Introduction to Stochastic Processes
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Lecture No. 16
Random Processes

So, now we are going to move to further higher dimension. So, we started with single random variable, then we talked about vector random variables, then we are going to talk about three dimensional already covered, like vector means.

Student: M dimensional cover (())(0:39).

Professor: Up to m dimension is covered, now we will talk about infinite dimension. It is like I have collection of unaccountably many random variables. And so you will see that in most of the application that is what it is going to matter to you. Because you want to understand like, how the suppose, for example, you want to understand how the stock market is evolving? You will be not interested in one day, you will be interested in how it performed in the last five years, and how it is going to evolve in the near future?

So, you have the collection of random variables, which may not be some finite number there, it could be unaccountably many and if you are trying to like a find a trajectory of a particle or whatever, at every point, every point of time, you want to understand how that is behaving? And there are so many point such times. So, suppose let us say something is taking a trajectory, and at every time you want to, every time its behavior you are trying to control, but its behavior could be random at that point, depending on so many others.

Now, you want to understand at every time instance you want to model it as a random variable. So, we will try to make this more precise. So that will lead us to something called random processes.

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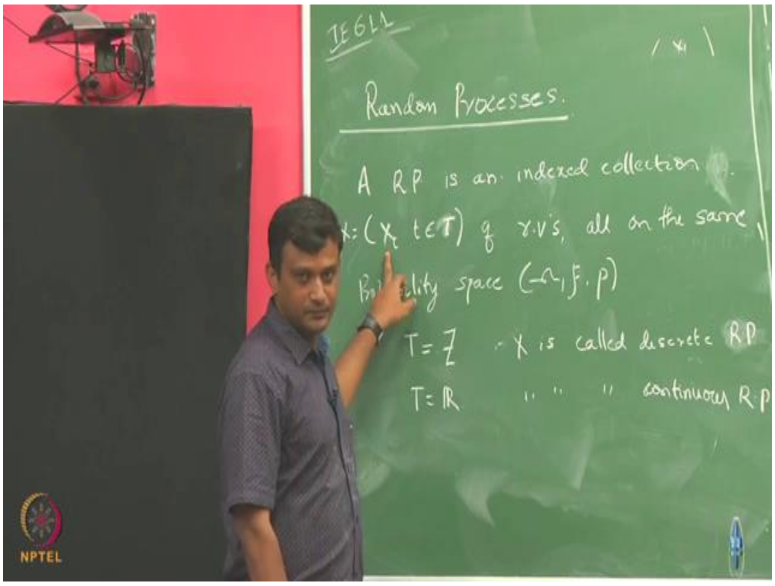

So, here earlier we had, earlier also when we talked about random vectors we had collection of let say finite number of random variables, but here that need not be the case. Here the index in set T could be uncountable, countable, but it could be infinity. So, and if in this case, if this T happens to be, let say integers, when I say integers, let say it is going to be like 0, 1, 2, 3, 4, all the way like that. Then we are going to say that then this random variable x is called discrete random variables. And if this T is going to be, let say, a real line or some continuous interval, then x is called continuous, sorry, discrete random process.

Student: Sir, Correction of random variable x 1, x 2, x 3.

Professor: Right.

Student: Then how they can be real between integers only x 1, x 2?

Professor: So it could be just like time in the interval 0, 1 you take any point in the interval 0, 1 there is associated random variable. So, for example, as I said, if in the stock example case you said, we took x 3 could be the value of the, what is that our stock, BSE stock index whatever let say on every day. So, on every day you can count day 1, day 2 like this and its value x t is going to be random, we cannot predict a priori what is that value? So, that will be denoted by this x t, if you give a day, x t will tell, x t is the random variable associated with that day.

Student: That is discrete random process.

Professor: That is discrete random process. But suppose you want to understand, let us say for example, let us say you are moving in a vehicle and you are trying to accelerate your vehicle, but you are you are you are in a very uncertain environment. At every point the velocity that your vehicle is attained, could be random variable. So, in that case at every possible time, in the time of interest, we want to understand what is the velocity at that time.

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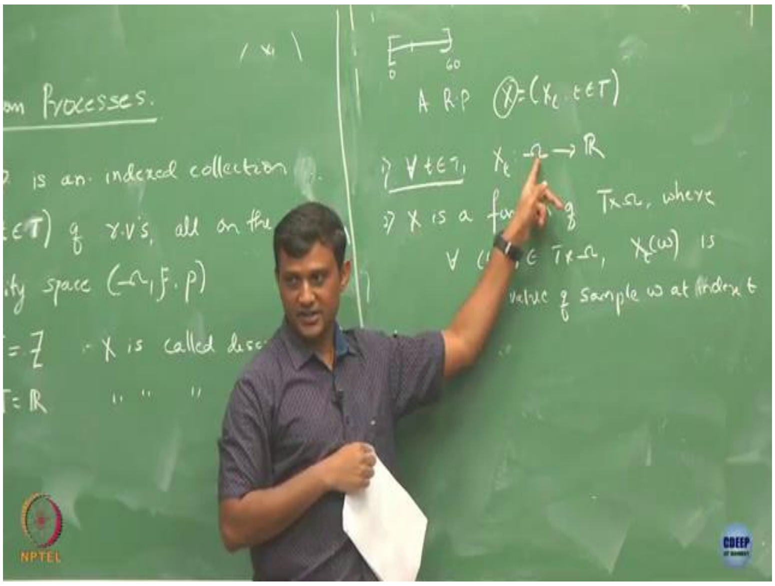

So, for example, let us say I have interested in one hour of time, this could be all possible seconds from 0 to, all possible minutes from 0 to 60 minutes. And you can think of anything like either vehicle moving or I am hitting some object. And I want to understand the temperature profile in that, at every moment you want to understand like, what is the velocity or the temperature profile of that? So, in this case you can more that T here could be all possible values between 0 to 60. Every instance we are talking about. So, then in that case, we are going to look at as a continuous random process.

So most of the thing terms will be dealing with discrete random process, but you will also encounter many examples in real life, where one has to worry about a continuous random process also. Now, we can interpret my random process in different possible way. Suppose, a random process, I have like this. Now, one possible way to interpret is as we already said, we can say that for all T in t, x t is a function from omega to R, this is just a random variable. So, these are collection of random variable.

So, for every t this is like a random variable which is giving value to each of my elements in my sample space. Alternatively, we can think it as, this entire x itself is a function of, so where it is going to give the value, for all t and omega in t comma y, x t of omega is the value of sample omega at time t, at index t. So, you can think of this x to be now a like this is random variable. So, now it has, you can think of it has two dimension to this, one is the index and another is given that index, what is the value it is going to take on a given sample?

So, for x that is why we are saying it is index as well as whatever value it is going to take on a sample. So, if you tell, x is collection of my is the random process corresponding to let us say, behavior of my stock exchange or what is the index on a particular day. If I say on day tenth about this particular sample, what is the outcome then this x is going to give this, if you have to look at the on the on that particular day on the particular sample, what is the value it took? And this is how we can interpret this random processes.

Other way you can think of about is? So, this is like fixing t you fix a pip and then look at on that particular day, how this value is going to change for each of the sample what are the possible values for each of the samples? Or you can fix omega as sample and then look at on this sample how it the value possible values on each of the days. So, for example, let us say, let us say you are interested in some, some 10 shares in a market, whatever that companies are?

And you have a random variable which assigns for each of the shares, whether it made a positive gain or a negative gain. Now, you can think of on a given day, what is the outcome for each of my shares? So, shares is a collection of this sample space, what is the value we took? Alternatively you can what you do? You can focus on a particular share and then look at on different days what it took, what is the value it took?

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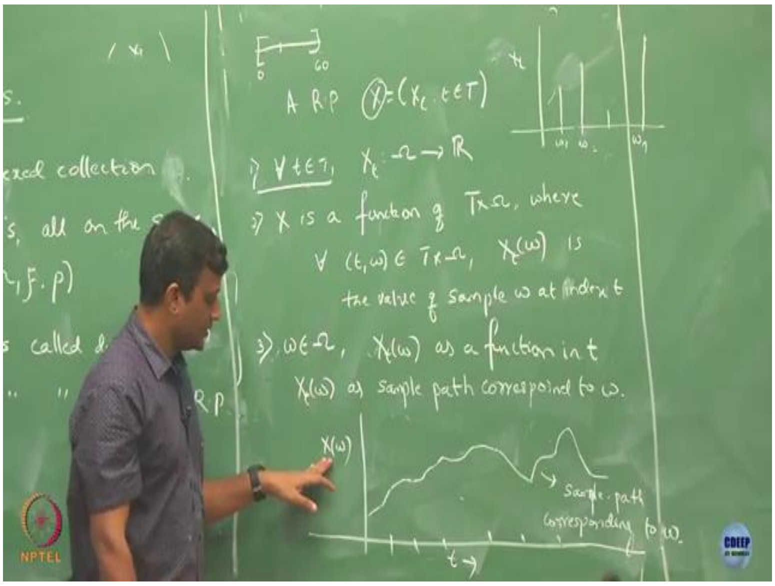

Now, you can think of x of t as a function in t and to call x t omega as sample part corresponding to omega. For example, let us say I have this, this is my time index. And I am interested in knowing my x of omega after particular sample. So this sample may take value like this, I do not know it is not, it may fall, it may rise, it may fall like this. And this is going to be, I am going to be, yeah this I am going to just interpret x of omega. So as for different value of t, it is going to give me what is the value taken on that sample omega here.

And this is we are going to call it as sample path. So is this different interpretation clear? So, let us now rework them, if I am going to fix an omega and then I am going to see like as a function of t how it behaves? So this is like I have done it for a continuous case, if it is discrete, I will have only certain points here, because this t only takes some values. Now, you are going to look into like as this aspect.

What you do in this case? If you want to draw this you are going to fix pipe and these are your omega’s, this could be let us say omega one, omega two, all the way up to like you had some n points. And what is the value taken? Let us say this is some value, here is some value and this is some time. So, this is like a single random variable that at a given time t and here it is given sample, how it behaves at different points of time or a different index and here as either you can now look it into in the joint space, this and this given an omega.

So, for each omega I can vary this t and get this graph. So, if you give me a t and an omega then I will come up with a particular point. So, suppose you give me some omega let us say you give me omega 2 and you also give me at t equals to 10. I look at this graph for t equals to ten and then I get the value, so, and that second interpretation will just give you.

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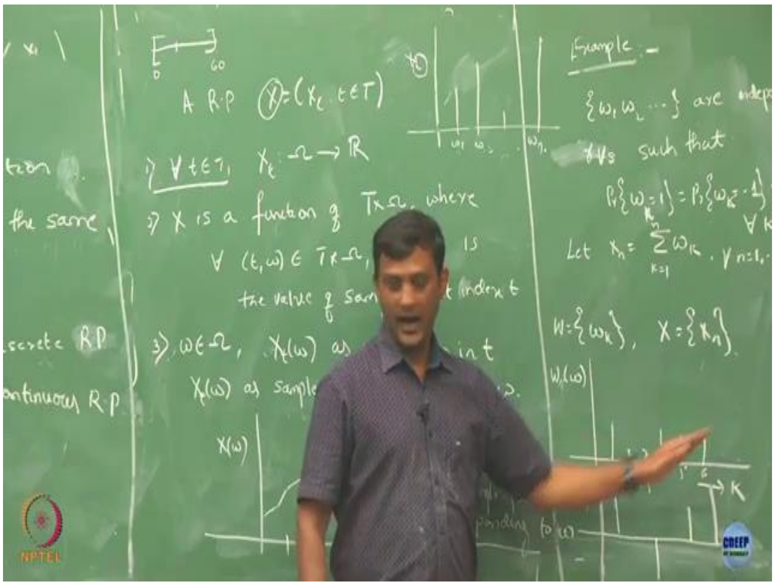

So, then, we have just an example. Suppose, I have this w 1, w 2 are independent random variables such that let x n equals to summation K equals to starting from K equals to 1, 2 let say n.

Student: That means P of omega is equal to 1.

Professor: Omega K so it is for all K. So, I am saying, I am giving you a collection of this random variable, index random variable, index at one, two all the way up to infinity, is this discrete or continuous random variable here? It is going to be discrete because now I have index which are discrete points. And now I am saying each of these other variables are independent and further each one of them is such that it is only going to take two values, one and minus one. And probability it takes one is half and now further I am going to define another random variable x n, way for all again, let us say 1, 2, infinity.

So, this is now sum of this random variables, is this x n, now I have these two random variables, one is this w collection of this w case. And another is this x which is collection of this x n. Are both of them are discrete random variables? Yes, because anyway w is only defined at one, two, three index like that. And the x n is also this is x n collection, sum of n w's. And that is also defined for each of n integer valued. So this both are discrete random variables. Now let us see how this looked like?

Student: W is random process.

Professor: Yeah, it is a random process, because each random variable I have defined like this. And it is a collection of so many of such random variables. So, now let us understand how this w K of omega looks? For some omega, let us fix an omega and now let us look it as a function of K that is in time here. So, K is the index, what I am doing is? I am fixing as sample point and for this sample point, it may happen that for the first one, it could be taking one, it could be then taking minus one, then maybe taking minus one, and then going plus one, and maybe plus one again, and like this plus one. It could be this is one sample path.

So, whenever I have defined a sample path here, for this particular random variable, I am now trying to try a sample path. Maybe I do not need K here. So, this is like one, two, three, four, five, six and I can I can go on. Each for the each of this K, my random variable is such that it is going to take either one or minus one. So, let us say when I perform my experience, in the first round it took value one, in the second round it took minus one. And it again took minus one and it took three consecutive one after that, and something happened subsequently. I do not know what is that?

So, this is going to give me a sample path of this process for a given omega. Now, let us try to draw my sample path for x. Now, x is a random process, which is a function of w. So, if I know this process, should I be also able to draw the sample for this, maybe, so let us say, what will the value of x of omega at n equals to one?

Student: 1.

Professor: It is going to be this and what will be a 2?

Student: 0.

Professor: It is going to be 0 then?

Student: Minus 1.

Professor: 1, then?

Student: 0.

Professor: Then?

Student: 1.

Professor: Then?

Student: 2.

Professor: Going to be like this. So, this is how we are going to get some picture of what is going to happen like on a given path. If you look at some fix a sample point and then we can visualize how on this point my graph is evolving as it takes different different values. So, as I said for example, if you are going to focus on a particular share value, you can now look at on each of the days whether it made a positive gains or negative gains and plot it like this. And this could be like the cumulative effect.

So, the cumulative effect is till date, it made affectively positive gain or negative gain. So, for example, this curve here could be like on each of the days it is making positive gains or negative gains. And here it could be till this point, the cumulative gain is positive or negative.

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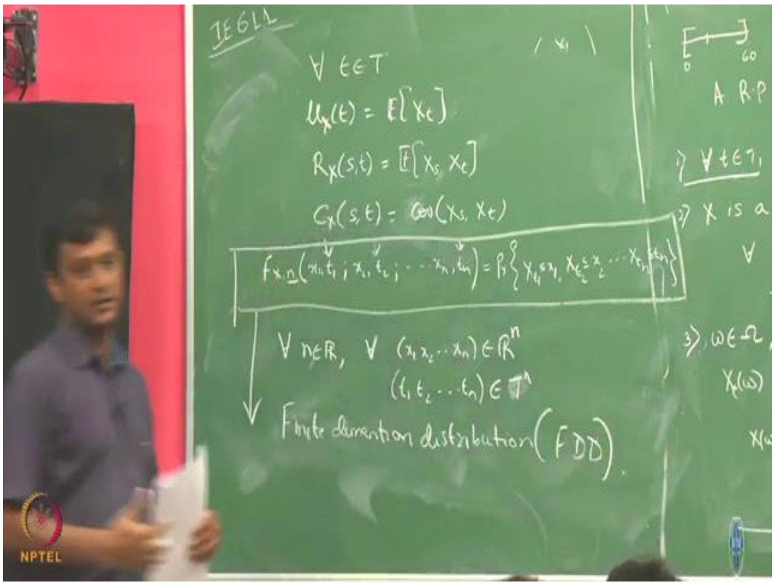

Then for such random process we are now going to define mu of x of t is going to put off t is T. So, now I have so many random variables, one define for each of the possible index. Now, I am going to define mean for each possible random variable I have. So, that I have one random variable for each of the index. So, for each of the index the mean value is simply going to be the mean value of that random variable.

And this correlation we have, now it depend on which two random variables we are talking about right. Now, suppose if you are talking about random variable at time at index s and t, then you are going to denote that is now you have to specify which time, which index you are talking about to calculate this correlation. So, if you tell me s and t are those indices that the correlation between that random variable is this.

And similarly, covariance of a random variables at indices s and t will be given by covariance of x s and x t. And then the CDF of this random variables and you are going to be defined as, now, we have to when I am talking about this random process, I have to tell which random variable I am going to talk about and that is going to be specified by its index. So, suppose if I am looking at distribution of n random variables, then I have to specify at what is the time, what is the index, you are looking at them.

So you are going to specify those indices and then if in that case, the Cdf this involving n random variables is going to be defined as the random variable at index x of t one taking value less than or equals to x one. And a random variable at time x two taking value less than or x two like this all. Now, to give a complete characterization of this random process, you need to define this for all n, n is what? Integer and for all x one, x two, x n belonging to R n because we have now a collection of random variables.

To give a complete characterization of this random process you need to tell me if I am going to look at these set of random variables what is the distribution? And I should be able to tell this distribution for any possible set of random variable you are going to ask me. That is why you tell me how many set of random variables you want to look at? And you tell me, which are the indices? What is the set of random variable? To decide that you need to tell me the indices. And then, you have to also tell this, so then for all this and also t one, t two all the way to t n that is coming from your, your t to the power n because these are this many indices.

So, you have to tell me which are those indices you are looking at and what is the value you want to and I should be able to tell what, is the probability that at that random variable taking value less than this particular number. So, I need to specify all of this to completely characterize my random process. And this kind of things if you can define your this CDF for all possible value of n and for all possible indices, and for it taking all possible vector like this, if you this is called finite dimensional distribution.

So, see like random process is a complicated thing, there are so many random variables there. And this could be potentially uncountable, but to define it completely, you need to specify how any possible subset of these random variables in this random process are going to be behave. If you are going to like if you cannot specify the way behinds at some indices, then you are not completely specifying me your random process.

That is why to explicitly completely characterize your random process, you need to define your CDF for all possible subsets, for all value possible values it is going to take and also for all possible indices you have. And that is, so this FDD is what going to completely characterize your random process.

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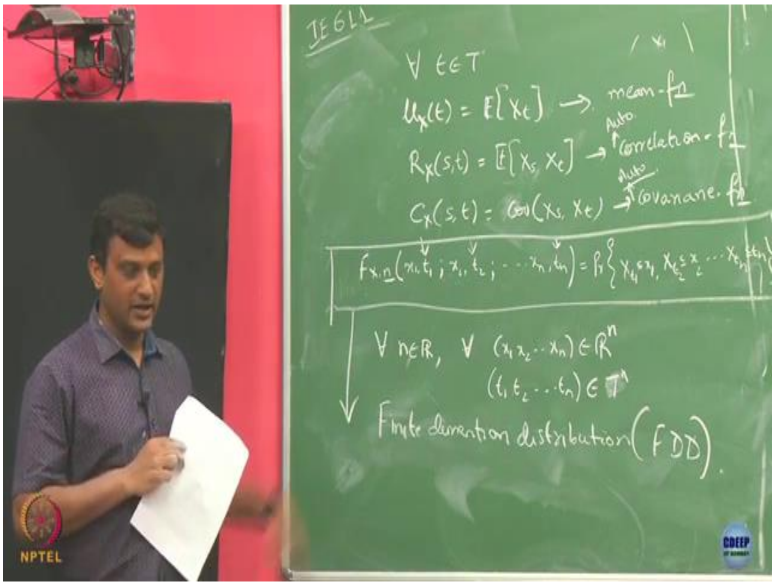

And now know this is the mean function here, I am going to call mu of x t as a mean function for my random process x, why this is function?

Student: It is a different.

Professor: So it is now takes input t, it is going to change as your indices…

Student: Changes.

Professor: Indices changes and this is so this is we are going to call a correlation functional now. Earlier, when we had a two, any two random variables, we know how to find the correlation. If you give me x and y, expectation of x, y is the correlation. But now I had, I have so many of them, not just x, y, I have x one, x two, all the way to infinity.

So you just tell me which two random variables you want to look at the correlation and I am going to come. So, this is going to be a correlation function now. And what is this? It is now a covariance function. So, sometimes you may be interested in, you may want to set s equals to t itself that is looking for a covariance of a random variable with itself.

Student: Variance of…

Professor: It is going to be a variance, but in this parlance you can if you are looking at the same you can add auto covariance. So, at all when I am going to look at the same time indices, the time indices are not two different things, same things.

Student: When s is equals to t then we will call it auto covariance.

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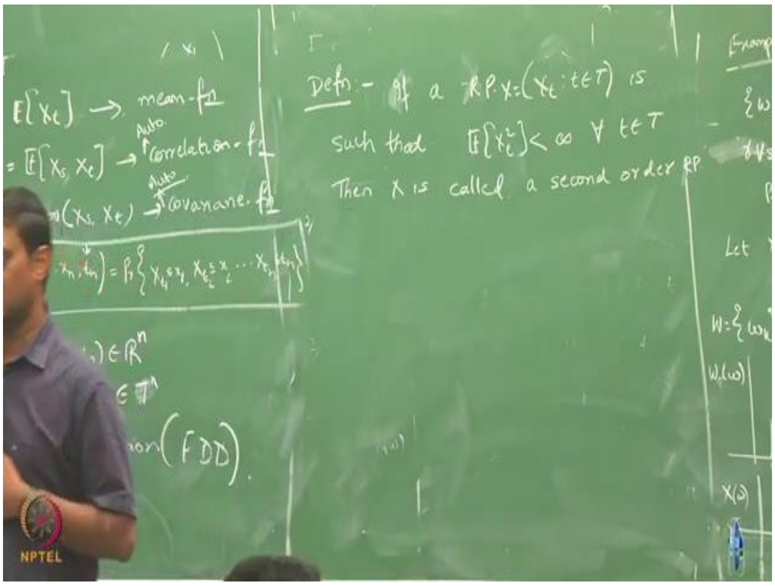

Professor: Yes, then we have this definition. So if my random process is such that each of the components in this process has a finite second moment, then I am going to call it as a second order random process, so just our definition. So, just one point I want to add here. So, this is I how this is a CDF, which I have defined for all possible subset. If I know that my process is, my random variables are all discrete. That may be then I may be interested in only probability mass function equivalent of this.

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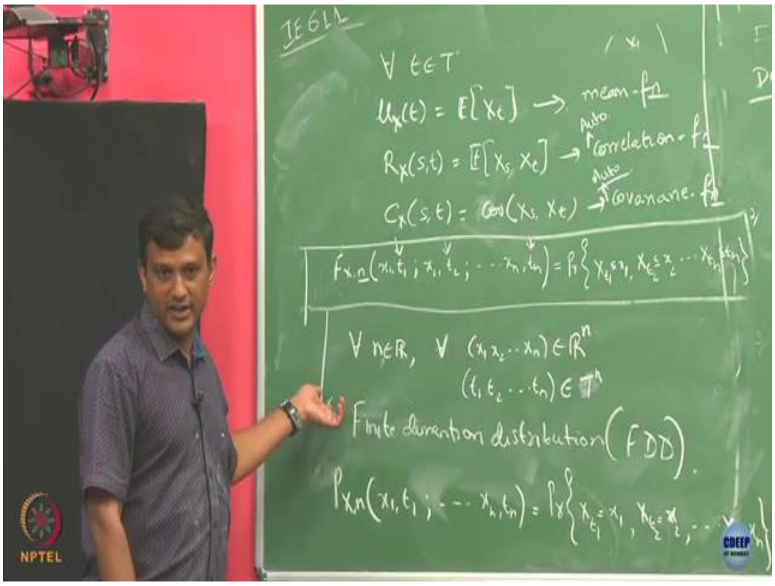

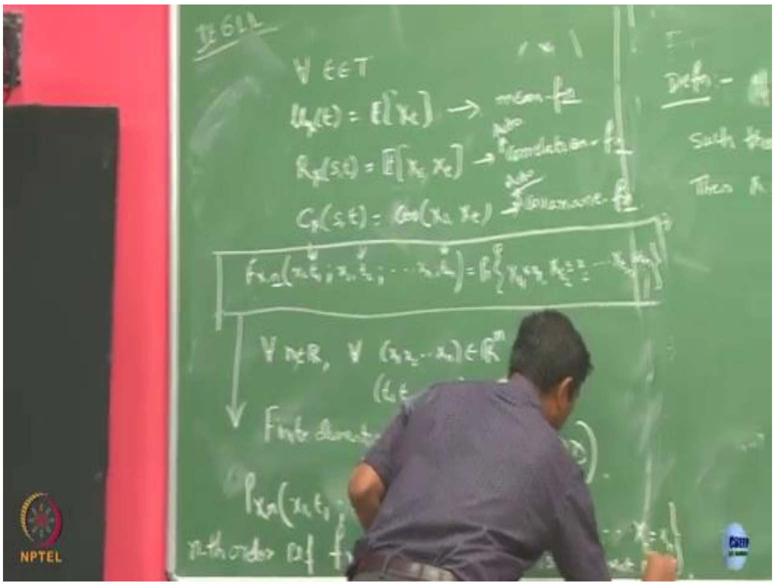

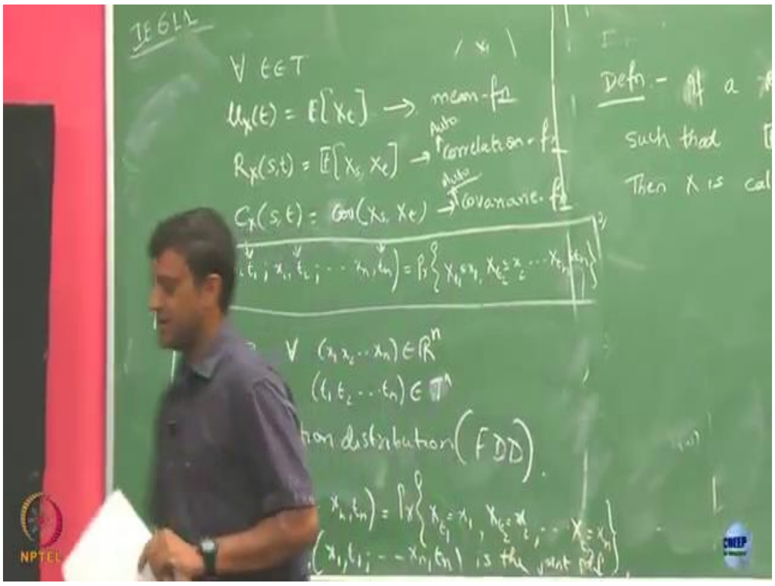

For example, if I have all my random variables are discrete then instead of looking for the CDF, I maybe just interested in x one t one, all the way up to x n t n to be just equals to probability that x of t one equals t x one, x of t two equals to x two, all the way up to x of t n equals to x n. So, this is just like a probability mass function version if my random variables are all discrete. So, I am talking about two discrete things here, my random variable itself taking discrete values, and then the indices being discrete. So my process is discrete random process if my time index is?

Student: Discrete.

Professor: Discrete, further if my each of my random variable is such that it only takes value from a discrete values, Then I am I will be interested in only further this probability mass function in that case, because this gives all the information. I do not need to go for this complicated CDF in that case.

And similarly, we are also going to say that my set of any n set like this are going to be continuous here jointly continuous if they have a corresponding PDF, the way we did earlier for a single kind of variable, if I can find, if I am able to express this in terms of some function f of small f of x n in in terms of integration where we did earlier, then I am going to say this set of random variables are continuous. So, we will just the nth order of PDF that is just like completeness.

Now, I want to just let me complete this one more time then we will move to the next class. So, further we studied some more properties of this in the next class, let us stationarity and wide sense stationarity. So, before that I want to just tell you what I mean by a Gaussian random process, I have talked about what is a Gaussian random vector. But now I defined the question process, then you may want to specify what we mean by a Gaussian process random process.

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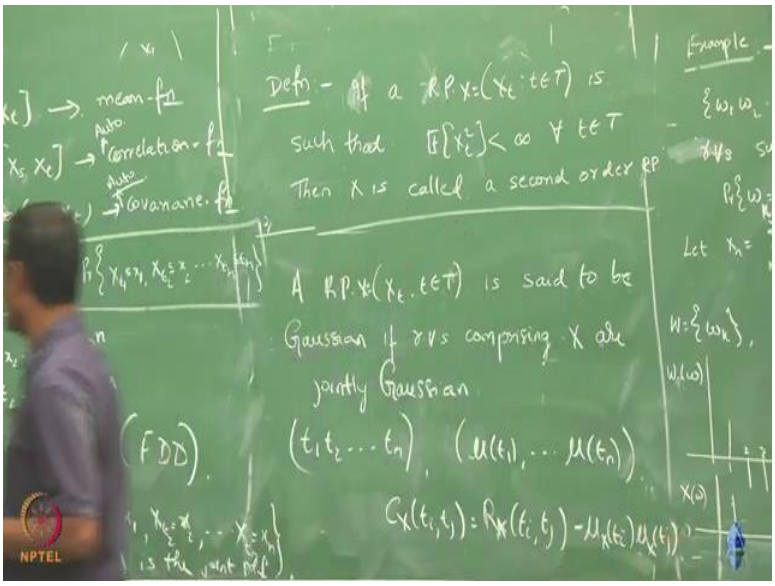

I have talked about what is a Gaussian random vector but now I have defined you Gaussian process then you may want to specify what we mean by Gaussian process. A random process, so now, we are simply extended the definition of Gaussian, Gaussianity from random vector to random process by saying that we have a, in the random process we have so many index random variable.

But from this index random variable if any linear combination, if the random variables this comprise of x are jointly Gaussian that means, if you take any linear combination of this random variable if they happens to be Gaussian, then we are going to call this process as simply Gaussian random process. And so, we already know that for this in the random vector case, we denoted that Gaussian random vector as mu mu K and its PDF there dependent only on mu and covariance K.

So to define this Gaussian random vector, I just needed to know the mean vector and the covariance vector. Now, what do you think I should know to define this Gaussian process? So again, maybe I just need to know what is the means for each t? And maybe the covariance for each possible time in this pair. So for this, so the good thing about the Gaussian vector was? It was parameterized but that parameters were just like the mean value and the covariance value.

Now, to define a Gaussian random process completely, so that I do not need to really look for all this finite dimensional distribution. So, maybe the parameters are just sufficient. So, what are those parameters I should be interested in to completely define a Gaussian random process? So, maybe one thing is you going to do this mu of x t for all t, and then so maybe like we can say that if you are going to take like take any subset t one of this time indices.

To define this joint distributions, what all the things you need to know? You just need to know their mean vectors and their covariance matrix, which is but as I said to completely define a random process, I need to define my final dimensional distributions. So for the Gaussian process what is this finite dimensional distributions? I know that if I take any n that will be jointly Gaussian or a Gaussian random process.

To define that joint random process, all I need is the mean vector and the corresponding covariance function and that covariance function can be expressed simply in terms of my correlation function and the mean value. So, as long as you give me the mean vectors, as long as the correlation function, I can I have the complete information about the finite dimensional distribution of my Gaussian random process. Because I know how to construct my probability density function for each of these n and I can do it for any possible n.

So, is this fine, so if you have a Gaussian random process, its characterize should be much simpler. All I need to know is, its mean vectors and its correlation function. But if it is not a Gaussian vector, maybe things are bit more complicated. I have to define finite dimensional distribution for all possible n, so let us stop here then.