going, going like at some point I come down. So, this violates the definition of my convergence, right?

So, if it is always increasing, increasing at some point either it goes to infinity in that case it is infinite is the limit or it saturates at some point then the finite in that case that saturation point is the my limit. Let me take this, I have this n; n greater than or equals to 1 this is a monotonically increasing sequence, right? Where does this converge? Converge like this, this is the extended notion of convergence.

Student: (())(41:38)

Professor: Yeah?

Student: (())(41:40)

Professor: No, this was deterministic case also. I am giving you a deterministic case, yeah, right? I have this n, why I like the sequence An, this An is simply n, this is convergence but it converges to infinity, fine. So, I am not defining the sequence in this form. So, I have my random variable sequence such that if you are going to fix a sample point on that sample point this forms a monotonically increasing sequence, and then defined my limiting random variable for each omega to be the limit of this.

So, now this Xn of omega is deterministic sequence, right? And since this Xn of omega monotonically increasing, this, indeed exists; maybe it could be, limit could be infinity. What is the term for that? With possible value of infinity. So, if this happens then I can always say my expectation I can interchange like this and here it really do not need to first check. I mean, I do not really need a convergence in probability, if this monotonicity property holds that is it, like if I can check this, then I can always…

Student: Infinity of omega become infinity.

Professor: Yeah.

Student: What will be expectation of Xn, it will also be inifity.

Professor: Yes, but provided that sample point has non zero probability. If that omega has 0 probability, even if it is infinity, mean, that becomes a bit weird to define, right? Like this has infinity, but its probability could be 0. But in that cases, we have to define its value to be the product of 0 and infinity to be 0, and then we can continue with that, ok? So, fine. So, these are 3 theorems called bounded convergence, dominated convergence and monotone convergence. You should know and understand when you can use them.