Introduction to Stochastic Processes
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Lecture 24
Cauchy criteria of convergence of random variables

So, yes, we have this sequence of random variables converging to another random variable in some different notions. It is not always necessary that if you have XN, which have some distributions and the limiting distributions of the random variable limit, random variable need not be always the same or even need not be the of the same nature.

For example, let us say if I had a sequence of random variables which are all exponential distributed or poison distributed or something and not necessary that the limit distribution is also of the same nature it is also either exponential or poison, okay? It could be something different.

But it so happens that when we have a sequence of random variables, where each of the random variable is gaussian, the limit distribution is also always Gaussian, maybe the parameters will change, but at least the nature of the limit distribution is still a gaussian random variable. Now I will just.

Student: (()) (1:34)

Professor: Yeah, right. So, it change right. Yeah, that is one good example. So, where each of the random variable could be binomial, but the limiting could be something different not necessarily binomial, it has change its characterization. So, but if all the sign of variables happens to be gaussian, then the limiting distribution is necessarily again Gaussian, okay? Maybe, as I said the parameters could be different, the mean and the variance could be different but it is still going to be Gaussian distribution.

Okay, next what are we discuss so far things are fine like to define this convergences I always define with respect to the limit, right? Like I said XM converges to X in mean squared probability or whatever but is the knowing the limit random variable is necessary to just a defined convergence.

So, for example, let us forget random variable and all probability just focus on our simple deterministic sequences. I know XM let us say I had a sequence of AN which converges to some A, that is fine but just looking without knowing the limit can I say my sequence AN converges? Is it necessary that always I need to know the limit before to tell what, whether it converges or not? So how people heard about Cauchy criteria, Cauchy sequence or Cauchy criteria what Cauchy criteria says?

Student: (()) (3:16)

Professor: That does not involve limit? No, right? So to define a limit, whether the sequence converges I really always no need to know the limit, limiting value. It is enough if the sequence itself satisfies properties, if the sequence satisfies some properties, I can say it converges, maybe I do not know what is the limit, but I all I can guarantees that it converges to something.

So same can be done here to a sequence of random variables, you may, most of the times it may be difficult to guess what is the limiting random variable or what is the limiting dash distribution but you are sequence may such that it is unable to cut some conditions which will tell at least it converges. What it converges is a later headache, first let us worry about whether converges and then once we convince yourself as it converges, maybe we can think about what is the limit it converges.

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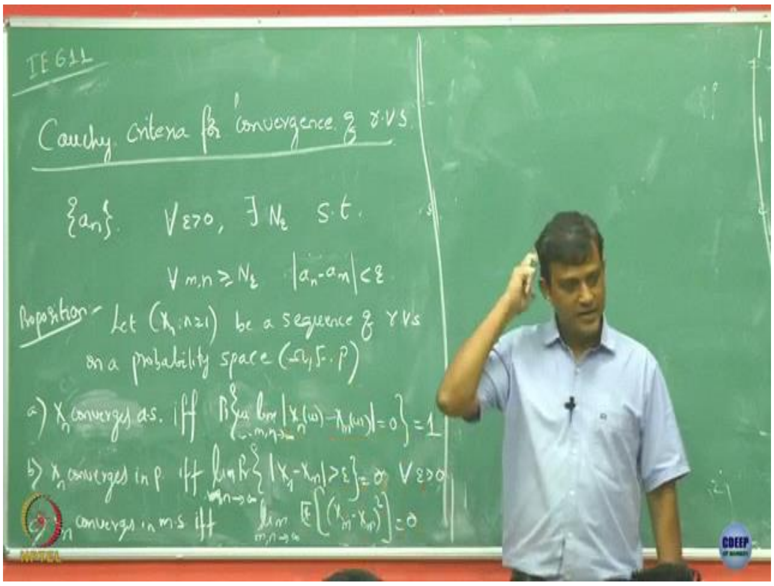

So far that we have similar notion of Cauchy sequences in this with a sequence of random variables. So, in other classic setting or with in a deterministic setting, let us say if I have sequence of AN's, what is the Cauchy criteria for convergence here? Let us say for all epsilon greater than zero there exists an epsilon such that for all MN, then epsilon,

Student: (()) (5:14)

Professor: Right, so if this condition happens, I know that this sequence converges just that I do not know the limit, what is that? Okay, now we are going to stage simply all the four notions of convergence we have in terms of this Cauchy convergence criteria. Okay, I am just going to state that as research. Yeah, so XN converges almost surely here is the same as testing this kind, this is equivalent of Cauchy criteria for our random variables.

Student: (()) (6:13)

Professor: Oh sorry, it should be outside, this is incorrect. So how to write this just constant okay. So, this is exactly analogue portion of the Cauchy criteria for my armature convergence. So earlier, I needed to know that limiting random variable X, but I do not worry about that here in the Cauchy criteria, what I look is take a pair of random variables and let them go to infinity and if that goes to zero, and the probability of all this omega that satisfy this condition is 1, then I am going to call this it converges almost surely, but I am not specifying which random variability is converging, it converges to something.

Similarly convergence and probability is again defined like this. So you see, what they have basically done is earlier XN we had X the limit, right? Now that X is replaced by XM again another point in the sequence, and same thing here also, we are going to look at XN minus XM the difference being greater than epsilon and that goes to zero then that if that limit goes to zero then we are going to call that convergence probability to some random variable.

So, same thing for convergence in means square sense also. So, notice that when I say convergence in means square sense, it automatically assumes that my XN's are such that their second moment is bounded less than or equals to, so that is already implied in this condition, okay.

So we will just skip the proof of this, it is just analog to how you prove, how in the deterministic sequence how the Cauchy criteria implies the convergence is same thing but you have to worry about all this constructing their sights, epsilon, delta business and also changing their limits appropriately. So, you can look into the book for details.

Student: (()) (8:53)

Professor: Here we cannot, that is the right. In the, so the when you go from almost sure to be, we could. So, what do you mean by interchanging probability and limit here?

Student: (()) (9:08)

Professor: No, we cannot of usually for to do that we have stated a specific condition like if you are looking at limit as n tends to infinity probability of DN, right. If the sequence BNR says that the monotonically increasing then I can interchange that was what we call as continuity of probability. If I do not have such a structure on the events, that is BN's in general I cannot do this, you have to be careful there.

So, it is not in general true that we can interchange limit and probability, you may also face a case with where you have to interchange limit and expectation, so you cannot do this? So, for when we can interchange, limit and probability we already stated, we are going to state later in the next class when we can interchange limit and expectation, okay? This can be done only under certain criteria, under some conditions.

So fine, we have for this convergence of random variables we have this Cauchy criteria now. Now, is it possible that we can similar criteria if we can, if you know something about the correlation of the random variables, so I have a sequence of random variables, right? I can look into their correlations basically, take a pair XN and M, I can look at the correlation XN and XN into XM, take the product and then look at the expectation that is the correlation for us, right?

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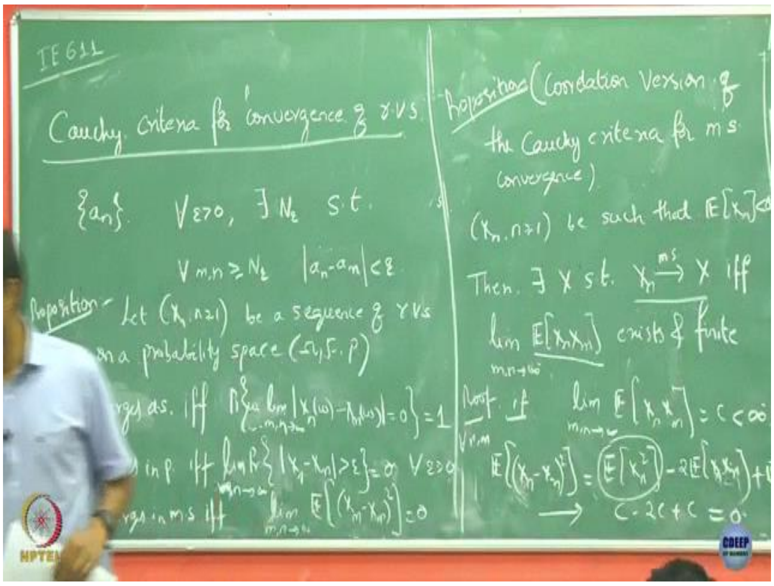

So, the next results says that there is there is some connection like that, so this is called correlation motion of the Cauchy criteria for means squared convergence, okay? The statement is, so what now it says is let us take a sequence of random variables which are finite second moment, then there exist an X which is the limit in of the sequence in the means square sense if and only if this limit convergence now.

Now what is this sequence? This is the correlation sequence, if this correlation exist and it has some finite value, then this is true, okay? Okay, now let us see why this is true, we will quickly argue this. So now, let us proof first if part. So, do you want to see the difference between part C here and this result here?

So part C here said that this XN converges, if this Cauchy criteria holds, but now this is replacing this Cauchy criteria by this correlation criteria where you are now testing something on the correlations of the sequences, okay? So as, so if so let us say this limit as MN tends to infinity is some value C and that is finite. So this is a hypothesis, right? Like it is exist and finite, let's assume that to be some C and it is a finite value.

Then what we will do is just let us apply this condition here and then this is, if you just expand this you are going to get XN square minus 2, this is for any NM and minus, plus expectation of XM square. Now, if I want to argue that this guy converges in means square sense I need to show that as NM tends to infinity, this quantity goes to zero, right? That is the definition of my means square converges.

Now, I know that if I let NM equal to infinity, this expectation go to infinity, this product this correlation term that is my hypothesis. What I can say about this XN square and XM square? So in this limit when I am letting NM go to infinity, right? I could as well set N equals to M and let that limit go to infinity, right? In that case is this, this, this guy will have a same limit as this?

So you can just check that they will have same limit, In this case as NM go to infinity, I can just verify that this is 2 minus 2C plus C and this is going to be zero. So that is why?

Student: (()) (15:38)

Professor: No I am talking about here expectation of XN square, so fine, see there are two things here, you are not setting N equals to M here when you are letting NM go to infinity, you can arbitrary let NM go to infinity, N and M could be different, but now this consists of XN square and XM square.

Now I am just talking about how, how this limit itself will go, on that I am inferring from this. So, if this is going to be C, even if I let N equals to M and both got infinity, that limit is also going to be C, that is what I am using C for these two, okay? So what we just said is from this, if this correlation criteria holds, this is same as verifying this Cauchy criteria, so that is this Cauchy criteria here.

We know that this is why, if we have this Cauchy criteria, already know that, that converges in means square sense. So if it convergence a mean square sense, it has to convert to some random variable, right? That is all you are saying there exists some X such that this XN has converged to that random variables, we do not know what is this X, but there exists some X, it has converged.

Now, to prove the other direction, we will assume that XM converges to X in some, to some random variables, then will try to show that in this case, this correlation limit will be finite and it will be, so this limit exists and it is going to be finite, okay.

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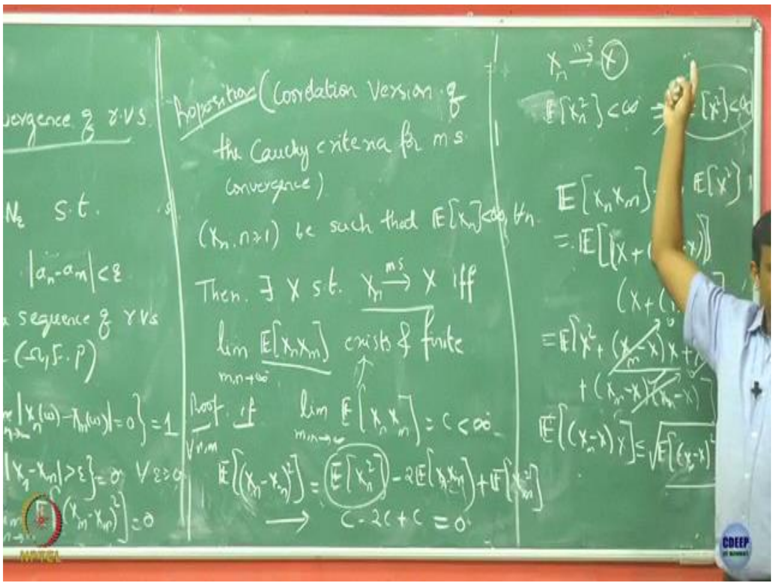

We when we try to show XN square converges to X in means square sense or assumption was XN square is going to be finite for all N, right? You can verify that this implies that whatever limit this guy is going to convert in the means square sense, this limiting random variable is such that even that is going to be finite. How you are going to do this? You are going to verify it by applying triangle inequality of the random variables, okay? Just apply this and you will check the get, you will get to, you can infer that expectation of X square is also going to be finite.

Student: (()) (18:37)

Professor: Yeah. Triangular inequality of the random variable. So we talked about this, when we talked about shots inequality, right? When we talked about shots inequality, we talked about triangular inequality of a random variable. So I do, I think we talked if I am not talk, just go and refer to that, it is in the same section where gaussian equality is defined. So by just applying that you should be able to infer this.

Okay, now we know this, we want to talk about this, where the limit exists. Now let us take this quantity, well this is some algebraic manipulations we have to do and then again we will invoke triangular inequality to conclude that this is indeed true. So this can be written as, so I will just cut shot this and just and you can just do this is all algebra. Now, just by expanding you will get this.

Now what we are going to do is, we are going to apply Cauchy shorts inequality in each of these and try to derive a bound. So let us try to apply Cauchy shorts on this. So, what did the Cauchy shorts inequality says? If I have two random variable product, what is the expectation is going to look like? It is upper bounded by in terms of there second moments, right? In what fashion? This was upper bounded by square root, like what is the first term?

Student: (()) (21:03)

Professor: Yeah, so expectation of this into squared, yeah expectation of square of this into expectation of square of this, so you can similarly apply shorts inequality to each of these terms, this term and this term here and you will see that as I let M go to infinity here, what is going to happen? Or here? So, notice that this is XM minus X whole square by the definition I have, I am assuming that XN converges to X in a means squared sense. So, if that is my case, what is this going to? It is going to zero. So, this term is going to vanish.

And similarly, what is this term is going to happen to this? It is going to vanish, and what about this? This going to be zero, what remains? Right. So, this is, now we are saying that what we have shown is this limit here as MN go to infinity is equal to expectation of X square. So, we have now shown the existence. So, the limit indeed exist, what is that limit? That limit is expectation of X square and this X is what I have assumed, that it converges to some X.

And now, the second part I have to show is it is finite, why it is finite? Why?

Student: (()) (22:54)

Professor: Because I said that means squared sense convergence also implies that the limit is going to be finite.

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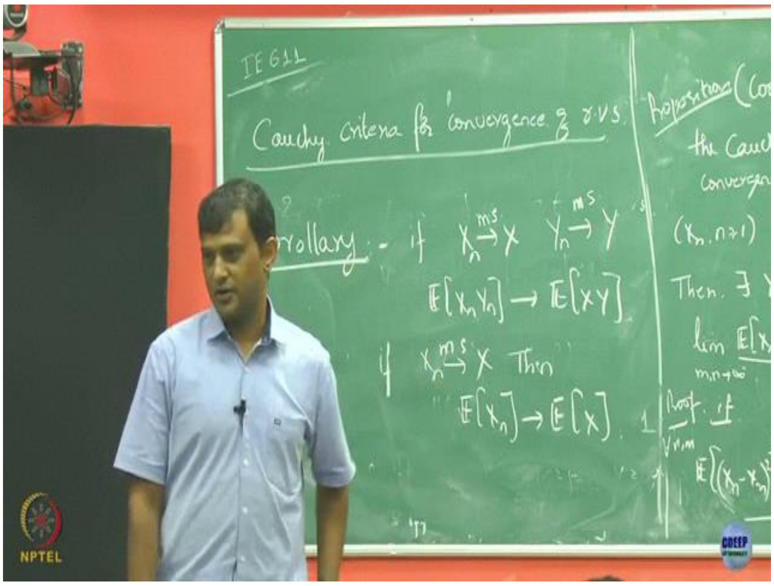

So, just one or two minutes, as a consequence of this theorem, we can derive many corollaries which I am just going to state which are very useful, you should know how to use them. So corollary, if XN goes to X then means squared sense and another sequence I have which goes to Y in means squared sense, then you can check that expectation of X and Y and goes to.

So, this is analogue version of about a deterministic case, right? Like for example, if AN sequence converges to A and BN sequence converges to B, what does the products sequence A and B and converges to AB, exactly similar thing we have here. And now then if XN converges to X in the means squared sense then, we have expect (())(24:25). So means square convergence also implies that the convergence in expectation, okay?

So, why is this true? You can just say that you can take this Y tends to be simply one all the case, in that case YN's are converges to Y. So, in that case this is already implying, right? This XN's, this expectation, this see expectation of XN is converging to expectation of X. So, these are pretty useful results like. So you should know how to use them, so for proof I am just keeping you can look into the book. So let us stop here.