Introduction to Stochastic Processes

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Lecture 01
Sample Space and events

So here today, we are going to start with simple stuff, some simple notation, some basic examples. So we will I will for the most of the initial lectures that I am going to give before the midterm. I am going to largely follow it from the book. The first book, I give as a reference last time that is introduction to probability models and by random process for engineers.

So as this we say, this course is about introduction to stochastic model. The book, the first reference I said that says introduction to probability models. So what's the difference probability and stochastic? They are the same. So what we are interested here is, so the book says introduction to probability theory and our course is titled as introduction to stochastic models, theory is us used to analyze models, develop models.

So, what is this probability theory we are going to develop to study models? Many things you are going to face in real life. They need not be certain, there is always randomness associated with them. So, is there a way like I systematically model them? First thing is I need to model them. If I can systematically model and then I can analyze and then whatever the analyses I have d1, I can tell that analysis to somebody else and that guy will understand it, if you can talk to him what is the model that you took it?

So fine there will be like if you give a realistic scenario like nobody is God like, you cannot perfectly model it. So, you are going to model it according to some set rules, which we are going to study in this rule. So, if you are going to do that, then it is possible for you to explain the model you have developed and what all the analysis you did, and then that guy may not or may agree with your analysis, depending on whether he agrees it to a model or not.

So, broadly, we start analyzing or we start to study some basic terminology that will be useful for us to start talking a stochastic model. That means yes, I am trying to model a system that is uncertainty but I want to still model it. And what is that notion of the stochasticity? You are going to make precise by going through sequence of terminologies now.

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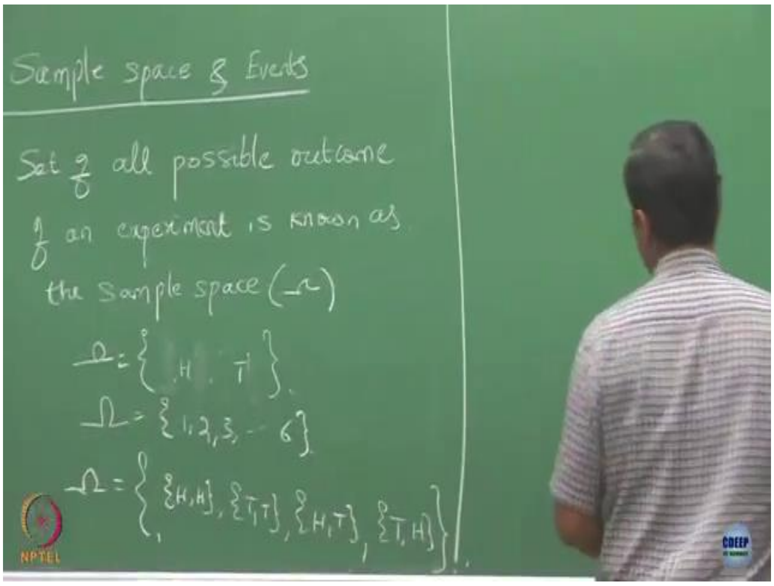

So the first thing we are going to study is, so I am going to call anything so when you are going to model something. Yes, there are set of possible outcomes possible with that. I am going to denote the collection of set of all those possible outcomes as sample space. And I am going to denote it by S. So whenever now I call experiment it is implied that I mean a random experiment, that is the outcome itself you cannot apply you know what is that it is going to take.

So, let us now try to distinguish what is a random experiment and what is a deterministic experiment? Can you tell me an, a certain thing in life, which you are sure short to know what is the outcome is going to be?

Student: Determining shape of earth

Professor: Determining?

Student: Shape of earth

Professor: Shape of earth, let us say when God has built something that nobody can change, these are things certain things are fixed. So for example laws of physics, they are fixed like, you know, that is how it has to behave. These are fundamental laws you cannot change. But now let us say you have a toss of a coin. Can I consider this as a random experiment? So the outcome could be heads or tails.

Student: Yes.

Professor: And each of head and tail can happen, mean I do not know like when the head I cannot operate when you toss a coin. I do not know whether it is already going to be head or coming to a head or tail. And that is why, so fine that is possibly the reason where is the coin tossed used to decide who is going to bat first or which court you are going to choose who is going to start first because that is possibly a very random event and maybe it is not going to biased towards anybody.

We will talk about this what I mean by bias, but at least temporary, we are not going to be saying, this guy is going to be favored, this guy is going to get whatever he prefers. So, we are going to do that randomly. So, we are going to denote sample space by omega and then depending on what is that random experiment, we can talking about, this could be a different set.

For example, if I am going to talk about coin toss, what is the set is going to be? 2 our coin side just it is going to be a head or tail. Now let not put them in flower bracket, just it may write this head or tail. Other you might have already talked about it thousand times like toss, toss of a dice, when you are going to dice. What are the possible outcomes?

Student: 1 to six.

Professor: 1 to six that is going to be your sample space. So this is I am going, this is a random experiment because if I throw it temporary, maybe I do not know what is the outcome that I am going to see out of this? So let say that is so when I asked you can you give me an experiments where the outcome is almost deterministic. It is deterministic, you have to think like it something immediately does not come to your mind, if I asked you, give me a random experiments, I mean, everything that comes to mind is possibly random.

Possibly like there will be a class today. It is random, I do not know what happens is that lecture gets out, you may not come for class or you you may for some reason you not attend the class for whatever various reasons that is not under your control. So, many things are random in life and if you want to model them or analyze them, so we need to have a systematic, you have to develop it systematically.

So, 1 concept we are going to use this notion of sample space. So, instead of 1 coin suppose I am going to take 2 coins and throw them, I mean in each trial I am going to toss 2 coins, what could be possible outcomes for me?

Student: head head, tail tail, head tail, tail head.

Professor: So I can see possibly head, head on both set on both coins. I am going to see tail, tail or first is head other tail or.

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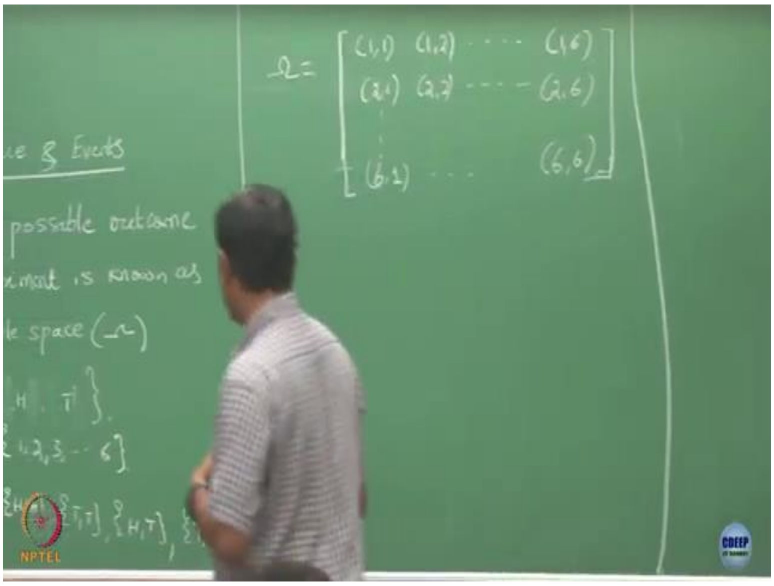

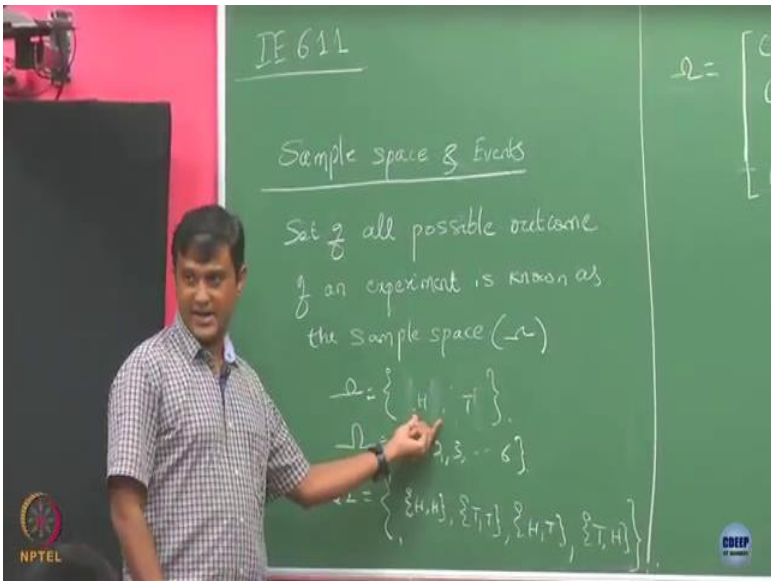

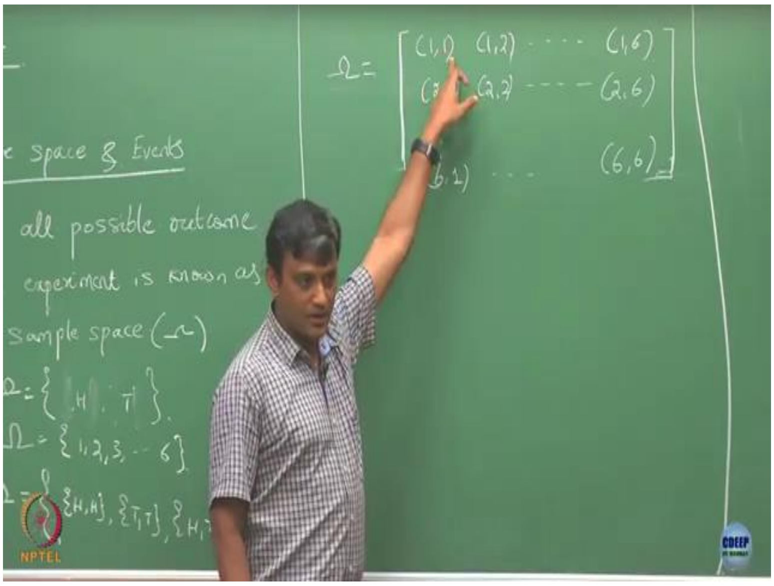

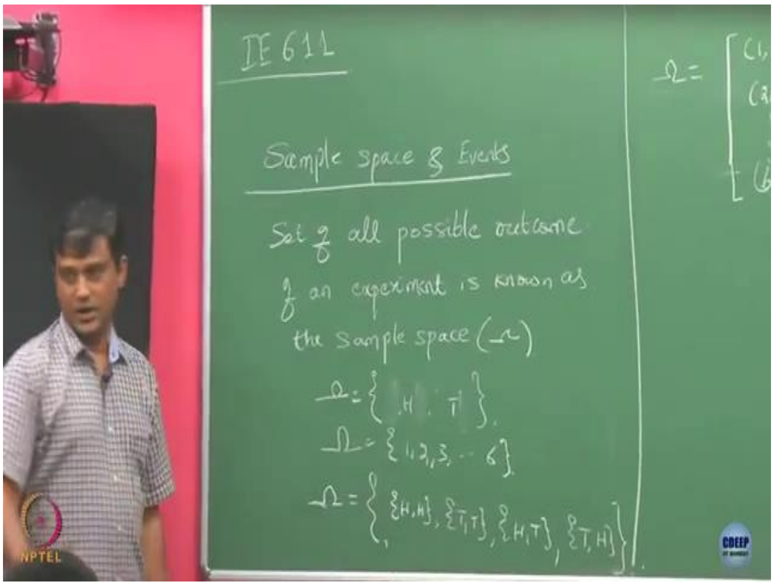

And similarly if I am going to throw through dice, what is the possible outcome?

Student: 36.

Professor: 36, right maybe the better way to represent that is from matrix where it is. So, these are very try trivial examples where as soon as I tell you what is experiment you right away know what is the outcome is going to look like. So, we will live up to our convenience how we are going to represent it when it is as simple as I write it like this, when it is more convenient for us to write it in matrix form we will write it. It is just like collect, so, these each 1 is an outcome here. So, they are going to say each element in this omega is an outcome.

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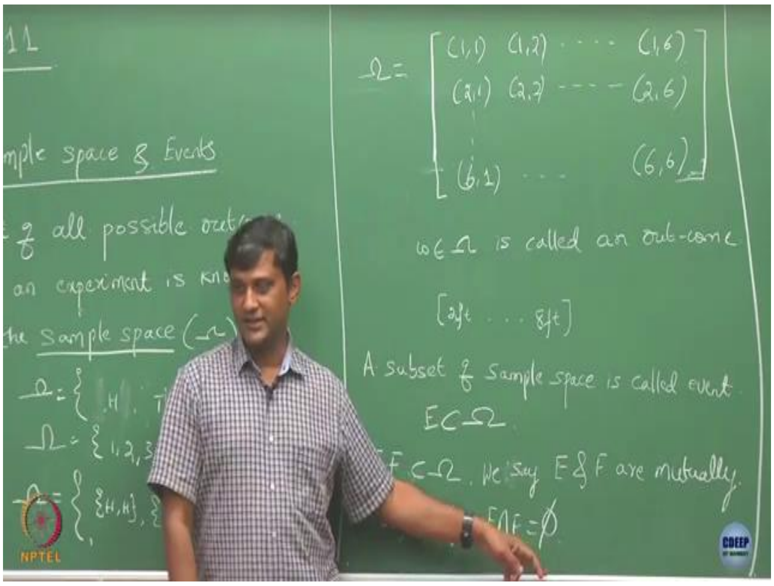

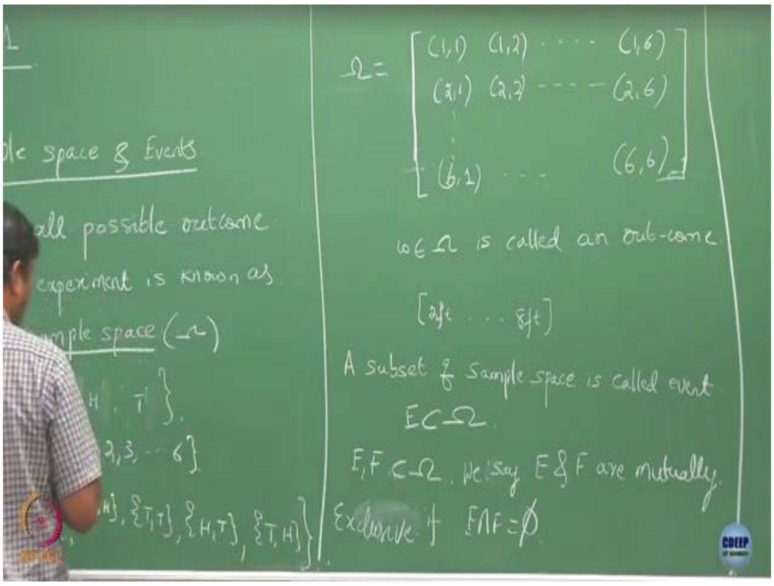

They are going to say, so this set of simple things, it is quite easy to write down but often I mean, you are not going to model what coin toss like you are learning this course to do much, much more complicated things. For example, you want to model how the weather prediction, you want to model my trajectory of a missile or like my trajectory of a satellite, what are the things? Or whatever like, if I send a signal whether my signal will reach my destination or if I am going to take a particular route, whether I am going to reach my destination within a stipulated time, many things we want to model.

So in that case, we always not worried, what is we are not always in rise it down and first right, what is the set of possible outcomes in this. So for example, if I am interested in let say find average height of all Indians. So then what would be outcome sample space in this?

Student: Random.

Professor: Yes it is random. I am asking about what are the possible outcomes?

Student: (())(12:19)

Professor: Yeah, any value like maybe the shortest person is let us say from 2 feet I do not know if it is a earth and I am saying about adults let us say some up average age of Indians who are 21 plus. And maybe like something like 7 or 8 feet, all numbers in this things are possible. So I am not going to just let maybe right al1 exhausted individual value but may be right arrange.

Or let us say you are trying to model weather forecasting. There I mean millions of millions of parameters will be involved in how the things evolve about and based on what is the value of this millions of millions variable, different possible outcomes are there? We just do not try to enumerate them but it will be on our back of our mind that what is the possible outcome is?

So whenever it is possible we just enumerate them but whenever say sometimes I may just say, this is sample space, but that sample space may be like an abstract we may not, we may have its visualization but we may not write it formally on a paper or on a board. Now comes event, so once we have given a sample space then we talk about events, what is the event? For example, in your context of problem, what are the possible events?

Possible event in a single toss of a coin either its head or tail and when it is a toss of 2 coins, the event could be like both showing up head, both showing up tails or 1 of them showing up head, tail and otherwise. And we are just simply going to denote an event like this and then they will be always interest in the questions like, what is if I want to do a 2 coin toss what is the likelihood or probability that you are going to see some value that the outcome is going to, the event is going to take 1 of this possible values. So, let us say E and F are 2 event, then we say E and F are actually disjoint. What is this?

Student: Null set.

Professor: It is a null set. What does null set means?

Student: It does not have any element.

Professor: It does not have any element. So we are say, what does the interpretation of E intersection F equals to null set means? So they do not have any common elements. So when they do not have any common element we exactly call these 2 events are mutually disjoint. For sets, we call it mutually disjoint, but these are events we are going to call them as mutually exclusive.

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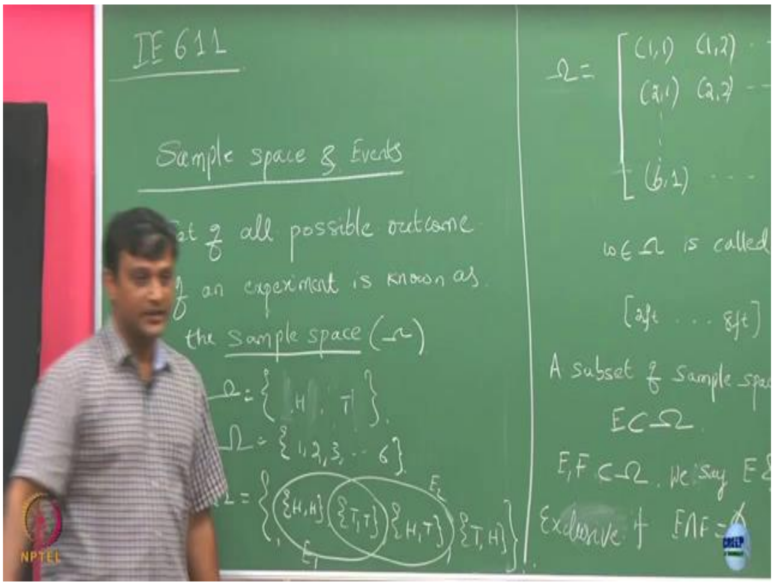

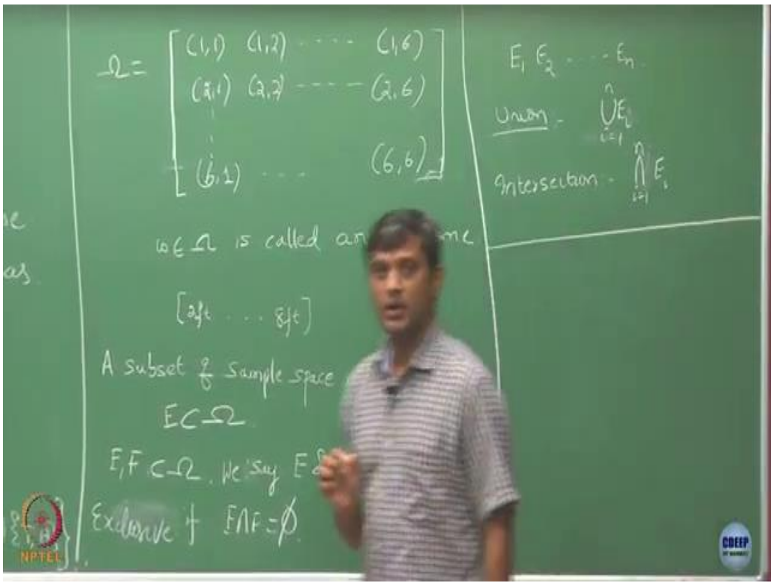

So similarly let say we are going to have E 1, E 2 that E n like let us say these are collection of events. They are all coming from the same sample space. We can take about their unions, we can take about their intersections and we can take about any combinatory as structure that is the union of some along with intersection with others and all. For example, if you want to let us say I want to take union, so here let i equals to (())(17:58), so what is this? So I am just taking union of all the sets. Now suppose let us say E 1 is H, H and E 2 is H, T, what is the union of E 1 and E 2?

Student: (())(18:25)

Professor: So union of H 1 and H 2 says what?

Student: (())(18:30)

Professor: The first outcome is head. That is right like you do not care about second, second can be either head or tail, but you want the first outcome to be head. So in that case, this E 1, E 2 could be some events in your space but you can combine them to derive more, more more events. And similarly, you can look at intersections.

So, I could write it as E intersection. So, what does intersection of set of events give you? The common element, so in the same example, this H, H and H, T, the intersection will be what?

Student: (())(19:41)

Professor: So, so notice that, when I said these subset of omega, one element can also be a subset. So, here these are the inclusion elements in the set omega. For example, E 1 could be this element and this element and E 1 could be or E 2 could be these three elements. So now, when you are going to take the intersection of let us say just two element in this there is nothing common between them.

But suppose you are going to define E 1 to be this set and equal to be this set, you can do that right like you can take this because both are subsets of my, my omega. What is that? Intersection is going to give you. It requires that both tosses give me tail.

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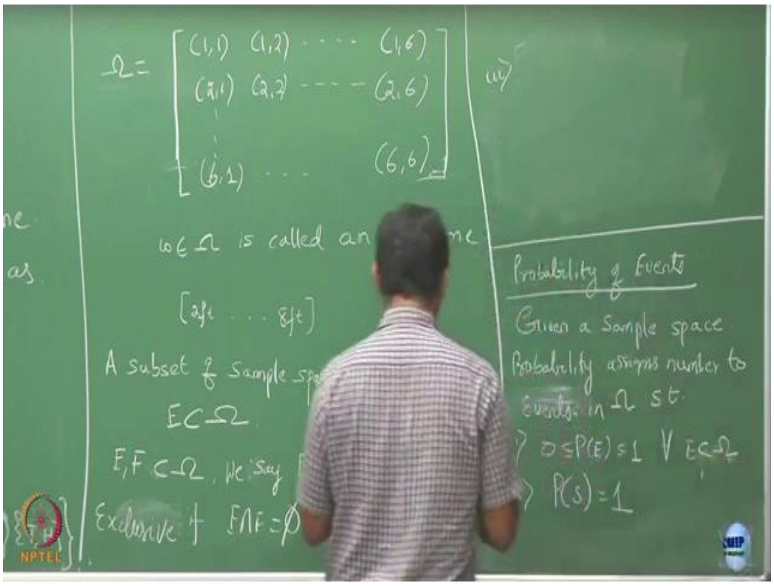

Now, we will get to the notion of probability and probability of events. So we will make the notion of probability precise more. But let us try to write down informally what we mean by probability of events? So you take an so hence not when I say an experiment, random experiment that will always come with an associated sample space. So when I said (())(22:51) going to take all possible, some possible outcomes, I am going to call it a sample space. And know on that I am associated event space. And I am going to say this probability is something that is going to assign a numbers to this event such that.

Student: (())(24:15) whether it need to be proper subset.

Professor: Here, not necessarily it could be full event space. So, I mean when I used the notation subset or it not necessarily the stick subset it could be entire set itself. I mean, if, if I want to specifically say that it is a proper subset, then I am going to use this notation. That means equality is not included. Otherwise, if I just write it, it will be just like this.

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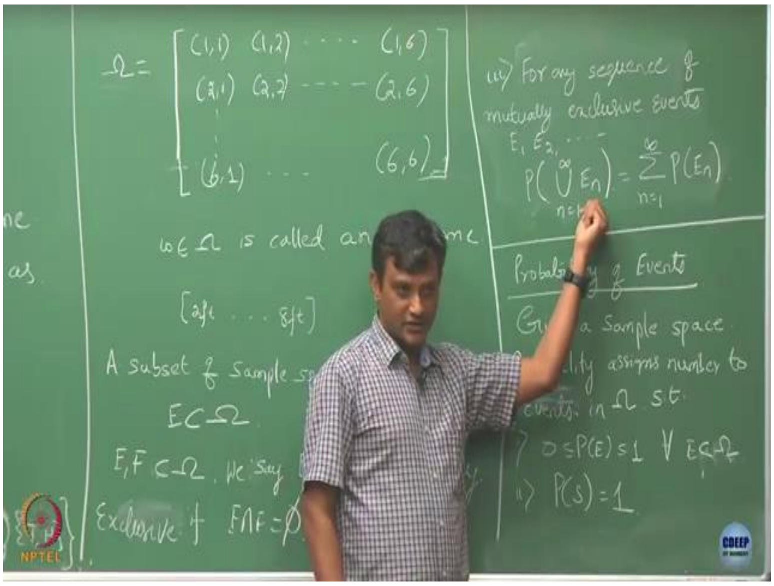

It says that this function P, P basically function. It tells what is the value assigned to each possible event in omega. It says that for every event will take a value which is between 0 , 1. And the entire sample space is going to give a value of 1. And if you are going to take a sequence of mutually exclusive event, the probability of their union is nothing but sum of their probability.

So, all of you understand what I mean by sequence. So, you all might have heard about sequence of numbers, sequence of real numbers, like x 1, x 2, where x 1 is a real number. For example, x n can be, let us say a 1 by n. So, what is x n converges to? If I said x n equals to 1 by n, but now I am talking about sequence of events. What do I mean by sequence of events? It is just like indexing like I have set. So, what I mean by sequence means I have a indexed set of values like you give me an any integer. I mean natural number that index is associated with some number. So if I tell me I, x i the corresponding number if you tell me n, x n is the corresponding.

The same notion we are going to use for the sets like this for every possible index, there is an association set if you have such thing, if you have such a sequence of this it is it is going to be just this. Now and we are going to refer to P of E, for any P of E we are going to just say is the probability of event.

Now let us see some of the things associated with this problem. And let us say we can come up with some such probability function. Suppose now let us go back to our basic coin tossing problem. So we have omega which consist of head and tail. Event can be head or tail. So let us say for if I say it is a fair coin, what is that P of H you want to assign?

Student: 1 by 2

Professor: 1 by 2? Right, like both are equally likely. If let us say it is a biased coin, let us say somebody has tweaked the coin and probably likely would heads is more tail then maybe you want to assign probability of head to be something greater than point five. And now if I go to dice problem, what is if I say it is a fair die? What is the probability you are going to assign to any number?

Student: 1 by six.

Professor: Just 1 by six, all of them are equally likely and if again, but that is not necessary. If you feel that, you design such thing, such that it could be biased so you may you make some face heavy so that it comes more likely it is a design is like that. You are going to assign what more than 1 by 6 further. And also note that this probability requires that P of S is equal to 1 that is.

Student: (())(30:00)

Professor: It is all possibilities, you are going to say. I am doing this event and these are the possible outcomes. When you do this experiment, one of the things that has to happen, otherwise your definition of outcomes sample space was incorrect. You can, it is up to you like you can define based on your model, how want to you want to decide this be. So you would like, if you want to model a fair, fair coin, you would like this P to be half for both the events. And the same thing if you want to model fair dice you want to set P of I to be 1 by 6 for all I, 1 to six. But that is not necessarily right.

This is up to you. This is your (())(30:53) function that you want to use it as whenever you want to model. Suppose let us say I have I say a fair coin and you keep tossing it, keep on tossing it. And after large number of toss, you count how many heads you got and how many tails you got. And divided by number of tosses you made. What do you expect that to be? So if it is a fair coin you expect that to be close point five.

So, in a way, you want to kind of interpret this probability as frequency of occurrence of that event when you do this experiment again and again. So let us try to… we will get back to this again.

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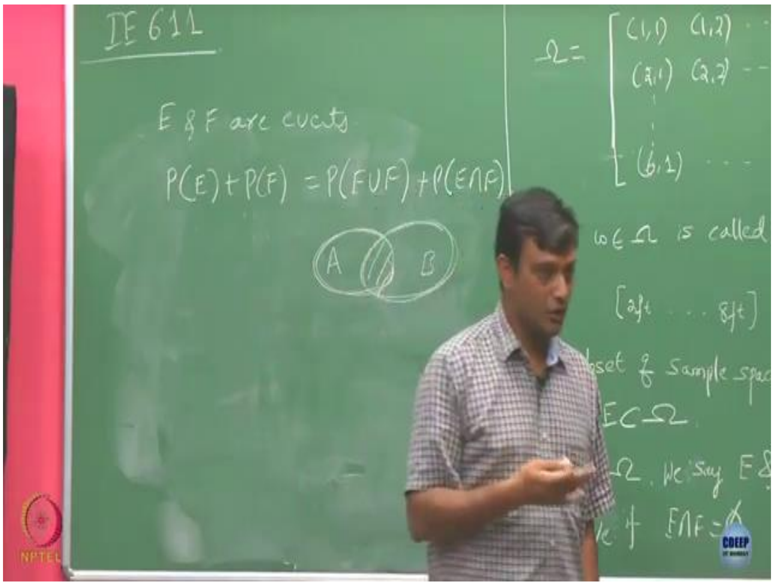

Now suppose let say E and F are two events, I am interested in finding probability of P plus probability of F. I am not saying that this E and F are disjoint. They could have some common elements. If that is the case, what is this sum you are going to be like? So, one natural way to do this is either take, so take it like probability of UF. You want to do correction for this and that is going to be and why is this correction here?

Student: They are not mutually.

Professor: They are not mutually exclusive. Like suppose if you have a set A and another set B, this set is entire this and this is B. If you are going to take this entire sets once and you take this entire set once, you would have added this region twice.

Student: (())(33:36)

Professor: Yeah?

Student: Added instead of subtracting.

Student: Because we have to find P E plus P R.

Student: Instead of minus sign here it has to be positive sign.

Professor: So I mean just by this Venn diagram geometric interpretation you will see that fine when I did this I counted the common part one. So, I need to do it again because that has actually been added twice on the left hand side. Again this relation is fine, it makes intuitive sense, the probability should be such that it should satisfy this conditions. So, now, you have introduced this basic notion of what is sample space? What is event? And we have introduced this notion of this probability and if we want this probability should be such that it it satisfy some intuitive properties.

Instead of going, I want this property to satisfy something, what will now try to do is? Partly some basic assumptions on my probability space or my event space and define probability in a systematic way. So that whatever the kind of things we want, they actually naturally follow by our basic definition. Instead of, we want this to happen, you start with our basic frame were such that that they are induced to happen.