Introduction to Stochastic Process
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Lecture 19
Properties of Poisson Process (Part 1)

So okay, let us continue our discussion on a Poisson process. So, before that, in the last class, we introduce what is a stochastic process and then we introduce some properties like what is a counting process and there what we meant by independent increments. And based on these properties, we defined a process called Poisson process. So, if you are going to look at some of the properties of this Poisson process and try to prove them as we will see that these properties you can directly derive from the definition and they are also very appealing in the sense when you want to do kind of a counting this property is what you desire.

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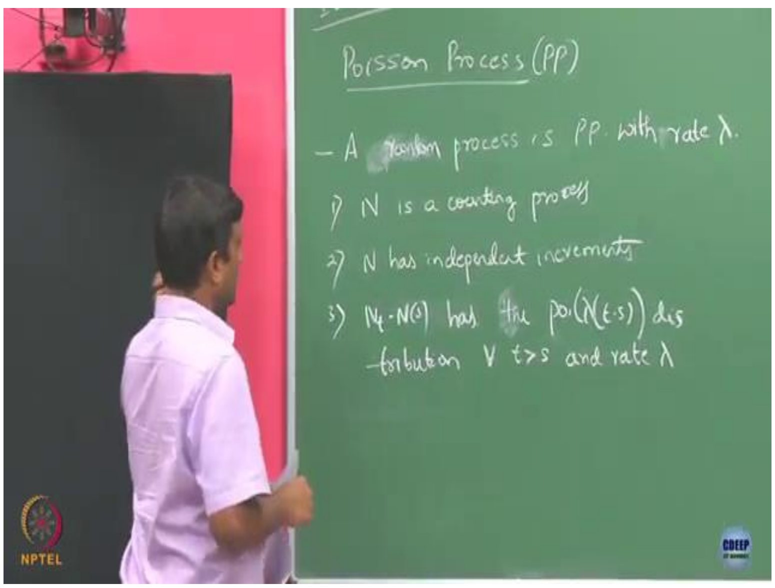

So, these are the properties we said let then is if a Poisson process, a random processes is a Poisson process with rate lambda that means n is a counting process and has independent increments and if you take any interval or any indices T and S then the number of counts. So we said that this is N t minus Ns has Poisson distribution with the rate which is lambda times the length of that interval minus t minus s. So, we are going to today see the following properties.

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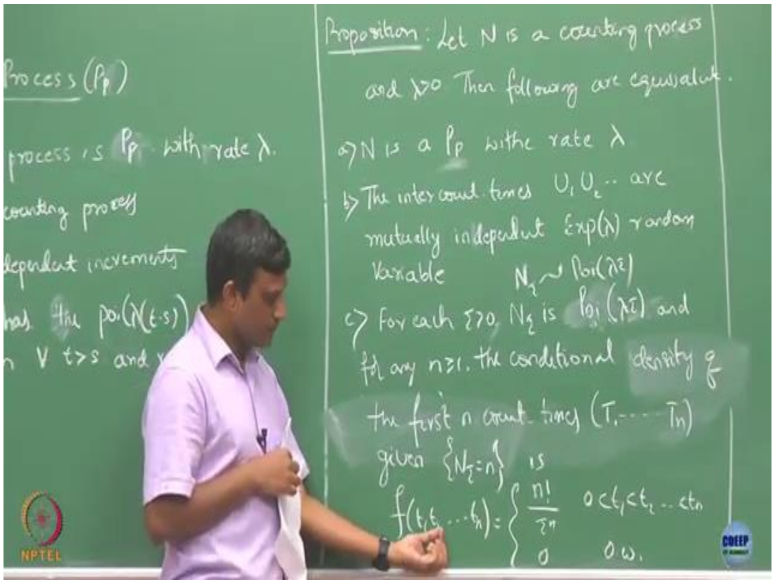

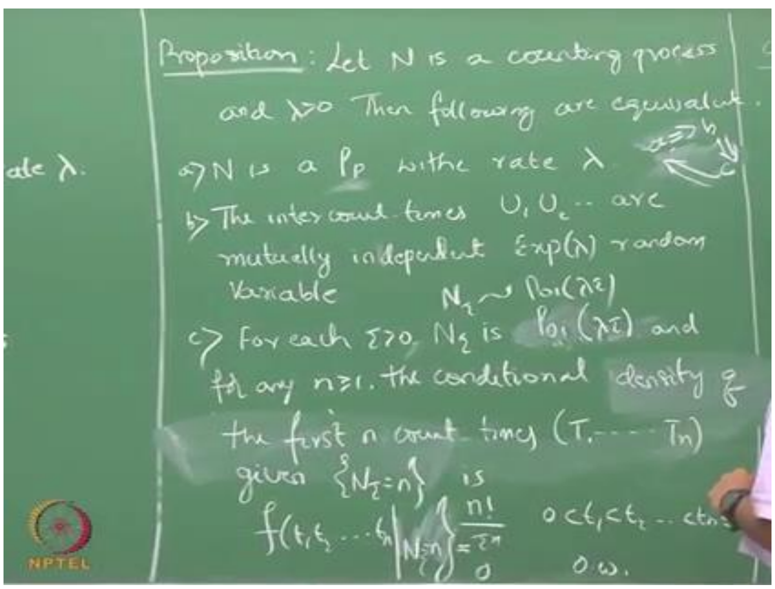

So, we had said some random process to be a Poisson process, now we are saying that it has other equivalent characterization, that is you can say up N is a Poisson process with lambda that is equivalent to saying that inter count times U 1, U 2, so remember we have already defined what we mean by inter count times that is between two count the time elapsed. So, that I am going to denote as this random variable U1, U2.

Okay, so they are random variables, right for a process, the time between two successive counts can be random so that is why I am denoting them as U1, U2 they are these inter count times they are mutually independent and also exponentially distributed with parameter lambda. So, all these

U1, U2 they are identically distributed further independent and common distribution is exponential distribution.

And this is same as saying if you are going to take any Tau positive then N Tau, what is this N Tau? This is in the process the random variable index that time Tau is Poisson with rate lambda T. So if you are going to take any tau, this N T is Poisson distributed with N T and if we are going to take any N the conditional density of the first n count times, remember earlier we have defined small t 1, t 2, t 3 like t 1 was the first count time, t 2 was the second count time like that.

But this count itself can happen at random times. So that is why I am now going to denote them as random variables with capital letters. And if we are going to look at the first N count times, and that joint distribution that joint distribution is given conditioned on the fact that this N tau has taken the value N that is going to be given us N factorial divided by tau to the power n.

So, we are saying this is A same as saying B, B is same as saying C or A same as saying C all these are one implies the other okay. So, okay just now from this property, we are going to look them try to prove each one of them today what is saying is if I have a Poisson process, okay then that time of arrival between any two counts or the interval between any two counts is going to be exponentially distributed with rate lambda.

And these intervals are going to be independent. The distribution of these (independent) intervals is going to be independent and further now if you look, focus on this. Okay, so I missed it, this is conditioned on the E range N tau suppose let us say N t if the Poisson processes that at the time T tau sorry tau N count is happened you are conditioned upon in this. Now you want to see that what is the joint distribution of this?

So, first count happened at time index T1, second count happened at T2 and let us say Nth count happened at time T n, these distribution is going to be expressed like this. Okay and I should also so notice that I am already saying till time small tau, N counts is happened. And the joint distribution of these times is going to be this. So now if you focus on this part, forget numerator. Now it is going to be like just now 1 upon T to the tau to the power n right.

So if I am going to look at the distribution of n random variables and if it happens that their joint distribution is 1 upon Tau to the power n, can you say something about what this joint distribution looks like? So just take N equals to 1, so I am saying 1 upon tau. So in the interval, tau, I am saying something like the PDF is constant 1 upon tau what that corresponds to? Any form, right something now I am saying that joint distribution of N random variable is kind of 1 upon tau to the power N, it what does this imply in a way, what? Yeah there are N uniformly distributed random variable each with parameter tau and they are independent right.

So, they are getting multiplied, 1 upon tau into 1 upon tau into further I have to look it such kind of ordering these uniform random variables. Further, I need to condition that the outcome of the first random variable is smaller than the outcome of the second variable. And outcome of the second random variable is smaller than the outcome of the third random variable. So, I need to further condition on this.

So, if I further condition on this, I am going to get this factor 1 by N factorial in the denominator. So, when it the N Factor goes in the numerator. So, this these N factor is coming due to the fact that I am looking for this t1, t2 be such that they are arranged in the increasing order. So in that way there are two distributions that are related to my Poisson distribution, one is what? The exponential distribution that is the inter arrival, the inter count times are exponentially distributed.

And now, if you look at the directly the time of the count itself and the joint distribution, there seems to be some kind of uniform distribution that is coming into a picture here. So let us try to prove. Now what we are going to do to complete this proposition theorem, we need to show that each one of them implies each right.

So, I have to show that A implies B, A also implies C and if I take B, it should be okay that B implies A as well as B implies C as well as if I take C it should be the case that C implies A and C implies B. So, instead of trying all this thought will try to show is A implies B and we ensure that B implies C. So, ensure that A implies B and then B implies C, so, that should be sufficient right to show that each one C implies B is already, C okay Oh sorry I had to do this and I did not do this, okay let us say.

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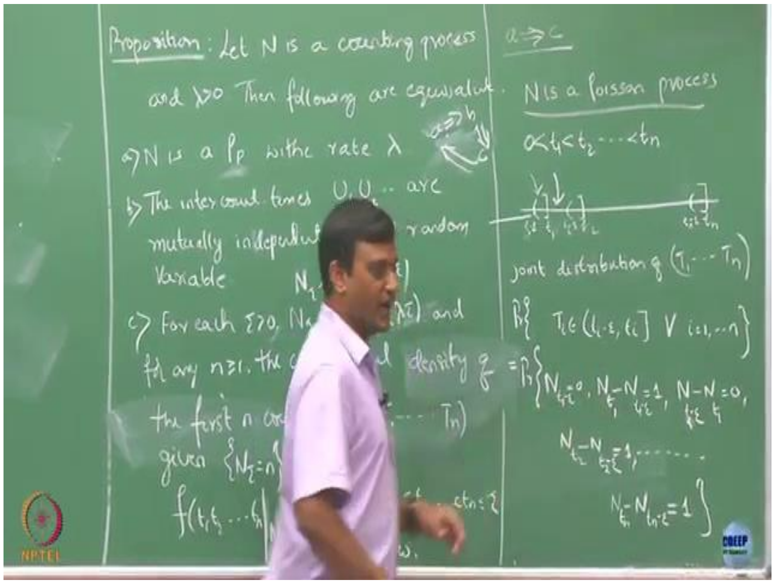

Let us try to show that A implies C. So let us assume that N is a Poisson process. So as soon as I assume N is a Poisson process, I have all these properties available to me, because that is the meaning of N is a Poisson process. So now, let us take any N real numbers such that they are ordered like this T1, T2 all the way up to T n where T 2 is going to be larger than T 1 and so on like this. So now think of them as. So this is T 1, this is T 2 all the way up to let us say T n. So let us take an epsilon.

And now what I am going to do is I am going to look at a small region. So I will just take an epsilon and just go behind T1 and get T1 minus epsilon and similarly around this, I will go slightly epsilon behind and get it to minus epsilon and this is like T n minus epsilon, okay now what I want to do is I want to look at this intervals here. So this means I am open at this point, closed at this point and similarly okay, now let us try to find joint distribution of.

So what is this T 1, T 2? These are the count times and these are the random variable, so let us try to understand. Now let us try to find what is the probability that so what is T1 here, capital T 1? The time when the first count happens, right? So let us assume I am interested in this Ti belongs to, okay, so what is this telling? I am interested in when I take T 1. I am asking the question, the probability that the first count happen in this interval T 1.

So T1, T2 they are given some numbers to me. Now, Ti is a random variable right, now I am basically asking the question, okay this Tith that means ith count happening in this interval, okay and now I want I am looking at a joint distribution and I want this to happen, this condition for all i 1 to N.

Now if I now I want to express this in such a way that I can go and apply my independent increment properties. So I want let us say I want to apply this, how can I express this probability to exploit my independent increment property? So one thing I can now do is what I am basically asking is the first count should be happening here, right? That is T 1, capital T 1 is T 1 minus epsilon and T 1. If that is the case, then there should be no count happened before that, right?

And the first count should have happened here. And then if I want T 2 to be again in this interval, they should know count I should have happened in this interval and the next count should have happened in this interval. Then then my T 2 is going to fall in this interval right. So then this is same as asking, is this correct? I can express this joint distributions like this.

That is nothing happens here and the first count happened here and second count only happened here. That means nothing I do not want any count to happen here and like that and the last count happens only in this region. So, you see that now I have expressed this quantity in terms like in terms of the increments. Okay, so now I want to expand this. If N is a Poisson process are this all of these events are independent or not? Because I am looking at a different-different intervals, right. And I know that for Poisson process, they are all independent.

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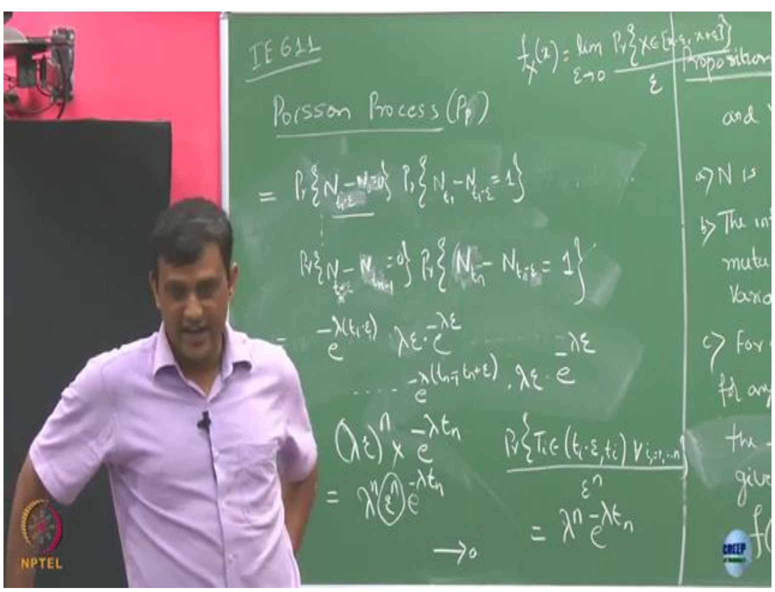

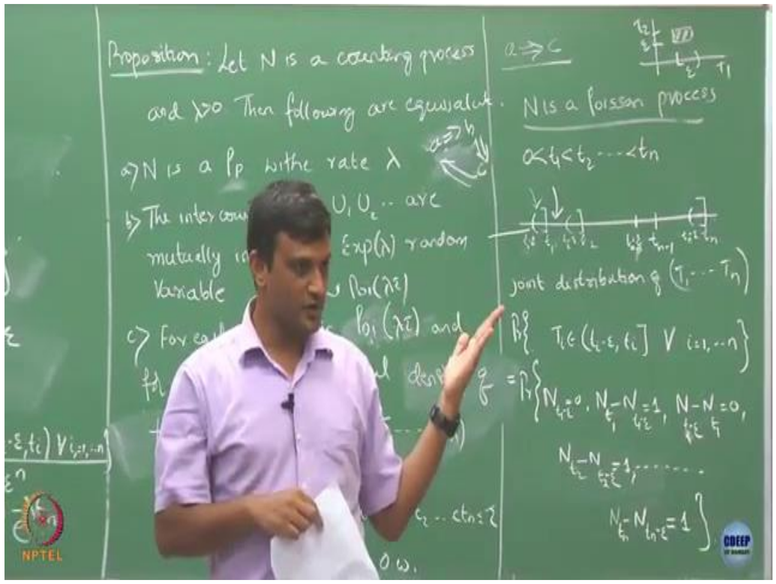

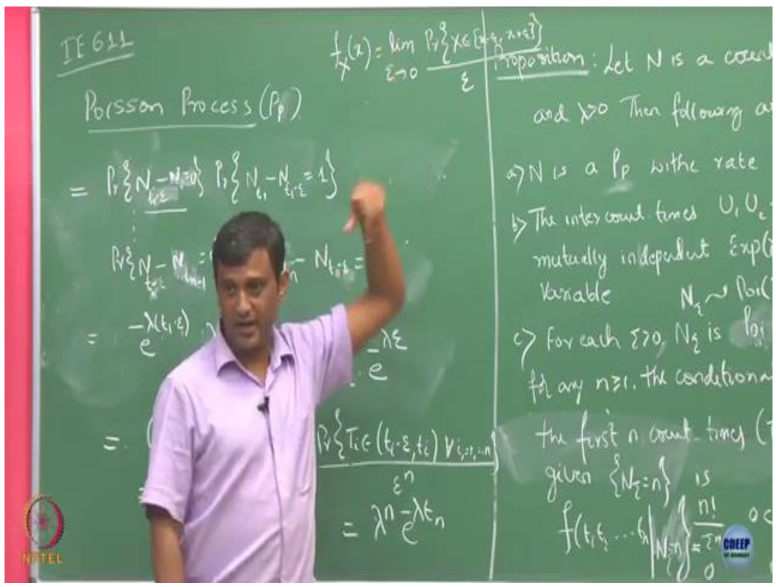

So, and now I am going to further exploit the property that, so now if I am going to look at. now first consider the case. So I can write them as probably okay, just let me write it for in terms of the probability first, probability this is for the first one and the last one is going to be right I simply applied my probabilities on each one of this, yes right.

Now, this is coming from what like as I said, this is coming from independent increment properties. Now further what is the third property of the Poisson process? It is going to be Poisson, right? Like if you are going to so I could as well, write this guy as like n minus 0 here at 1. And I could just take n 0 as 0 itself.

Now what is this probability is going to be? Sorry, this is 0. What is this? What is the length of this interval? T 1 minus epsilon and I am asking, so this is going to be Poisson process with what rate? Lambda into T 1 minus epsilon right. Now, what is the probability that a Poisson process with rate lambda T1 minus epsilon takes a value zero? E to the power, here the length of interval is simply T 1 minus epsilon right?

Now, what is the probability that it takes value 1? The length of the interval here is it is simply epsilon. So here it is a Poisson process with rate what lambda epsilon right. And what is this value is going to be? Lambda into lambda epsilon and there is a one factor in the denominator which we can escape, right. So, like this, you can keep writing for all of them. And just let us write what is this looks like e to the power lambda, this is going to be what? T n minus 1 minus T n plus epsilon and the final one is going to look like lambda T n minus epsilon here this is going to look like epsilon e to the power lambda epsilon okay.

So if now I am going to simplify all these things that are how many lambda epsilon multiply getting multiplied here? There are going to be N one of them and if you go into simplify all of them, you see that some subtraction is going to happen because T 1 is happening it will also come as minus T 1 later. And if you going to simplify that, what it will end up is simply e to the power minus lambda T n okay, so whatever this probability I was looking at this joint distribution it is now lambda epsilon.

So this I am going to simply write as lambda n epsilon n into lambda Tn and now let us say I am now want to take this probability that Ti belongs Ti minus epsilon divided by power epsilon n this is going to be equals to lambda N, A power E to the power lambda Tn, right that is what I have shown. So now if you are now going to, see what I am doing here for a given T1, T2, Tn I am looking at for each of these Ti around an interval of epsilon length.

And now everywhere these lengths are of epsilon. So what I am basically looking at for a given this T1, T2 I am looking at the mass of this random variables in a ball, which is of what volume? So for example, let us take a simple case I have now taken, let us say this is my T1 and this is my T2 and T1 I have looked at interval of epsilon and also on T2 I have also looked at and volume of epsilon.

So what is this region is going to be? This region is going to be I was basically looking for my T1, T2 to be in this region here, right? Where T1 is of length epsilon and T2 is also of length epsilon. So what is the area here? Epsilon square, right. So if you are going to extend it to N dimension, what is this volume is going to be? Epsilon to the power N, so now this is what I want to do. I was basically looking at this joint distributions to lie in some volume which has, which is epsilon square.

Now, that probability I am dividing it by epsilon N square. And now I have what is what is lambda to the power N, e to the power lambda N. Now, this epsilon I have chosen is arbitrary right? I could as well, let this epsilon go to infinity here, even if I take epsilon arbitrary small, this ratio is kind of independent of lambda and right this value is going to remain the same. So if I take this ratio and let epsilon going let epsilon tend to 0, what this ratio will convert what is this ratio according to our definition?

So, you remember the way how we defined our CDF, sorry PDF and we corrected it to the probability? So, what we had said? We had said that f of x at a point x can be thought as limit as this is in all one dimensional case as epsilon tends to 0, probability that x is let us say, in this interval x minus epsilon x plus epsilon divided by epsilon, did we say this? Then we are try to argue the relation between a problem when we try to do interpretation of what we mean by media PDF of a point, at a given point.

So we are dealing here with what like we are dealing here with continuous random variables right, this T is here, are the continuous random variables? T is the count time right, they can take any number on the real value. So, here we are dealing with continuous random variables. So, when we try to give interpretation for our PDF, we had said that, PDF of a random variable x at a point x can be thought of its probability in the neighborhood of x, where that neighborhood is defined in terms of epsilon.

And we can make that neighborhood small by letting epsilon go to 0 and when we take this ratio that is nothing but a PDF, yeah.

Student: (())(29:02)

Professor: Ti, yeah these are finite, but each one of them can is a continuous random variable I am talking about individual one right. So each one of them itself is a it can take any (position) any real number. So, if you have this probability and if you just like epsilon go to 0 in this fashion that is nothing but the CDF. Now, we can apply the same definition here, right. But here is sort of a real line we are in a N dimensional space. That is what instead of just epsilon we are looking at epsilon to the power n which corresponds to area which is the volume in the N dimensional space.

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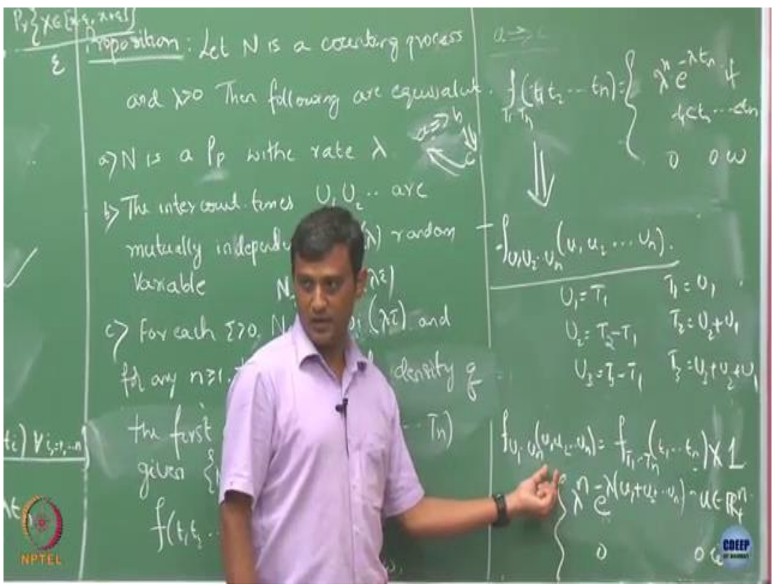

So, due to this we could now write a f of, f of T1, T2 all the way up to Tn the joint distribution of this should be what, lambda to the power N, t to the power lambda Tn provided what is this, this T1 is greater than T2 and all the way up to Tn and now we have 0 otherwise. This is correct? Are you convinced? And what is this? This is the joint distribution of these random variables. So, this is what we just derived by using the properties of my Poisson process.

So, it is clear that we use our second property here of the Poisson process that independent increments and then we use the properties of poison distribution, the third property here. So, did we use anywhere the property, the first property which said that the poison process is a counting process? Whenever use it right because one of the, what is one of the properties of the counting process? N of 0 is 0, right?

I mean, we use part of that counting processes much more, but we partly used some part of the definition it being a counting process here. Okay, now we have but this is not what we are interested in right? Fine, we have come up with the joint distribution of my count times. But what I am interested to show property B is, the distribution of my inter count times. Okay now how to drive a joint distribution of the inter count times using the join distribution of this count times themselves?

So, if you want to drive what property you can exploit now? Is there a way you can think of? So, from this I basically want to know go and write what is the distribution, the join distribution of this random variables? So, if there is a relation between this Uis and this Ts I should be able to apply my transformation of random variables properties and should be able to drive this right. We already studied that, but for now the question is what is the relation between this Us and Ts?

So we already know that right like Ui Ti minus Ti minus 1. So, we know this relation, now how can we write this joint distribution in terms of f of x, f of Ts here? What is the property, what is that we know about this? So, we know that right when we have to write this distribution, we need to compute the Jacobean. So, what is the thing? So, we have this so, let me write it more explicit, we have U1 equals to T1, U2 equals to T2 minus T1 and like that and also we know the other way around right.

So, T1 anyway equals to U1 and what is T2? U1 plus U2 and T3 is U3, U2, U1, all this we know right. So, how to write this then? So we already know that this is nothing but U1 to Un of U1, U2 to Un is equals to f of T1, T1, T1 way up to Tn what was this and what we have here? A Jacobean matrix, right, determinant of a Jacobean matrix and what was that? So, can somebody quickly compute what is the Jacobean matrix here? Is it going to be 1? Why is that?

You are just going to get a, so what I written just compute that and finally, whatever we want here we will have a unit determinant. So now how to express this? I have this T1 but T1 is U1 and Tn is sum of all the random variable, but this distribution only depends or it should be Tn here and Tn here So, what is Tn in terms of Us? Tn is nothing but some of Uis right.

So, because of that you could write it as lambda n, U to the power minus lambda and this is for any U and this is going to be 0 otherwise. Now from this, did we conclude what we wanted to say here? I am looking at the joint distribution of U1, U2 all the way up to Un, now what I have done is finally able to show that give me any U this can be express as lambda to be power N, E to be power lambda U1 plus U2 up to Un.

So, what is this basically? This is nothing but the product like lambda E to the power, lambda E to the power lambda U1 into lambda times E to the power minus lambda U2 like that right I can split it as a product of N such terms and what is each terms there, what is each term there? Is a exponential random variable with parameter lambda. So, that is what exactly the claim and why I am saying independent? Because this joint distribution I am able to express is a product of N distributions, right. And each one of them now corresponding to exponential random variable with lambda, so that is how this claim holds.

Okay, so as you see, I mean, if you highlight Gosh, if I have Poisson process that means nothing but if you just look into the entire count times, the collection of inter count times they are nothing but they are Poisson distribution with the same rate lambda. So, if I say, I Poisson process with rate lambda, or I say I have a bunch of random variables, which poise a random process where each random variables are independent and everybody has a Poisson distributed with rate lambda. That means I am basically referring to the same poison process with rate lambda.