Introduction to Stochastic Processes
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Lecture 21
Convergence of Sequence of Random Variables (Part 1)

We are going to now start understanding what we mean by convergence of a sequence of random variable. So, we already known like when you have a given sequence what it converges to and the limits are very important because many of the times like integral differential functions are all defined in terms of the limits. And also when I have a process, let us say I have modeled it as some stochastic process, which are indexed by time. I want to understand how this process evolves over time.

So, I would be interested in knowing as I let my t go to infinity. How my process evolves and behave? Suppose, you are you are investing your money in a gamble, every day you are going to win or lose and based on that every day you have some money left with you. And you want to understand eventually if I continue to play the same game, eventually as the number of place becomes large, will I end up with a positive amount with me or it will all go to zero or it becomes negative.

So, you want to understand as time evolves in my stochastic processes as the index becomes large, how my process looks like, is there any limiting behavior in that. But then the question is; fine, we understand what we mean by convergence of a sequence of random variable. Sorry, convergence of a sequence of numbers, then what does we mean by convergence in for random variables? So, we need to make that notion of convergence of a sequence of random variables precise and that is what we will focus on maybe in the next two to three classes.

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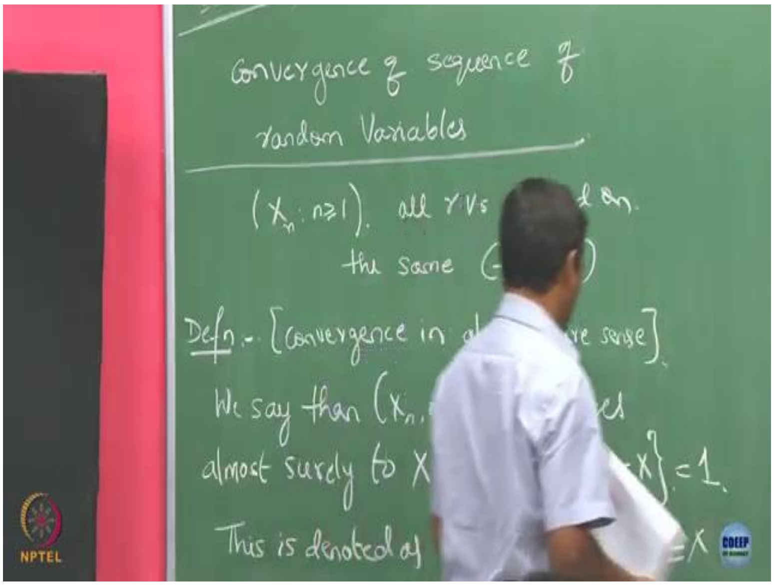

So, let us say I have a sequence of random numbers, random variables, all random variables defined on the same (())(2:42). So, another thing is I am going to only focus on are random processes here or a sequence of random variables, which are indexed by discrete, which are indexed by discrete numbers. So, here n is one, two, three, four like that.

Now, the first notion of, sorry, convergence in almost sure sense. We are going to say the sequence of random numbers xn they converge almost surely, If probability of limit extends to infinity of this x is going to take the value x. So, what I mean by this limit here, I have already said that limit of x n is equals to x.

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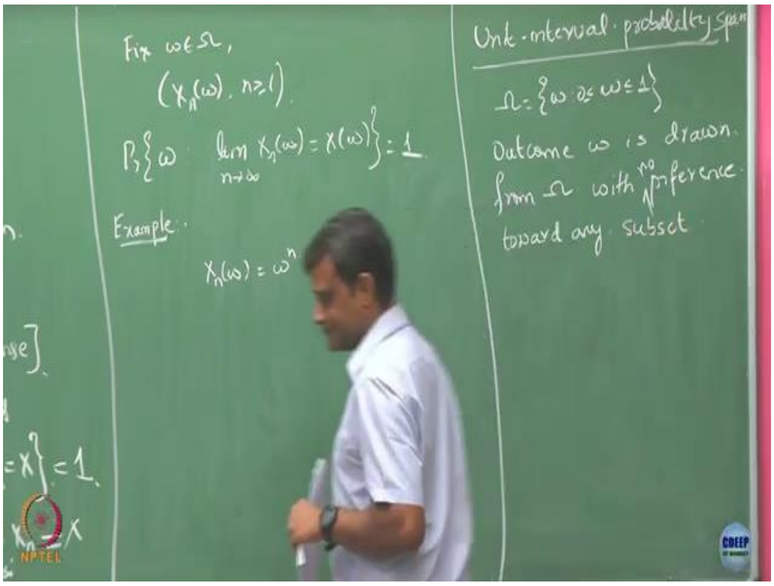

So, what I mean by this here? Suppose, you fix omega belongs to omega and then you are going to look at x n of omega: Is this is a deterministic sequence x n of omega? Yes, if I fix an omega this will denote, because xn is a function of from capital omega to r. So, if you fix an omega, this is some real number. And I have a sequence of such real numbers. And I know what, what I mean the convergence of this sequence of real numbers, that is a standard convergence and whatever is the limit.

Now, what we are saying is if the meaning of this is if a limit of x n of omega so, so this is the short form for this. I am going to look at omega and see whether the sequence of xn omega converge to my limited random variable and then look, if wherever that convergence happens, I am going to look at all this omegas and then look at whether that probability is equals to one. If this happens, then I am going to call my xn converges to x almost surely.

So, let us look an example. So, let us say I am going to define a sequence of random variables like this. So, before that, I want to introduce this notion of unit interval probability. So, earlier we have already defined the notion of what is event space? What is sigma algebra? And what is what we mean by probability space and all? So, now I want to like define a special probability space, which is as following that omega is that omega is. So, omega is basically interval and then this outcome, omega is drawn from omega, with preference, with no preference towards any subset.

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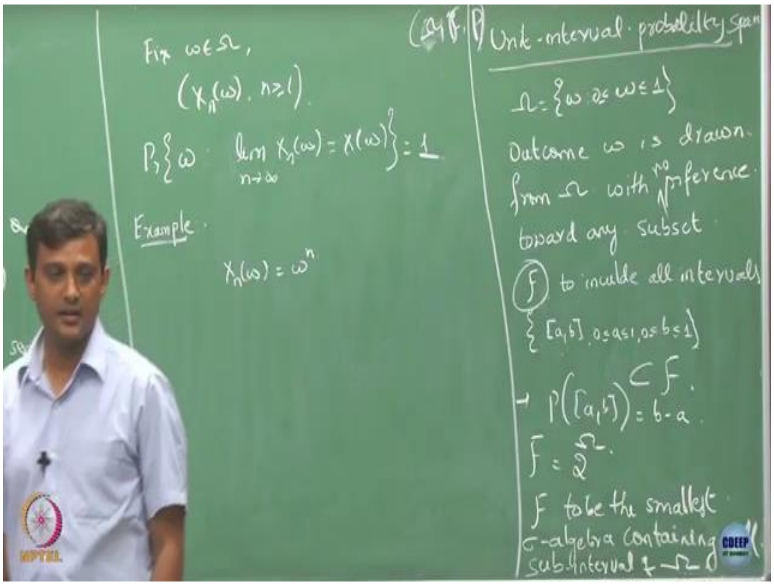

In a way like I am saying that I am going to draw omega from this sample space in a uniformly, without giving any preference to any of this subset. Then I am going to look. I am going to now start constructing my event space on this let us say F, to include all intervals. What I mean by this? I am going to say that let I am going to take. So these are all possible intervals right take a and b and which is a is between 0 1 and also b is between 0 1 this is going to define one interval and let us all possible intervals be contained in this event space okay.

Now, that I have been trying omega from this capital omega without any preference to any of this subset, then one natural value I am going to assign probabilities to this interval is b by a. I assume that b is going to be greater than our equals to a. Now, I have defined omega, I have defined my F partially here because I only said it includes closed and this intervals here, but if it has to be a sigma algebra, then I know that all its, it has to satisfy the properties of sigma algebra, which says that compliments should be there and their unions, finite, finitely many and countable unions should all be there.

So, if intervals are there then the open sets are also there in this sigma algebra F right. I will have open intervals in this, I will have closed intervals in this by taking the unions I will have sets which are open at one end and closed rate other, other end and I will have all such kind of combinations, if I have to look at a sigma F.

Now, when I have a such a sigma then to make this completely probability space, I also need to say how I am going to define probability for each of the elements in my F. For intervals it is easy. I have defined like that. But if I am going to look at all possible elements that are in F, how I am going to define it.

Student: (())(11:02) because of probability of a will be a minus (())(11:05).

Professor: Yeah, say if you are to take b equals a, then you have only one element and for which are saying 0. I am saying you take any interval here for which you have defined like this and you can go like this and if you want this F to be sigma algebra, it will have all possible subsets of my 0 1 interval. So, the worst case, what you can take one, we can take this F to be a upper set of this that means it includes all subsets of the interval 0 1.

But here is some technicality here when you do this it is so happens that we will end up with certain sets for which we will not be able to consistently assign probabilities. I mean that comes from some complicated analysis or some better understanding of the real sets but we will not going to that, but what we are going to take is when this F to include, what we are going to take this is F to be the smallest sigma algebra containing all sub intervals of omega.

So, we take f to be the smallest algebra. So, you can come up with many-many sigma algebra which will contain all these intervals. But we were going to take that sigma algebra to make this, to define this, which is the one which contains, thus, which contains all sub intervals of omega. Is this point clear? So, we are going to take all, so the way we are defining F is let it include all the intervals to make it sigma algebra, we also has to allow it to possibly include all open sides, their unions their intersection, so many combinations are there.

So, you may end up with so many sigma algebras, which so many F which may it on its own satisfied the properties of sigma algebra, but among all them, we will take the one to include the smallest one, which includes all the sub intervals. So, this is just to make this bit more formal and then such a, when we have such an F which contains all the smallest, all these sub intervals, we know how to assign probability to them and using that we can up try to define the probability for each of the element in that F.

And such probability space we are going to call it as unit interval probability. So, for all practical purpose what we mean by unit interval, probability space is my sample space is unit interval and my sigma algebra is such that it contains all possible intervals and on each of the intervals there, I am going to assign probability like this if I a interval. So, we only need to take, so our understanding of unit probability space will be just this. So, but it has something more to it in terms of how this F is defined in terms of the smallest sigma algebra containing all the sub intervals. Fine. This one, just a detour and this is what our understanding of unit interval probability space.

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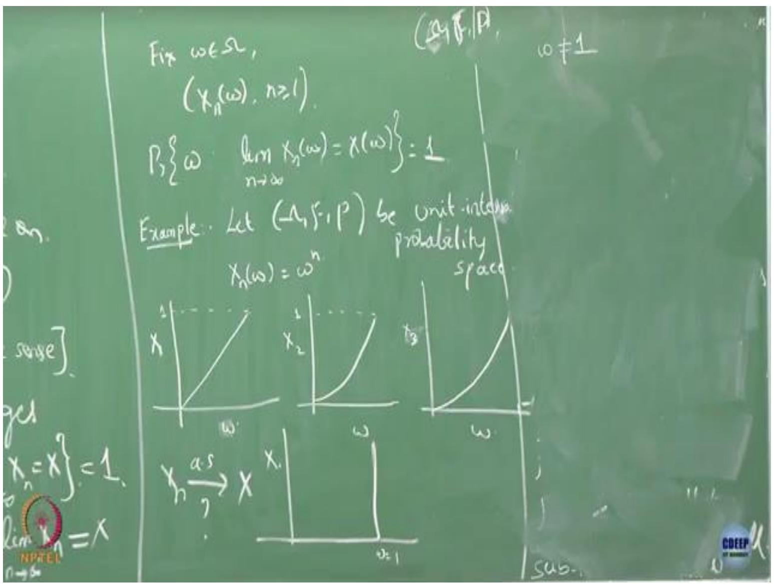

Now, to understand our notion of convergence, we will be looking at examples are random variables defined on this unit interval probability space because this is going to be easier for us to understand. So, I am going to take let omega FP be what I call as unit interval, probability space. Now, I am going to define my random variables like this, fix an n and that n is going to be such that for any omega that is coming from capital omega, it is going to be defined like this. Now, let us try to map this.

So, let us say this is x 1 and this is my omega. So, how does x 1 look like? It is going to be linear curve? It is going to be linear and going hitting at one. And how does X 2 look like? It is like quadratic. And how does X 3 look like? So, its curvature will open up this side or that side?

Student: (())(17:28)

Professor: It is going to be like this. Now, let us try to apply this definition here and see where it will converge. So, let us before this, as we saw that as we move from one two three, the curvature is opening towards the right, and you as n goes to infinity, how does this curve look like.

Student: (())(18:06)

Professor: So, it is going to almost 0 till this point and that omega equal to 1 it is going to be 1. So, in this case in a way as n is tending to infinity we see that this sequence of graphs here are converging to a place where it is all 0 and then it suddenly shoots up to 1 at omega equals to 1. So, let us take that to be our limiting x.

So, then let us try to analyze whether this property holds and in this case can be call x n converges to that x. So, so claim is, we want to check whether xn converges to x in almost surely, where x is at omega equals to 1 this is my x. So, now can you verify and see whether this guy satisfies this property? Whether this is true or wrong? Yeah?

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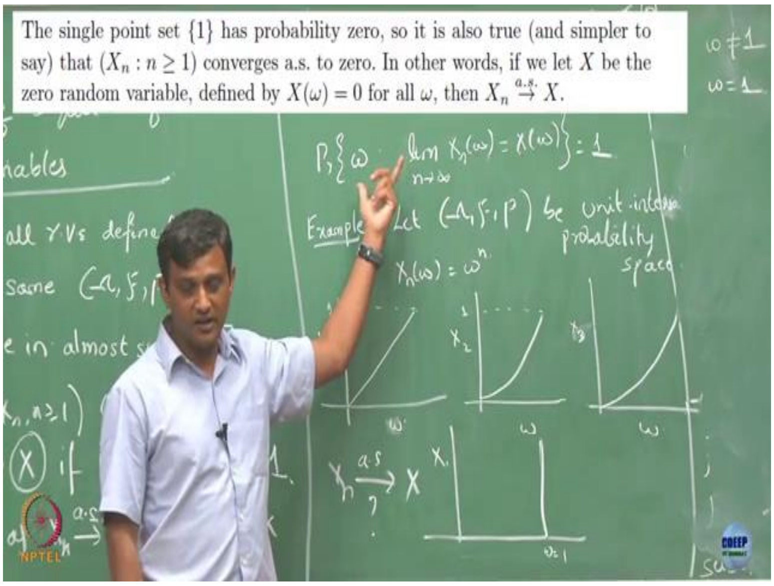

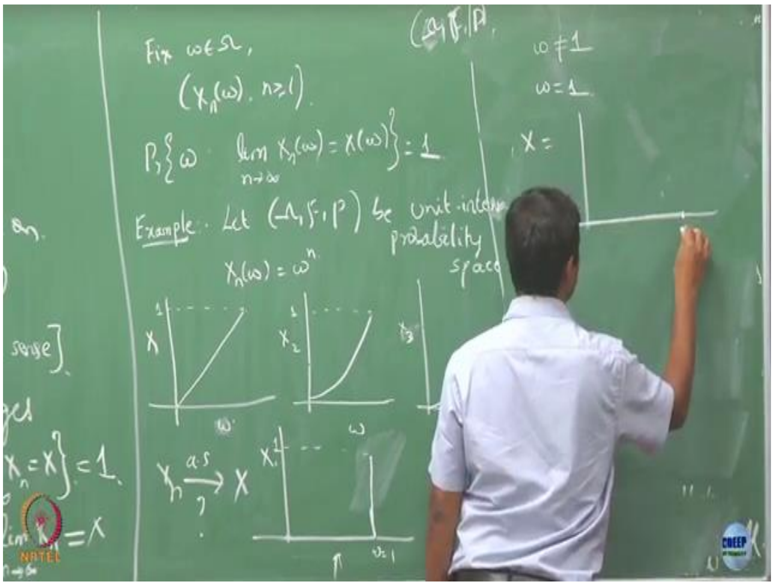

So, let us take our omega which is not at, so let us take omega which is not 0, not 1. If you take any omega not 1 that means it is strictly less than 1, what is going to happen xn of omega will converge to what? It is going to converge to 0? And X is also 0 in that range and if you take omega equals to 1, what is this going, guys are going to convert xn of omega, they are going to be one and what is this guy is, this is going to be 1.

And then it looks like curve, so all omega which are between 0 1 are going to satisfy this property. So, they are included in the set and what is the probability of that set? What is the probability that set, 1 because every point in guys, America has satisfied this and So, probability of that set is 1, because every point in omega has, omega has satisfied this and the probability of big omega is one that we already know.

So, by this thus, by this definition, we already know that this example converges to x which is like this, almost surely. Just let me check this at the point omega equals to 1 at omega equals one this guys, these guys are all 1 1 1 and so...

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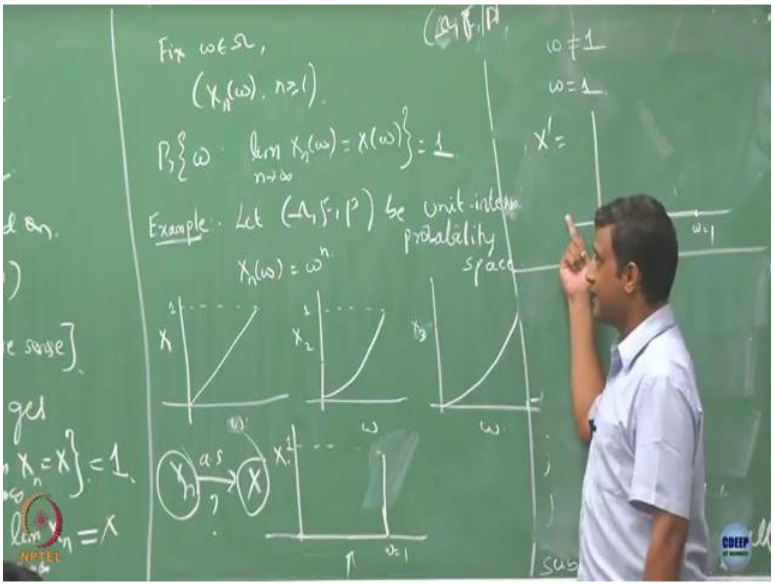

So, fine if I have defined my x equals to like this, which is 1 only at omega equals to 1. So, let us say I am going to define my x to be in a slightly different fashion. I am going to take my x to be 0 all the way even at omega equals to 1. It is not jumping at all here. Is it for that in this case? Let me call this as x prime here. Is it true that my x n converges to x prime almost surely?

Student: No

Professor: Why?

Student: (())(21:48) equals to 1.

Professor: Yeah, but I do not care about one point, what I care about this probability. So, as you said, other than omega equals to 1 everywhere this holds, only omega equals to 1 not included. So what is the probability of this set? We all omegas are included except omega equals to 1. It is going to be still 1? Because probability of that single time 1 is going to be 0.

Is it true that in that case my xn converges to x prime also here because that singleton values did not have any mass in our example here. So, both like, both are valid, like I can say that both converge to this random variables. So that is fine. Another thing we will likely notice is, when we have a deterministic sequence of random variables whether my limit was always unique… Was it? Like, is it possible to, let us say I have a deterministic sequence a n, can it have two limits? So, then limit is always unique.

But when you are talking about convergence of this random variable that is not the case. So, here x is the, when I said xn convergence to x, this is my limiting random variable and these are my sequence of random variables. So, it is fine that is not the case that I have only one unique random variable. So, but as you will see that this is only at the points which carry 0 mass. So in that way, this random variable and this random variable on this probability space they are identical, because they only differ at points which has 0 mass.