Introduction to Stochastic Processes
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Lecture 11
Continuous Probability Distributions

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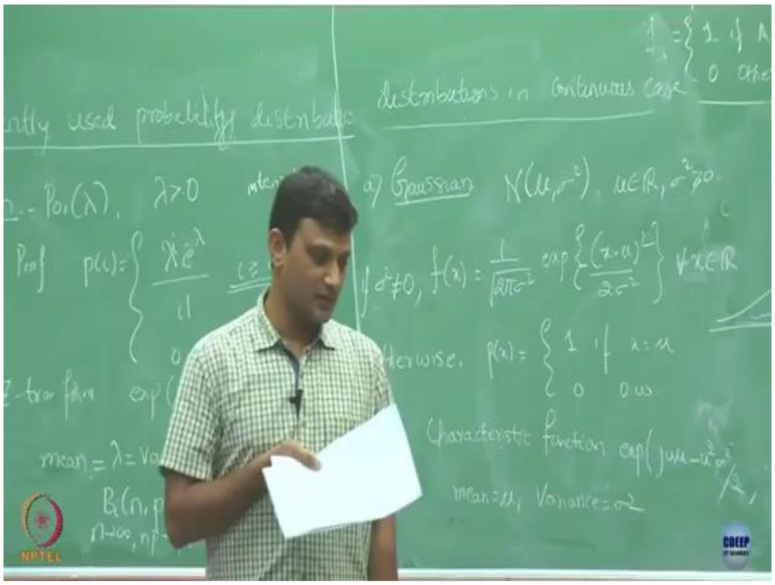

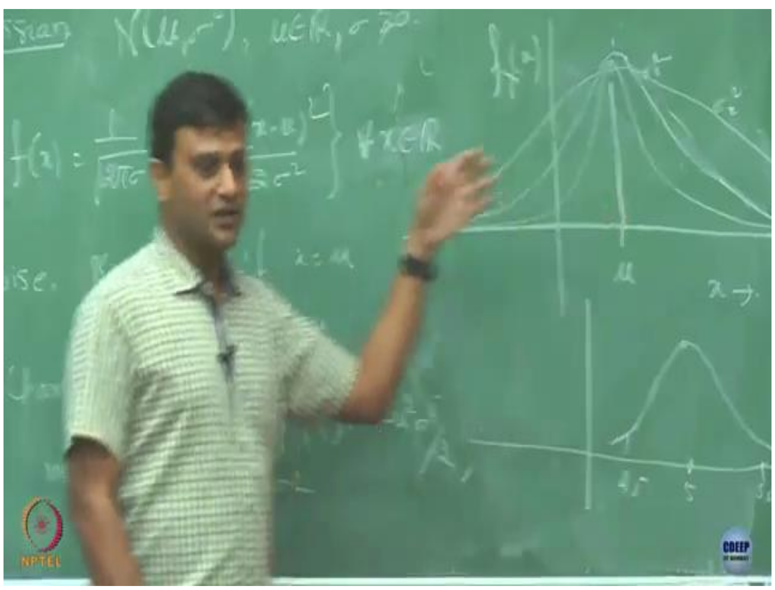

So, next we are going to look into something called, the first thing is. So, I am sure like all of might you already heard Gaussian random variable right many many time. So, let us formally define that. Gaussian distribution you are going to denote it as with a symbol meu, sigma square where, meu and sigma square are the parameters like, like earlier we had p, n as the parameter for the Gaussian distribution meu and sigma square are the parameters. And here meu is a real number and sigma square is a positive quantity and it could be also take 0 value.

Now, this is we are talking about continuous random variable. So, we are now going to define the probability density function of this random variable. For a Gaussian random variable we are going to define its density as, f of x and this is provided my sigma square is strictly greater than 0 and if it is a sigma square equals to 0 than it takes the degenerative format in that case it is simply going to be of this form 1 if x is meu. So, this is if sigma square greater than 0, otherwise we are going to it takes a degenerative form where it will have only one possible value at x equals to meu and 0, otherwise.

And its characteristic function is given by function exp. For its so happens that for this question random variables with parameters meu and sigma the mean turns out to be the parameter meu and the variance is turns out to be the parameter sigma square. So, we are just going to call our random variable Gaussian with the mean something and variance something. So, the mean is nothing but that parameter where meu and sigma square is the other parameter.

So, here if you look into this, this is our pdf and this function is valid for all R. For all R, for all x belongs to R and now let me, if sigma square is not equals to 0 than my CDF is going to be define like this for all x. If not Sigma square is 0 then it has this format and its characters functions are given like this.

So, let us just because this is one of the most popular distributions that is used let us see. So, let us say this is my meu here and this is my x and this is my x. So, just looking at this function here in variable x, is this function symmetrical x?

Student: (())(05:22).

Professor: About what?

Student: Meu.

Professor: About meu just by looking at this you can see that, because this is a square term here it will be symmetric about x. So, whatever and it goes on like, goes on like a it is defined for it does not look symmetric but it is symmetric.

And suppose let us say, this is for some sigma square and meu. So suppose, I increase this variance sigma square how do you think this lobe will look like? I will keep the mean same, suppose I want to have another random variable, Gaussian random variable with a same meu but let us say it has a variance which is larger than this.

Student: Which will decrease.

Professor: It is going to be the this lobe it become fatter like and it will like a, it be like a something like this. This sigma 2 square is going to be larger than that. But now if you have a some sigma 3 which is going to be smaller than sigma 1, how it is going to be? So, in this case it could be like it is going to get thinner. And eventually, if you trying to make it sigma square to be 0, it will be just this, just this our delta. At x equals to meu this is going to just take the value of 1 and it is going to be 0.

So, if it is a sigma square equal 0 this height is just going to be 1. That is the only point where it will have positive (())(7:48) and 0 everywhere. Fine. So, I have a Gaussian distribution like this where you want to use it? You know it is very popular. Where it is used? Or where you think, its going to be of natural use. Are you feel that I can not imagine anything where this can be used.

Student: IQ distribution (())(08:19).

Professor: IQ distribution of humans, animals, humans. So, what is the range of IQ? But, range kya hai 0 hota hai kya negative hota hai?

Student: 0.

Professor: 0 maximum, kuch bhi.

Student: (())(08:41).

Professor: Is there, so is there an upper bond on IQ? Or anybody, somebody can have infinite IQ?

Student: Theoretically infinite.

Professor: Theoretically infinite, but then one side is fired and I like, negative toh nahi hora hai so, 0 to infinity. Toh this one is by the way have defined it has also negative values here. So, if you have only positive values maybe you do not want to use it. What could be other example?

Student: Sir that white one which you are taking is an example.

Professor: Ya.

Student: (())(09:23).

Professor: Ya, but I can if you just a if you want to begin with you know that the height is going to be some in the range let say some 2 feet to 6 or 8 feet and it is never going to take a negative value, but still ya.

Student: (())(09:54).

Professor: You want to cut it off and.

Student: (())(09:59).

Professor: You can condition but before conditioning I am talking about.

Student: (())(10:05).

Professor: Ya, ya, ya fine. So, maybe that is one possible thing like let us say you are like a big, big, big angel investor or maybe you can make billion, billion dollars or if you are back your investment goes this like you may make negative billion dollars and you have a big, big range maybe.

Student: (())(10:38) tolerance design for the.

Professor: Tolerance design. So, that can you make this more concrete example.

Student: Sir suppose we are manufacturing (())(10:57) we can say the a being value to be something suppose let us take 50 microns, so we just (())(11:02) want to you know increase the tolerance, taking that to be 0.1 or specified it to be you move as there is not a manufacturing (())(11:13) or something like that.

Professor: So, another example I can think of it like let us say, some kind of a deformation with temperature. Like I can apply temperature can be both positive or negative. In some critical applications temperature can have a wide swing. So, if you are going to apply this positive temperature how your deformation or whatever the related quantities look like an when the temperature goes below negative how its look like.

So, there are again like it is just about that as you can see that it finds many, many application but it has this need to have this both positive and negative value but that is not a constraint like you can always, you can adjust about your mean and (())(12:03).

For so, as long as you feel that there is something heavy concentration about a particular value and then things slowly died on around it maybe this a good thing. Even in case of a population or a height distribution like height lets say this is like about a 5 feet, there is maximum number of people around 5 feet, most of them. Like then other guys maybe like just die down.

Maybe if I go here itself like around 5.5 and if I just take it like the 4.5 here already significant reduction is happening. So, this is in that way at least in this region it is still like looking more like a Gaussian distribution. You can still continue to use that but provided that you appropriately.

So, you will right away will it the Gaussian should come to your mind whenever you see that there is something like, there is something going to be symmetric around some value and there is a going to be like as I said diffusion or dispersion about it both on the left and right hand side. Then you will do so fine. You may feel that, this is a good but I do not know how, how concentrated this lobe should be? This should be thin or it should be big that you find it from your data. You do lot of experiments and see that which, what how you going to fit your data so that you get the right parameters meu and sigma square. So, fine so this is one thing.

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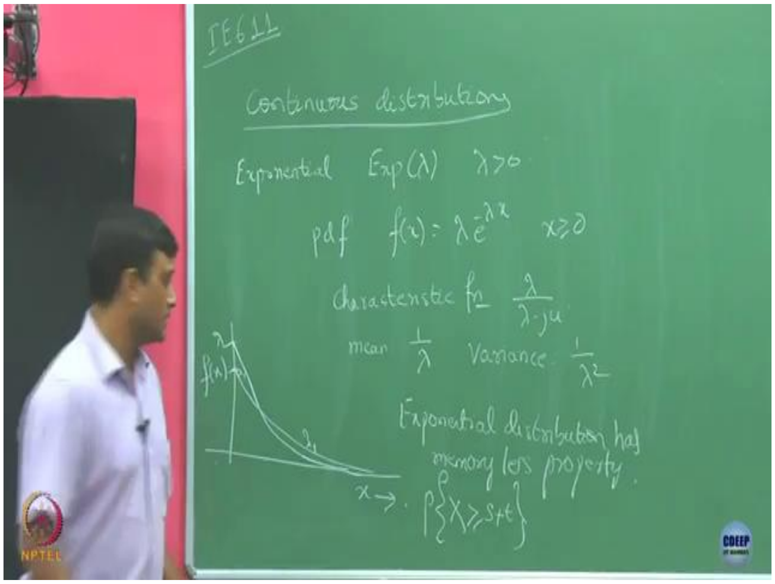

Now we are going to see another distribution called exponential and the pdf of this distribution we are going to take it as f of x equals to lambda times e to the power lambda x and this is going to be for all x positive and its characteristic function. So this distribution is going to look like if you give me lambda so this is going to start at x equals to 0 this is going to be what simply somewhere it is going to start from lambda and where it is going go?

It is going to go to 0 as x increases. Let us says my f of x. Suppose in this I increase my lambda lets say this is for some lambda and if I increase my lambda that you think how, how this another function is going to look like? So, at least I know that if I am going to increase so let us say this is for a some lambda 1 and let us take a lambda 2 which is bigger than this lambda 1.

At least I know this parting point was lambda 1 the other point is going to be lambda 2.

Student: (())(16:18).

Professor: It is going to fall much faster, because this is coming in the exponential term. So maybe it will start faster but it will decay much, much faster, fine. So I have a such a arbitrary density function which is taking positive values I mean which is defined on positive real half. So, where you think such a and you see that as x increases this is going to like fall rapidly it is falling actually exponentially fast.

So, where do you think such a distributions can be helpful? What kind of things you would like this to be used to model? For example let us say, some component inversion what machine? Like component what? Let us say I have a component I want to predict a life time of this machine, like find lets say I have a bulb and I want to see how long this bulb is going to survive before it breaks up.

So, it may last lets say it count lets say lets count a life in terms of second it may last 100 second, 200 seconds or maybe 1000 seconds and maybe like 36000 seconds or whatever. Like after that it may break. So, you feel as you see that if you are expecting it to survive longer and longer that probability is going to be falling rapidly.

So, that is what in that case maybe this kind of exponential distribution is going to be much useful. And in that case if you 2 let say bulbs one is of a better quality than the other and so which one you want to assign higher value of lambda? The better one or the poorer one?

Student: Poorer one.

Professor: Poorer one because with that it is going to decay much faster.

So, another good thing about this exponential distribution is it has the memoryless property we already discussed. So, we discussed memoryless property with respect to which distribution?

Student: Geometry distribution.

Professor: Geometry distribution. Geometric was a discrete distribution. So, it is so happens that this is a continuous distribution but it also satisfies that memoryless property. What does that mean? What does that mean? Suppose you have the random variable x is going to take value larger than s plus t, given that it has already taken value larger than t. If x is memoryless what you expect this quantity to be?

Student: (())(19:29).

Professor: You expect this to be just probability that greater than or equals to s. So, do please verify that if you have CDF like this it is indeed like if x has a CDF like this, this satisfies this property.

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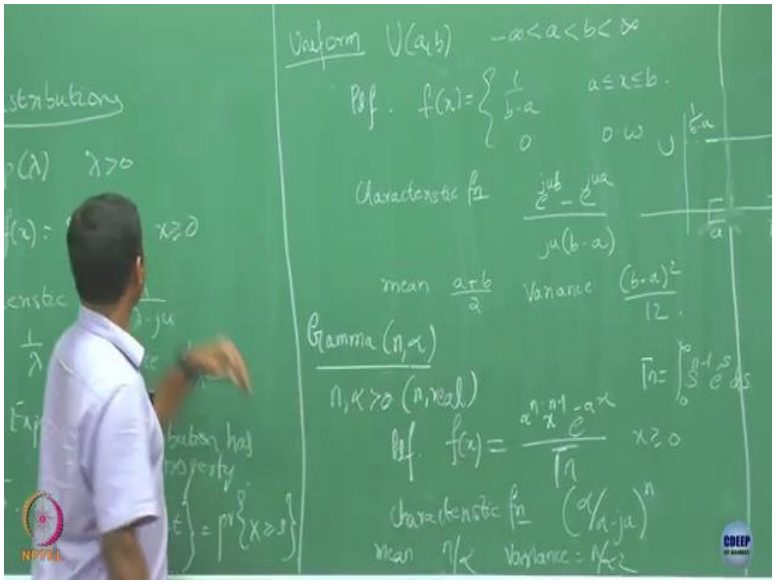

Next is uniform. Uniform I am going to denoted by b where and pdf of this is for all ab to (())(20:34) going to 0 otherwise. And its characteristic function is, is given by e to the power j u b minus ju a by ju minus b minus a. And again its mean is going to be you, you can work out that its mean is going to be simply the average of the points a and b, and its variance is b minus a whole square by 12.

So, what it is saying is like suppose, if I have a lets I have, I want to assign lets say I am interested in some interval a and b. I want to say that anything in this is equally possible. That means what it is saying that for all x in this range all of them are going to take the same value. So, that is this the function U here is going to be simply this quantity where this quantity here is 1 upon b minus a.

So, this pdf is a like a flat in the interval a and b that means if you interpret in terms of the intensity at every point the intensity is kind of a same here. And anything outside you are not assigning any probability. So, you can just the see that the mean of this is going to be simply the average of this 2 points and the variance you can compute like this.

So, where do you want to use such at distribution? Like us say, let say in terms of the temperature like often like what did you we can say that the temperature of a day maybe like a uniformly distributed in so, so not necessarily like a temperature let say anything you want to, up want to apply like lets say pressure on something the amount of the pressure that this guy can sustain could be any uniformly distributed between the two extremes.

Let us say some minimum amount of pressure and some maximum amount of pressure it could be taking equally like it can break at any of this amount of pressure equally likely. So, this is one possibility other anything other useful thing you can think of like maybe like, like fine you have grades like A, B, C, D these are discrete but think of instead of this think of these are numbers between 1 to 10.

I can think of like the distributions could be uniformly like any distribution of the marks could be uniformly in between 1 and 10. Can I use that?

Student: (())(26:54).

Professor: Yes good. So, you want to be uniformly distributed the marks between 1 and 10? Fine that is a then come up. And then n alpha (())(24:19) and real so the pdf of this. So, actually when you have this Bernoulli toss we have only head and tail. Suppose, you assume it is a fair coin that is taking head is Hoff and also tail is Hoff.

So, then we can say that both my outcomes are equally likely or my head and tails are like uniformly distributed that is in the discrete version. So, what we are now talking about is uniform distribution in the continuous domain where any value in this interval is like equally likely. So, pdf is alpha to the power n and you can find that its characteristic function is, let us move happens to the n (())(26:09).

So, another useful distribution is gamma distribution so it has again define on for n integer and for any x which is non negative valued.

Student: (())(26:29).

Professor: Gamma n, this is a gamma function where a gamma is defined like this. So what is the useful of this function? So its so happens that gamma function is related to exponential distribution in what manner? If you are going to take gamma to be n, alpha this is related to exponential distribution through its so happens that this distribution can be expressed as sum of n exponential distribution with parameter, with parameter alpha.

So, if you are going to take this gamma to be alpha and add such n exponential distribution then you will end up with gamma distribution. It needs another requirement that these are independent further we will define what we mean by independent today, later in the class. So, if we add n independent exponential distributions with parameter gamma then you will end up a distribution which is gamma and alpha.

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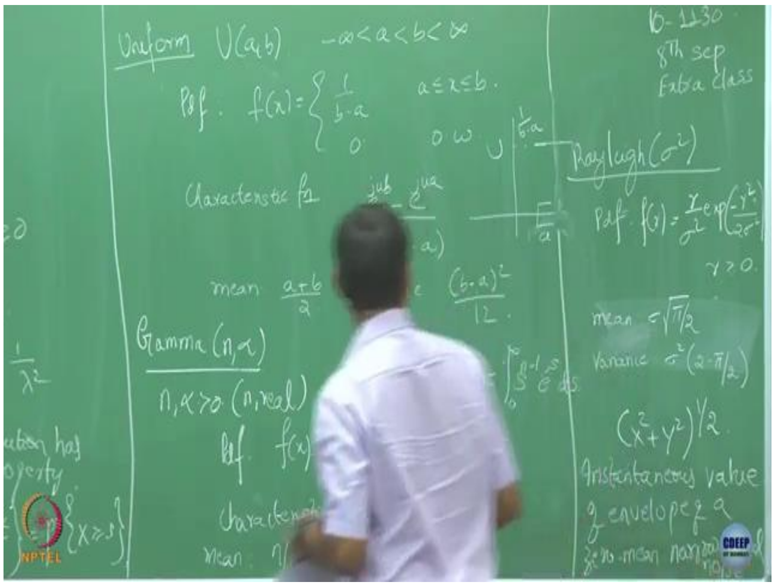

Last one is, its pdf is so its characteristic function is bit complicated I am not expressing here. So, what is this Rayleigh distribution? So this Rayleigh distribution I think we are not going to use this much but this may be useful to this communication network people its so, happens that it comes very handy when you have to deal with multiple Gaussian distributions.

Its so, happens that this gamma distribution this Rayleigh distribution with parameter sigma square is nothing but the annual of pause so it is going to be so, so, so first let me explain this so, if x and y happens to be Gaussian distributed with parameter meu and sigma square where meu is set to 0 that sigma square is going to be same as this sigma square so then its so happens that the distribution of this turns out to be this Rayleigh distribution. Suppose you are dealing with lets say 2 quantities random variable x and random variable y which are both Gaussian, independent, mean 0 variance sigma square and if you need to happen to deal with such a quantity so, instead of looking at each another separately you need to end up with so this is like a what?

This is x square plus y square will give what to you?

Student: Circle.

Professor: Kind of circle and you want to take a square of that and you want to understand this is kind of radius of this circle you want to understand if at all it has a constant radius this radius could be varying you want to understand how that radius is distributed its so, happens that, that is can be characterized this Rayleigh distribution. So, its so happens that for a it can be thought of as instantaneous value of envelope of a 0 mean narrow and noise, fine.

Now we just quickly coming back to this uniform distribution. So, uniform distribution when it is going to be useful, uniform distribution you want to use when among the possibilities you do not have any prior information. For example let us say, let us say tomorrow Mumbai's temperature is going to be between 20 to 30 degree Celsius.

But I do not have any prior information like what it is going to be like let us forget I do not have, I just born today and I have to choose a what is going to be a temperature tomorrow and I was told that it will be something between 20 to 30. So, of I do not have any prior information what I am going to do is, I am going to think that everything is equally possible.

I do not have any prior information in that case when, when you have to deal with a situation when you have to model some randomness about which you do not have any prior information you would go with uniform distributions. For example in this case, exponential like when I take a lifetime of a bulb I know that as time progresses the, the probability that the bulb will break down is going to increase. But in such case, like suppose I do not have any such information lets say I have a bulb and I do not know like it can break down any time within a 0 to lets say 100 seconds if that is the case then I would go and like to model it as a uniform distribution. If I do not have any prior information maybe that uniform distribution comes very handy to us.