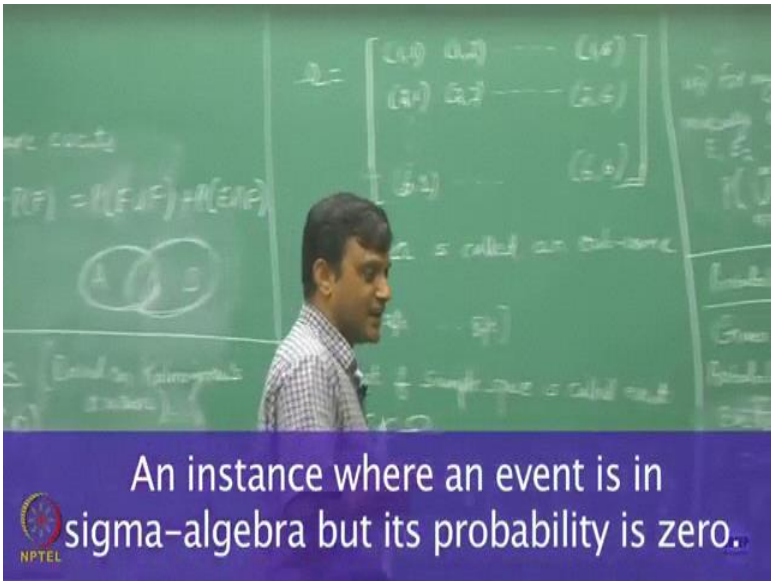
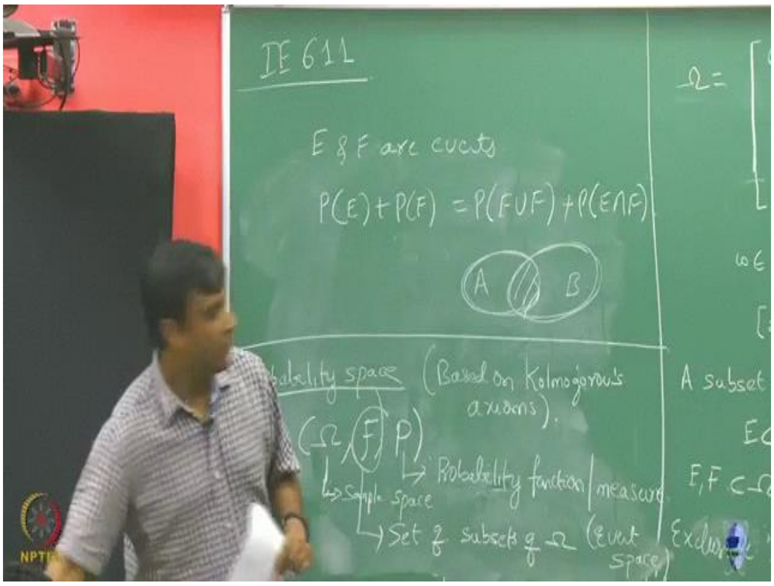
Introduction to Stochastic Processes
Professor. Manjesh Hanawal
Industrial Engineering & Operations Research.
Indian Institute of Bombay
Lecture no. 02
Axioms of Probability

Okay, so far this long, long, long back, people try to put all this probability theory in a very, very systematic manner. So that all of us are on the same page, when we are talking about these term things. And also they lead to kind of intuitive properties that we want them to satisfy.

(Refer Slide Time: 0:40)





So, that is given by some we called some, someone called like (())(0:46), you might have heard this name. So, he built this all very systematically by introducing this notion of what we call as probability space and this is. So, to do this, he introduce a triplet called denoted as omega script F and P omega is same as earlier it is the event space, yeah?

Student: Sample space.

Sorry sample space. So, what is the difference between sample space event space did we say that? The same right like, I mean this is a collection of, so okay. So that so what is, so, I just by mistakenly said sample space sorry event space, but I have only introduced this as sample space. So this, what does the sample space constraints? It contains only possible outcomes, event space could be collection of events and that is where this, F come to picture okay.

So now it is not necessary that every subset of a sample space is a event of interest to you okay? I mean for example let us say two guys they are making some decision. Let us say just making yes and no decision one guy says yes, no or other guy also has to say yes and no and these guys are just like dead opposite guys. I mean they will never say the same thing. Then you already know that yes, yes and no, no option is out of the picture right? You do not want to consider that event at all.

So and in that you are, your events of interest are only now restricted. It is not that all possible subsets of your sample space okay. So that is where now this will differ, what is that possible of events that you are interested in. And now you are, once if restrict yourself to possible set of events now on that you want to assign the probability that, that outcome is going to happen and that is what the third term is going to define clear.

Student: Sir, odd term is probability of a particular event that learned.

Professor: Particular event that is coming from this event space I am going to make it now more precise. So, like now, this is your sample space. So this always happens right. Like do you see that why this should be done like why this? I will not always consider all possible subsets of sample space as event. Why I will be only taking only some of the events? And so you can imagine right.

Like you, when you are going to do an experiment, best, whatever outcome, and when you have to do, going to do this experiment. Let us say 2 or 3 of you are going to do this or whatever you are individually doing, you know that some, some outcomes are not possible.

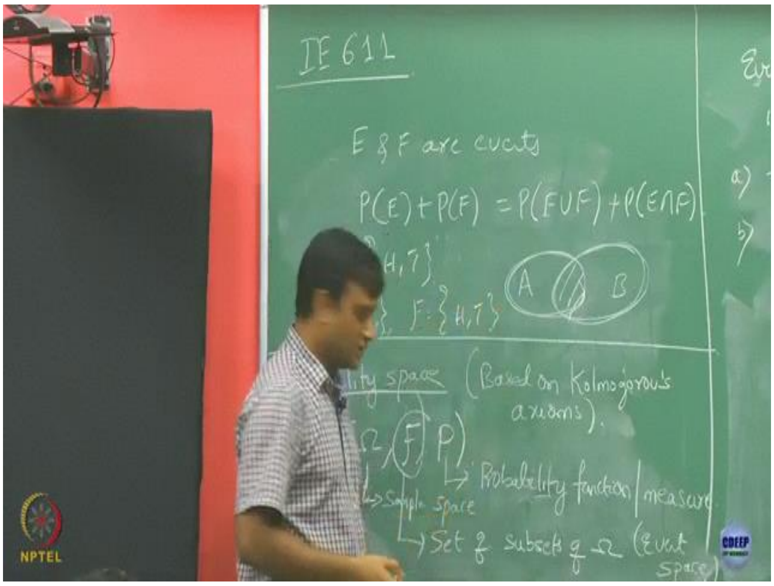
Even though that the outcome is there in the sample space, but when you are looking at that particular event, that may not happen.

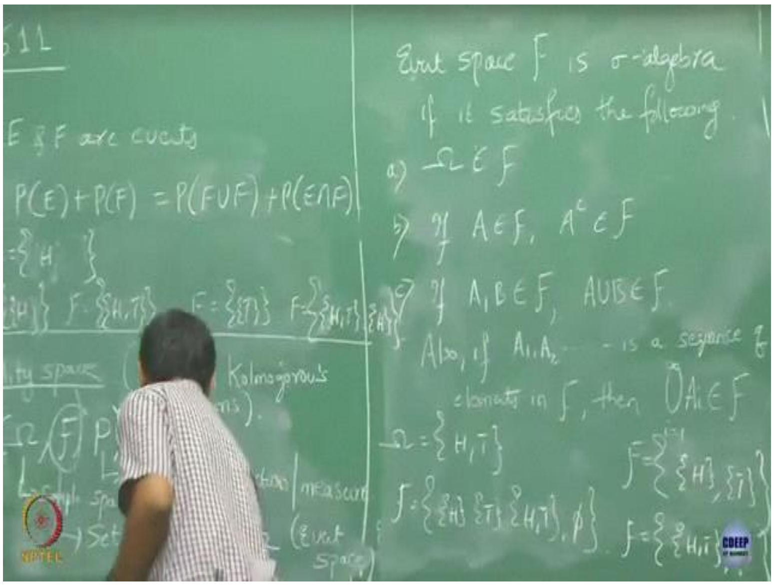
So, can anybody think of a example that you have seen in your daily life where this can arise? (())(5:53) Let us take one hypothetical example. Like let us say, you are, you are reaching airport and your friend reaching airport and let us say you are another close relative reaching airport. What are the possibilities all 3 of you reach, none of you reach, 1 of them reach, 2 of them reach like that right, there are so many possibilities.

But you know that 1 guy is already in the airport right. So, that guys never going to be, so that guys on already in the airport right. That possibility that that guys is not airport out of this option is already eliminated. So, there are always such among all the possibilities, such combinations can be eliminated right. So, that is why you want to restrict yourself to some events which are not, which is not necessarily always a subset of your omega. Maybe you can think of better real life example it is not on top of my mind now.

So fine and this one, they are going to call it as event space. This is basically set of subsets of we are going to call it as event space, and this class term, we are going to call it as probability function or probability measure, can now, let us see.

(Refer Slide Time: 7:56)





So the script of here is basically a collection of events right this is an event space. Now, as I said it could contain either all set of subsets of, all subsets of your sample space or it could contain only some subsets of your sample space. But how that should be? Like if I want to do a consistent modelling, how this F event space should be? So, for that we are going to make some axioms that is going to say, how this F is supposed to be? So for that we are going to say something like, you are going to call, event space okay, event space.

So let me write the properties of this and let us then discuss whether they are natural, are you expect this thing to be at least as necessary to happen. So, we are going to say some event space F to be sigma algebra if it satisfies the following. First thing I want is this omega to be included in my event space that is entire collection of my outcomes should be included in F. And then if some event if belongs to F, then A compliment belongs to F?

Student: (())(10:20)

Professor: F is what?

Student: (())(10:31)

Professor: F is a collection of subsets of omega. That means F is a set of sets, you understand, that is what I said right. Set of subsets of omega, you can think of power asset that is the largest possible event space you can get right? So okay, before I write this may be an example is good here. Let us take our trivial thing omega is equals to H and T. What could be a event space? It could be simply H that is it, only 1 element or F could be H and T itself, like it has the both elements.

Or let me just has H equals to T or it could be null set. Anything is possible, right? That is what I am saying. I am going to focus on certain sigma algebras. I want, now, I am telling you, what are the properties of that event space? Okay, what do the properties? I mean when I call it as sigma algebra.

So first thing I am saying omega, omega is itself a subset of itself, right? Omega is a subset of omega. So it can be possible candidate in F. So I am saying okay, when I say sigma algebra that omega belong to F. If any, if suppose let us say some subset of omega. Let us call it A belongs to F its compliment also belongs to F then.

Student: That is the set of subset so would be just (())(12:38)

Professor: So okay, so in this case it makes more sense to write them as. So okay fine, so this should be like, this is 1, this is, this just 1 and maybe the yeah this is one or other possibility here just to make it more clear here is, it could be like H and T and just H. All of you get this. So, I mean, so earlier this is what? Now the distinction, when I say it in a flower bracket, this itself is a set. And here this is a set, this is a set and now this is F is a collection of these 2 sets.

So, A is such a subset right, because this an 1 element in F, 1 element and F means. This is one subset of omega, is that clear.

Student: Other matters (())(13:46) first H then HT or?

Professor: Set you call A, B or B, A how does it matter? It just a collection of sets.

Student: They are same same.

Professor: Yeah, you just say 5 people, you do not say who is who. That is why I add order. I mean, if you want to say order, you need to define what is order, right? When, when, when you say order, okay, this guy is greater than this that guy greater than this guy, What I? That way is not getting, just write it. This is 1 set, this is 1 set.

And the third point we want is. Okay, so we want is F, A and B belongs to this F. Their union also belongs to F and as an extension to this, also if A1, A2 is a sequence of F, then F. So, I am saying, if I have a collection of subsets, which I am going to call event space and I am going to call that event space particularly events means to be sigma algebra if is the following hold.

So, first thing is omega belongs to F. So do you think this is natural I should include omega to be in event space. So simple question is okay let anything happen, that means enter omega right. So, let omega be in F and now if you say some event happens, there should be some meaning to that event not happening also right. So, that is why, if A belongs to A. A compliment belongs to A, you should say A can happen, B cannot happen.

Then you should be able to answer whether both can happen and that is what A union B that, that has to be there in script F. And also you like, like, like that instead of just 2 or like finite number of unions to be in F, you can have n sequence of events from my script F. All this A1, A2 they are belong to F. Then I want their union to be also in script F.

So, so this is other one I am conducting, trying to model something, some random thing and if I want to define the set of outcomes in, in the set of outcomes. I would like the outcome omega to be included. If A belongs it is compliment to be included. And this unions should also be included.

Okay, let us take again, let us go, let us go back to our H and T right. These are just outcomes. I said these are all possible subsets right. These are collection of subsets, which I call it F. What is the best possible, what is the largest subset of the set, number of? What is the largest number of subsets from the sets I can get?

Student: 4.

Professor: 4 right, What are those?

Student: HT HT.

Professor: H,T. H,T and null set that is like a power set. But I had here considered only some, some subsets here. That is what want to know like that is. So, okay fine let me write it here. If this there is one possible ways, F could be H,T. H,T and null set. These are the only possible subset of omega I cannot add anything more here, this is the adjust.

But I, Why should I look for the largest? I could also look for the smaller than this in this case this could be on possibility, this could be on possibility, this could be. But all are this could not be sigma algebra okay. Now let us say, if I am going to define F to be like this for this omega is this a sigma algebra?

Student: Yes.

Professor: Why? Because now just best thing is, just go and apply plainly these axioms. So this is why we are going to call them as axiom but because these are fundamental things which you need to hold and these are definitions. When you are going to talk something system algebra, the only thing that is common to both of us is this definition.

Now let us take another H and T, is this a sigma algebra?

Students: No.

So right away the first condition is violated, right?

Student: Sir what is the meaning of sigma?

Professor: Meaning of sigma is in definition. I am telling this is what? Then you are asking what is that?

Student: (())(19:50)

Professor: So okay, telling it in other way is sigma algebra is some sensible set of events. I want such that they satisfy some property. That means reasonable sense that is what I said right? It is something which should include all possible outcomes as one possibility. If something happens, naturally the question you are going to ask is what is that event does not happen?

So, both happening and not happening should be there. And the second condition is what that. So I am going to say okay, that happened, this happened can both happen. This is what you guys always asking right? I mean, if I did that course, can I do this course? If I drop that course can I do this, if I want to shift it can this be possible, where all permutation combination can happen? But as we all most of the time reject that right? There are only going to take some possibilities and this is what those some possibilities they are going to define okay.

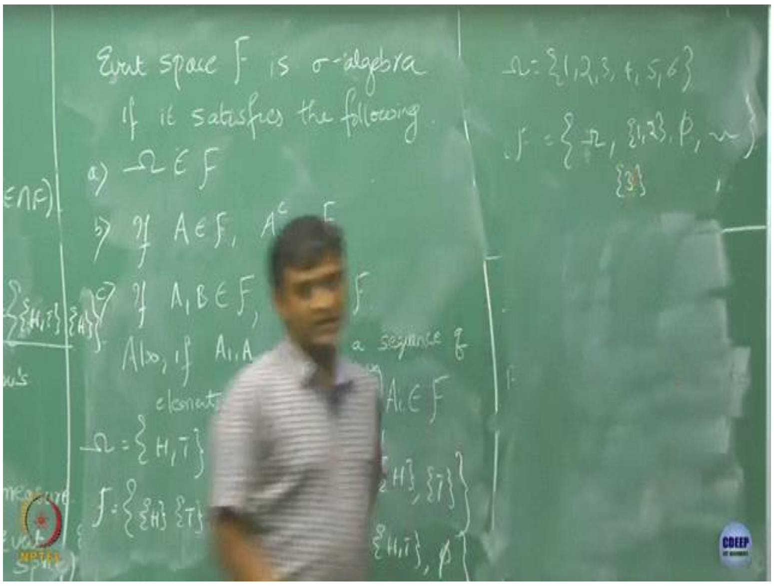
So this is not a sigma algebra because it does not satisfy the definition. Suppose I take a simple thing in this. So, what is the, and (())(21:12) simple thing this is, is this a sigma algebra. Yes quickly apply of all the 3, all the 3 condition fine right, it satisfies. Omega is anyway there. If A, suppose if I take this to be the A, A compliment is 5 that already belongs, if I take A to be phi, phi compliment is omega.

So here, we are going to talk compliments with respect to the sample space. So complementation is happening with respect to the sample space. So when I say omega equals to 1, 2, 3, 4, 5, 6 and let say A equals to 1, 2. What is A compliment? 3, 4, 5, 6. Suppose now, I redefined my omega to be just 1, 2, 3, 4, 5. If I take it A to be 1, 2 what is a compliment?

Student: 3, 4, 5.

Professor: So just this guy, so always my complementation is with respect to what omega I am talking about okay. If my omega changes, my compliment can also possibly change. And here that is the exactly thing, my complementation is with respect to my event space.

(Refer Slide Time: 22:50)



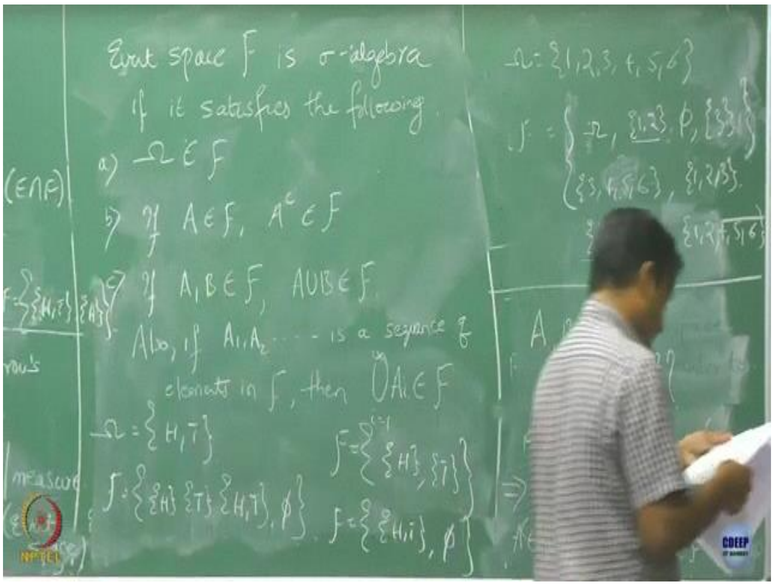
Okay, these are like, this is a very term model right? You are not getting that more, less space here to construct a nice event spaces which are not sigma algebra. Let us take bit the a (())(23:05) problem. So in this let me construct a simple thing, so some possible I am going to take you. So first I will going to include omega that entire thing. I will take 1, 2 and phi like this. And let us say I will ask you, give me 1 element here that will make this F sigma algebra. We just want to take the compliment of 1, 2 to make it, any other elements required.

Students: No.

Suppose I have added one more thing. Now tell me how many terms I should include in this so that it becomes sigma algebra.

Student: 4.

(Refer Slide Time: 24:10)



Professor: What are those? So, okay fine. So now I have these elements this, so now tell me what are the things I should include in this?

Students: Complements (())(24:22)

Professor: Compliments of this okay 3,4,5,6

Student: Their union and its (())(24:29)

Professor: Tell me one by one.

Student: 4, 5, 6.

Professor: So first go systematically right. First thing is their union should be there. So 1, 2, 3 and then its complement. So I have the compliment of this. Now, what is this? I have to bring a compliment for that right what is that value?

Students: (())(24:54)

Professor: Enough or I need more, more what.

Student: 1, 2, 3, 4 (())(25:13)

Professor: Am I right?

Student: 1, 2, 4, 5, 6.

Professor: So the union of this should be there right, it is already there. We are done, done. So, no more addition is required. So fine, so what we are doing is, we are just sticking to the definition of this, so that we are, we call it a sigma algebra.

What we just said is? In this condition in this axioms, basic axioms, you will see that these are the bare minimum things I need. In, in my events space. So, in this I did not say that if A and B belongs to script F. A intersection B belongs to script F, they are need to explicitly saying this or that is implied. So, A, B belongs to script F. I know that their union belongs to script F. Supposes if F is a sigma algebra. Does A intersection B belongs to script F.

Student: Yes.

Professor: Why?

Student: Because A union B is inverted and then (())(26:41).

Professor: Okay, tell me A union B belongs to F, this implies, then?

Student: (())(26:56)

B compliment and belongs to F. So does this imply A intersection B belongs to F? Start wherever you want. What do you want to start with?

Student: A, B belongs to F. So A compliment and B compliments belongs to (())(27:34).

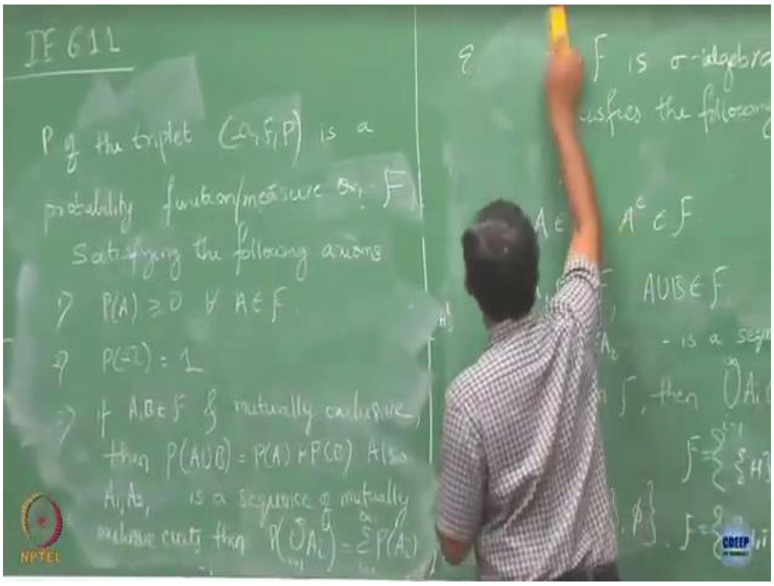
Professor: Fine, then I want to reach here.

Student: A complimented union B complimented (())(27:47)

Professor: You want to take their compliment, I mean union, then you take. So if this belongs I know that this compliment has to belong to this. Then it is, then once I have this I also know that A complement union B complement of compliments belongs to F and you can see that this is nothing but A intersection B.

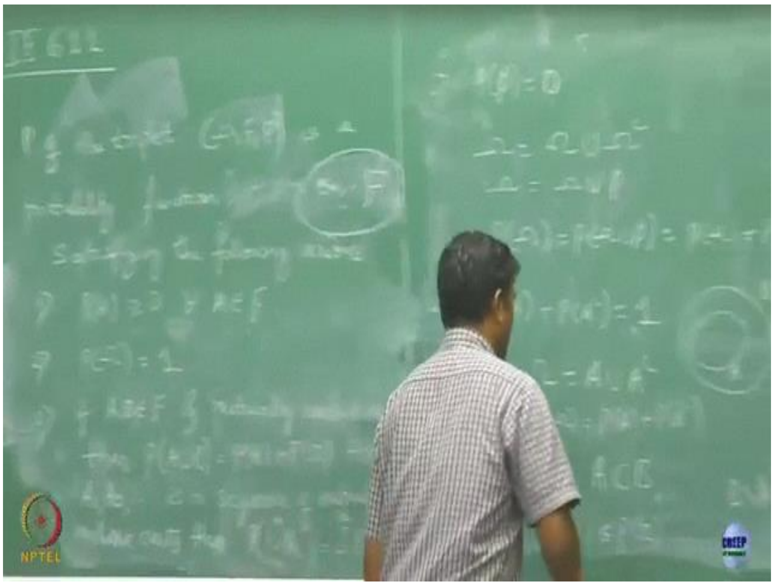
And once you have this, similarly you can go and extend if you have an sequence of elements in F such that their union belongs to F, you can show that their intersection do also belong to F, same set of arguments. So, make sure that you work out that.

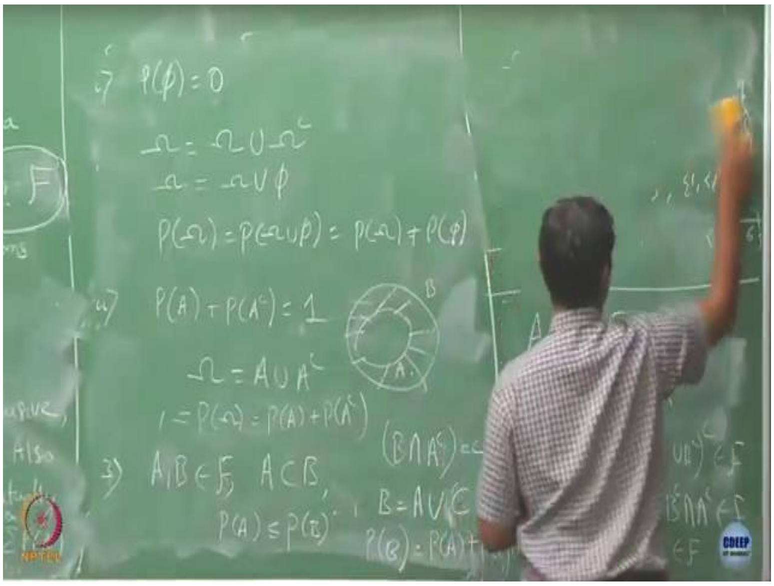
(Refer Slide Time: 28:49)



Okay, so last thing. So, we will going to now revisit what I have already defined P function. Now I am going to make define this. So P of, now this P is defined on F, so the way we have defined this triplet omega F sample space. F comes from the sample space and now this P is defined on this event space. Once you give me this event space, I do not care what is your sample space? To define my P.   So we already written this, let me rewrite, P of A is equals to 0 for all, is equals to F.

(Refer Slide Time: 32:13)





Okay, let us see now what is the implication of?

Student: (())(32:19) formal probability and probability measured on (())(32:21)

Professor: So, there is not any difference, we are just making this the same whatever we have defined earlier. Now we are trying to define it on my event space right? Earlier I have just defined it on any possible event, right you just take an event. Now I am saying that fine I do not need to define on any possible event. I am going to take this, take this bit more systematically for a given and then experiment. I am going to represent it using these 3 omega, script F and P.

Omega is outcome, if the sample space and script F is possibly the events of my interest and of that this probability is not defined. I know it was the same thing. Earlier A was any event now I am making it more precise to make this triplet clear that this probability is now defined on the elements of script F.

Now, from this is it true if P is a probability function, we are not going to use this word measure. Measure is for a bit more abstract, more general things, I mean in the space we are looking, lets simply keep calling it probability function or a probability function defined in this fashion, is it true that probability of a, it is a probability function. It is defined the probability function is defined on F right. F includes null set. So I should know what is the value this probability give some a null set?

What is this value is going to be? Can you show this or if you have to show this, what should be the argument?

Students: (())(34:29)

Professor: Okay, let us take in this case, I know A. Let us take A to be omega itself. Omega belongs to F right. And by this if omega belongs to script F. I know omega compliment also belong to script F. So omega is equal to omega union, omega compliment. And in the space of omega what is omega compliment? It is a null set by definition. And this omega null set disjoint, means right there is nothing about this. Now, what is this I can now? How can I conclude?

Student: P sigma plus 2 pie.

Professor: Why is that?

Student: (())(35:39)

Professor: Because of my third part. So this concludes that my P of phi is equals to 0. So do you expect P of A plus P of anything to be 1. So can we is that, it is necessary that I have to include this in as a fourth condition or this has be implied from this 3 conditions. How?

Student: (())(36:18)

Professor: So what is omega?

Students: (())(36:21)

Right? And is A and A compliment or mutually exclusive.

Students: Yes.

So then we are done right. And we know this guy is 1 by definition. And now, suppose let us say, A, B belong to script F and let say A is a subset of B. So, do you expect the probability P to give more value to this event or this event.

Student: B

Professor: More value, do you expect it to give more value to B right, that is, So do you think I should include this in this condition? So, this is a natural right, if we want this. If 1 event is has a more element, this probability of likely is more right. Should I include this here or so this is a requirement right. I like this to be there. Like I want P to satisfy this, is this implied by this condition or I have to include this?

Student: Implied.

Professor: Why? How? Construct.

Students: (())(37:49)

Professor: So what? Can one of you say what could be the arguments to show this?

Student (())(37:57) the intersection being another set.

Professor: B?

Student: Minus A intersection B.

Professor: B minus A intersection B.

Student: Yeah.

Professor: Okay, I will write whatever you want to say B minus A intersection B. What does this?

Student: (())(38:24)

Professor: Okay, let us make our lives simple. Let us take this to be set B, bigger one and lets take A to be something contained in it.

Professor: So I mean, there could be many possibilities. Can you think come in, I want to do the full thing. Fine, let us try to construct. So the thing we know that you want to apply this I want to like somehow bring in mutually exclusive sets and complement. So lets say if I know that this set A and whatever that in this anyone.

Student: (())(39:16)

Professor: They are disjoint right. How can I represent the guy in this?

Student: (())(39:21)

Professor: One, okay you tell.

Student: B intersection A compliment. Is this correct, what is this B intersection A compliment will give me?

Student: Shaded area.

Professor: The shaded area only. Now how can I write probability of B how can I write B?

Student: A union B.

Professor: A union right. Now lets call this C, some set right. Does C belong to F? You add, does C belong to F? Why?

Student: Two things are belong to F.

Professor: So I assume A belongs to F. So is A compliment belongs to F or assume B belongs to F. So therefore this belong to F. So this intersection belong to F as I said intersection of 2 elements belonging to F also belong to F right. So this C belongs to F. So now I have this I am going to just write it, I have this B written as A union C. I have argued that both are disjoint and both belong to F and on both of these I know P is defined. Now, what is this? I know P of B is equals to P of A plus P of C. Why this?

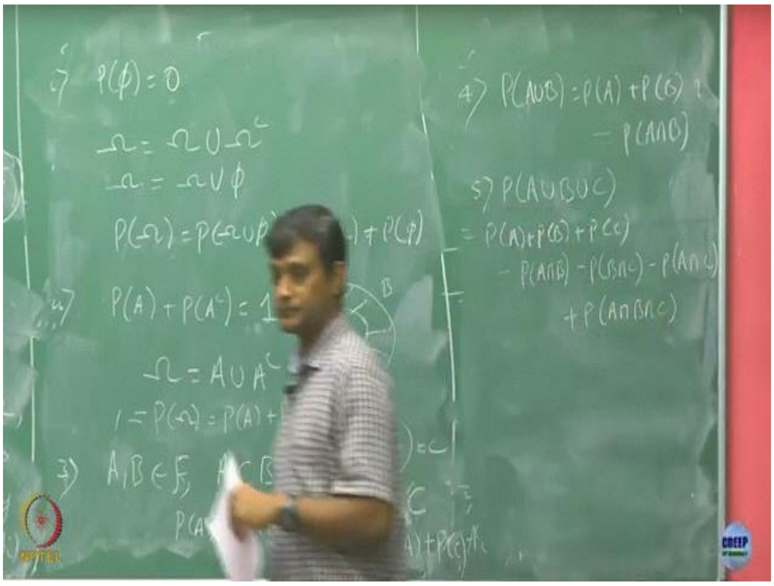
Student: (())(41:13)

Professor: My third property. Now, with this am I done. Now, I go back to this and I know that P of C has to be?

Students: Positive.

Non-negative like it has to be greater than 0. Then it must be the case that P, B is greater than or equal to P. So like this, you can, these are like that is what these are called axioms, these are very, very elementary conditions are required, they are going to imply lot of thing. So, they imply many things which I am going to just write and I expect you to work them out.

(Refer Slide Time: 41:52)



And some of them will be also in your assignment. So, these are 3, may be fourth is. So earlier I have said that I expect this property to (()) (42:11), but you can show that if these 3 conditions false this is indeed true for any A, B belong to script F. And you can generalise this by taking, okay let me first. The things were derive and you can derive more properties which I will skip. So, some of these more properties that you can derive out of this axioms we will see in the tutorial session.

Student: (())(43:06)

Professor: Okay the last one. You can also visualise all this through this Venn diagram right. I mean you should whenever you, is possible just draw Venn diagram that will help you to understand all this relations instead of memorising the Venn diagram should help. So let us stop here.