Introduction to Stochastic Process
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Lecture 18
Poisson Process

As of now the way we define this random process they are very general notion, right? Very abstract general, it just said okay it is going to capture something. But often to model something and to analyze this we want this random process which have bit more structure in them. So for example, when we said okay, we said random variable and distribution, but any random variable if I just take any arbitrary distributions, maybe it is hard for me then that is why we defined some special distributions, right.

We said uniform distribution, then said Gaussian, Geometric different-different distributions which are helpful to model the reality to somewhat extent but they are also a bit more structured in the sense we can analyze them system when we model using these distributions. Now, similarly when we are talking about different random process, any arbitrary random process, if I take maybe I may not able to analyze my system at all.

So I would like to now look for certain random process which are more structure in it, but still capture my reality than those using those random process maybe I can try to understand my system and get some useful insights. Now we are going to now start focusing on special type of random processes, okay, so we will see like.

Today and tomorrow, we are going to talk about one special random process called rise on process. And later in the second of this course we are going to focus significantly on a particular random process called Marco processes, okay.

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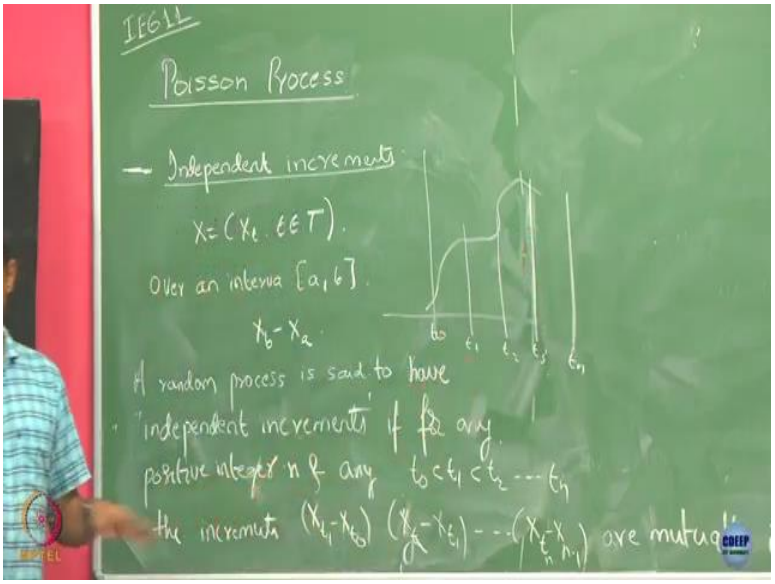

Now, let us say Poisson process, okay so to define this notion of Poisson process we have to understand two properties called, some notion called notion of independent increments and another called accounting process.

Okay, so let me first talk about independent increments. I mean, most of these things like you will find out, start relating these things actually when you get into modeling business and try to analyze, if you just get into some coding business, you will never appreciate any of these things. If you want to really system, you want to analyze? Yes, you need all of these things like.

You need systematic tools to understand any system and you have some common language of talking to other guys who are also doing or understand what we are talking about, right. So you need all these things. If you are not going to do any system analysis or modeling, maybe in your life, you will never appreciate this or you will never find the use of these things.

Okay, so what I mean by independent increments? Suppose let us say I have a process xt and I look at over an interval let us say a b. So for time being, let us assume that this is continuous process, so my T is continuous here and I am going to take some interval then I am going to look at this difference. That is the difference in my random variables at this on this interval, okay

So I am going to look at my random variable at this time index b and at the beginning of this interval at a and then look at what happened in this. Suppose you have processes like something like whatever it is. And I took this interval and I want to see that okay between this interval what happened actually I went, I increased or decreased, whatever you want to understand. So let us say you are interested in these intervals and now we are going to say that random process okay.

So what we will do is if you have a random process let us say and you take any n and you take n indices such that these indices are arranged in increasing order. The first index is t0, second is t1, t2 to tn. And now if you look at these differences between these intervals, the first is x t1 minus x t0, then x t2 minus x t1 like that. Now, if these are mutually independent okay, what I mean by mutually independent? You understand the meaning of mutually independent?

If I take this it is independent of all others and similarly for every point. If they are mutually independent then I will say that it is, has independent increments, okay for example here, if I let us say this is my I will take another interval, another increment here and then so this let us say t0 and this is t1, t2, t3 and let us say this is tn.

So what if I take the increment in this? And look at increment in this interval like each of this interval if that is going to be independent then I am going to call this as independent random process with independent increments. So notice that these intervals need not be of the same length. One could be larger here, one could be smaller. So that is why like this is any set of indices. It is just that they are increasing order. And if you are going to look at the difference, they should be mutually independent, okay.

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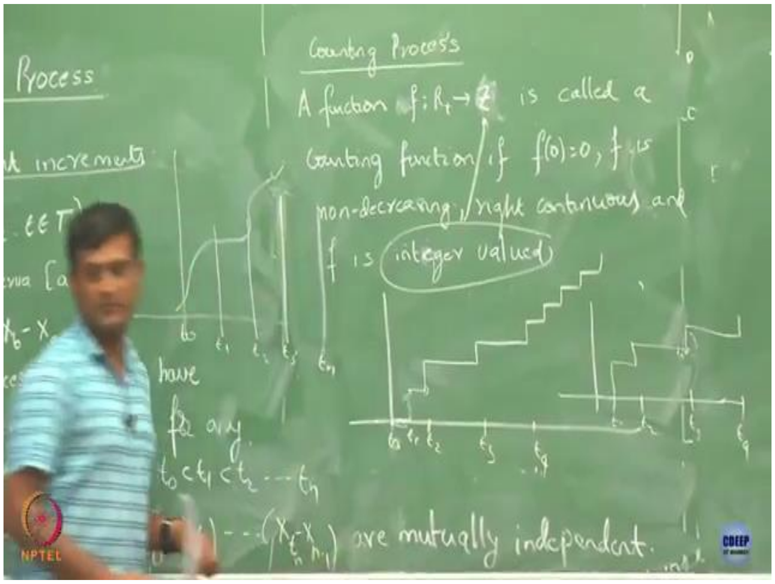

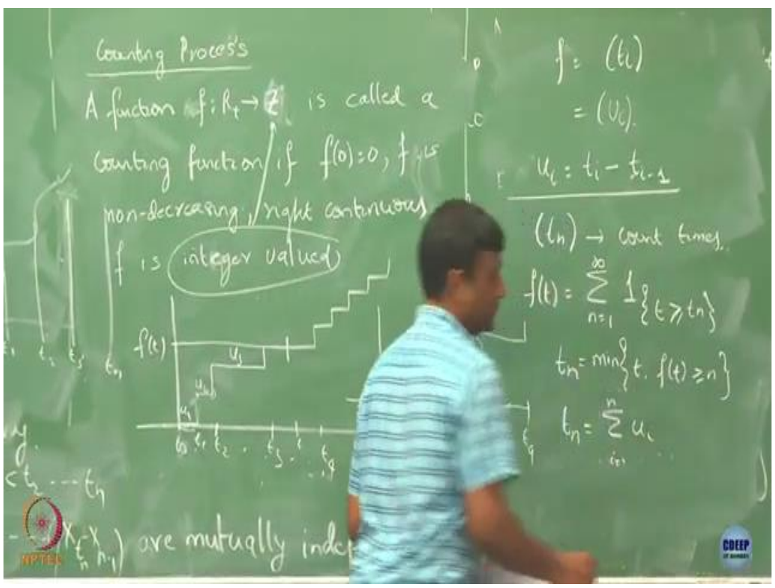

So now something called, we look at something called Counting Process. So counting process, as the name indicates it does the job of counting. So for example, in this graph I have shown here, it could be like a plot or something like the number of people that have entered at this time.

And let us say nobody leaves once they enter. So this is going to be increasing curve and this maybe let us say If it is an increasing curve, so they maybe like you this if I learned to look at this point or this point you can think of like how many people by that time have entered or already there. And that could be like called as a counting. So that we will try to make it some more formal. So this is this like a general graph.

So let us say, we are going to say that a function, it is actually already said this. So you take a function F which will give you at any time any positive real number integer valued outcomes, and it is such that it is non-decreasing that it keeps increasing. And it is right continuous. They are already understand what we mean by right continuous such a function is called as counting function, okay.

For example of a counting function could be. Let us say I am counting number of students that are arriving into the classroom. Okay, let us say all my arrival happens between 5:30 p.m. to let us say max 5:35 or 5:40 p.m. and let us say you guys enter one at a time okay and then at each of this time second, maybe I will have jump like this, it remains like this, maybe like this, here that nobody comes, comes and then like this then some people start coming dhat-dhat-dhat.

Okay, so such a process, such a function can be a counting process, right by our definition. It is non-decreasing and I can make it right continuous always by defining the point to be and distribute this point. And I can also make this to be a 0 at f of equals to 0, okay fine. So now to such a function counting function can be represented in many-many ways.

One possibility is suppose I am going to denote this okay this is t0, t1, t2 this is t3 and this is t4. So t1 is the time instant when the first entry happened or when the first count happened. So t2 is the time when the second entry happened. So I can on my X-axis I could denote my time like this right. So whenever a entry happened that time index I will look into it then I will call appropriately if it is a fourth entry then I will just denote it as t4.

So, now then my F function can be represented in terms of this ti's right? So and same thing right. Like if you tell me this exact points t1, t2 on this x axis, I know that on this point side, a jump is happening. And then I can construct this function from that. So, for example, like if you just tell me a thing and just tell me okay here, here, here, here you just give me this t1, t2 t3, t4 then I know my function looks like here-here, here-here, here-here, here-here. So I could reconstruct my function.

Another possibility to represent the same function is instead of giving this time at which it is arriving, just give me the entire counting time. For example, okay after how many times, how much time from the origin the first arrival, first count happened? Let us call this E1 and since the first count, how much time it elapsed before the second count happened? So that could be U2.

And after the second count, how much time it elapsed before the 3rd could happen? This is U3 like this, right.

So you could also express if you, I could also write this in terms of you Uis where ui are this inter count times, okay. So how is these Uis related to Ti's? So what is U2 here? T2 minus T1 right? And by default T of 0 is going to be taken as 0. So when I said, U1 here this is going to be T1 minus T0 right. T0 is always going to be the origin for us, okay. So the same F can be expressed either in terms of this call times or in terms of this inter count times, right.

Now, so how to make this explicit? For example, if I suppose if I am given let us say I have been given this Tns count times. How can I construct my F function from this? So I want to construct F means to suppose let us say you have been given this time slots and I will ask you to tell me what happens to this function at time T? So give me the value of this F of t. So that means you have to basically count how many counts have happened before that and that is going to give you the value at function f of t right.

So that formal way of writing that is indicated at T is greater than n equals to 1 to infinity. So in this case, let us say your T is here. We know that this T is going to be greater than T1 that will add 1 here. It will also going to be greater than t2 ,this will add another 1 here, this is going to be greater than t3 will add 3 here. But T is not greater than T4, so they are all going to add 0-0. So here you will recover that the value is going to be 3 in that case, okay.

Now If I have only given my FD function like this, how to recover this count times? So I want to now find out Tn from my f t function, so how we going to represent this? This is going to be minimum value of T such that f of t is going to be greater than or equals to n, so is this correct? Try to digest this. So let us say I have been given this function f t now from this I want at understand when the n count happen, right.

So just look when this guy f of t is going to take value larger than n but look as the smallest value of that t. So that is going to give me my Tn. And I already represent, represented Ui in terms of this count times, but I could also write Tn in terms of my inter count times, how can I write Tn in terms of Uis? So let us suppose if I want to get T4, all I need to do is this, this, this, this 2is I need to add, this is Ui, i equals to 1 to n, right?

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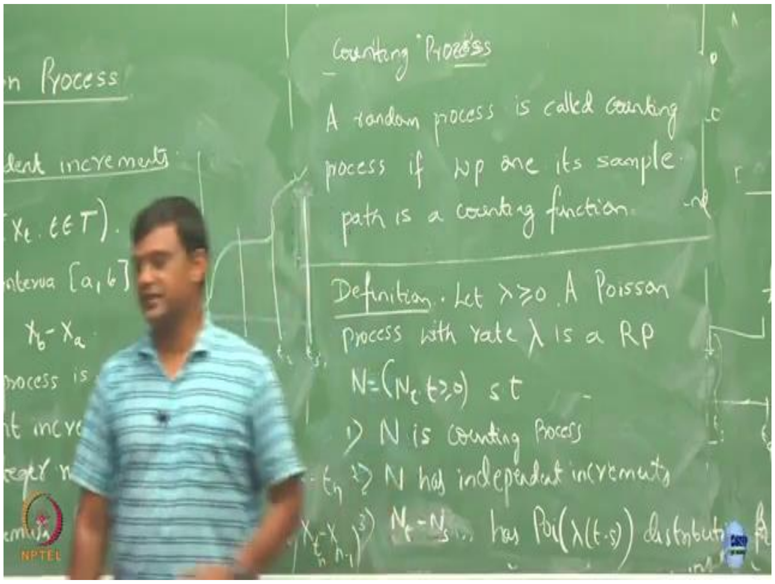

So, now coming back to counting process, so all of you understand what I mean by a counting functions, right? Now we have already defined what is a counting function. It has to satisfy like non-decreasing property and right continuity property. And I already defined you, what do you mean by sample path of a random process? So sample path of random processes if you fix an omega and look at as a function in t that graph is we call it as sample path, right.

Now what we are saying is, if the random process is such that if its sample path is a counting function then we are going to call it as the counting process. Is this clear? So each of the sample paths has to be a counting function then we are going to call that process as a counting process. That is why I like, you can when you see example it will be more clear. This is just a definition. So then we have defined two properties, one is the independent increment property and another is a counting property.

So there is one distribution what I already called as Poisson Process which is based on these properties and it comes very useful when we want to model many-many things. So let me define that process now, which is based on these two properties (greater than S). Okay, just what random process means is for a given lambda, it is a counting process. That means all of it sample path has to be a counting function. Then it has independent increments.

So if you look into difference number of counts between different distant intervals they have to be independent and third f you look at this random variable for take some T and S and look at the difference. So the way I defined is okay this should be n t minus n s that is the random variable at n t and n s they have Poisson distributed. We all know that is Poisson distribution right. But with what rate? That rate depends on the length of the interval multiplied by lambda. So if these 3 properties holds we are going to call this process as a Poisson process and this Poisson process have very interesting properties which we will talk in the next class.