Game Theory
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Lecture – 03
Combinatorial Games: Zermelo's Theorem

In the previous session we have seen take away game and discussed the optimal play of players. We also introduced the formal definition of combinatorial games.
In a combinatorial game,

- There are 2 players who make moves alternatively.
- There are a finite set of possible moves.
- Rules specifying movement from a position to other positions.
- The game ends when a player cannot make a move.
- The game eventually ends.

These 5 rules specify a combinatorial game. Note that, we need to understand that these five rules do not say who wins.

Winning in general, is specified by 2 things: One is called *normal* play, the other is called *misere* play. In a normal play, the last player to make a move is the winner. In misere play, the last player to make a move is the loser. Despite the fact that they seem to look like the negation of one another, they are different and we will see that later.

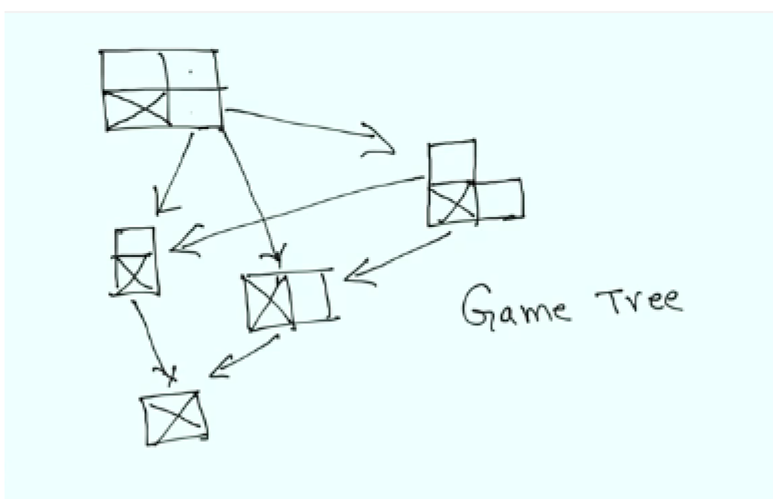
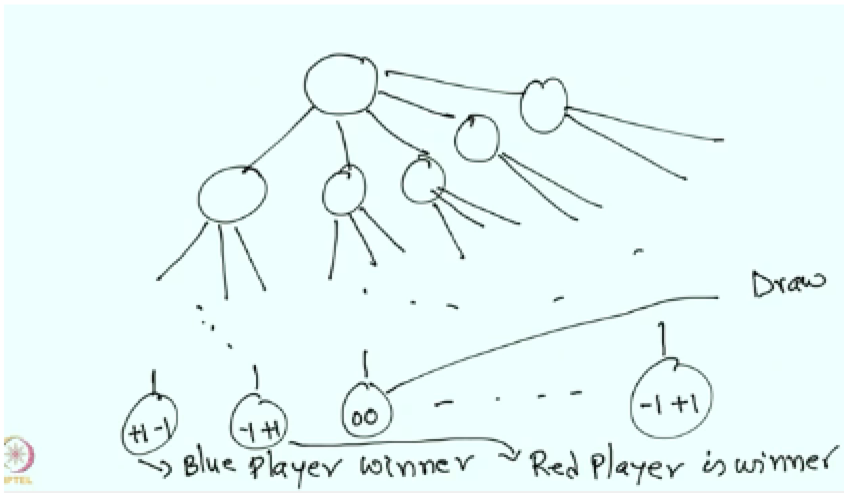


Figure 1: Refer Slide Time: 04:21

We have previously looked at positions in a combinatorial game. There are two types of positions: the *N position* and the *P position.* A position in a combinatorial game is called an N position, if the person who is going to make a move from that position is going to be the winner whereas, in a P position, the previous person who had made the move is going to be the winner. Note that, these are mainly for win, loss games. Here, we are not considering that draw option. Now, let us look at a simple game called *Chomp*.

Consider a 2×2 game of Chomp, where the bottom left cell is the poisonous cell. From the initial position, there are multiple possibilities that can happen. These possibilities are captured in what is called a *game tree* representation. In this representation, the terminal positions are the positions which say who won the game or if it is a draw.



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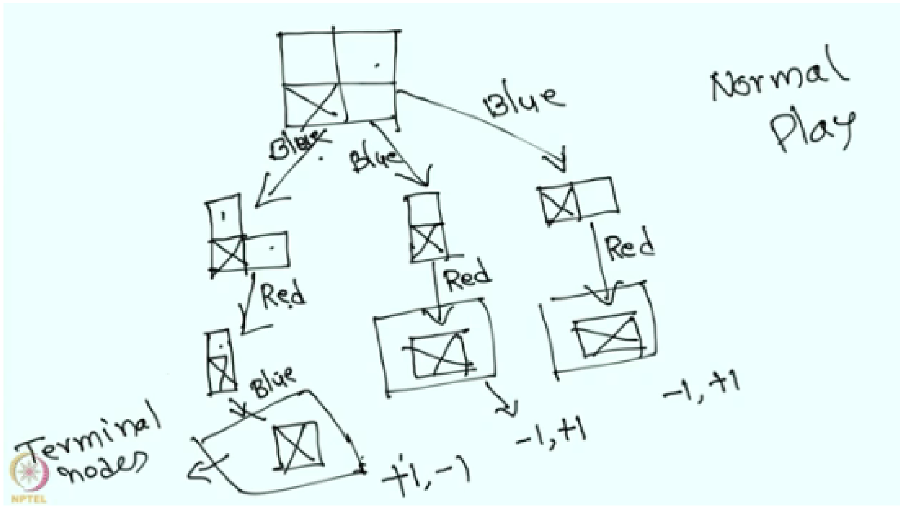
The game tree is a very powerful idea. In fact, we will explore this game tree in a later part of the course. Now, we will present a very important result where we use this idea of the game tree. In fact, if we recall we have not said anything about the following fact: Given a game, how do we know whether a person can always win or not? This is a very famous result due to Zermelo. Zermelo's theorem says that in any combinatorial game, either the first player has a winning strategy or the second player has a winning strategy or there is a draw.

Theorem(Zermelo). *In any combinatorial game with the possibilities win, loss and draw, one of the three holds*:

- *Player 1 has a winning strategy.*
- *Player 2 has a winning strategy.*
- *The game ends in a draw.*

Zermelo first discussed this theorem in connection with the game Chess. Even though chess seems like a finite game, one can actually repeat certain moves making it infinite.

In order to make it finite, people impose rules, for example, if you repeat the position thrice, the game ends in a draw. With those restrictions, Chess becomes a finitely terminating game and Zermelo's theorem tells you that in Chess, either the white player has a winning strategy or the



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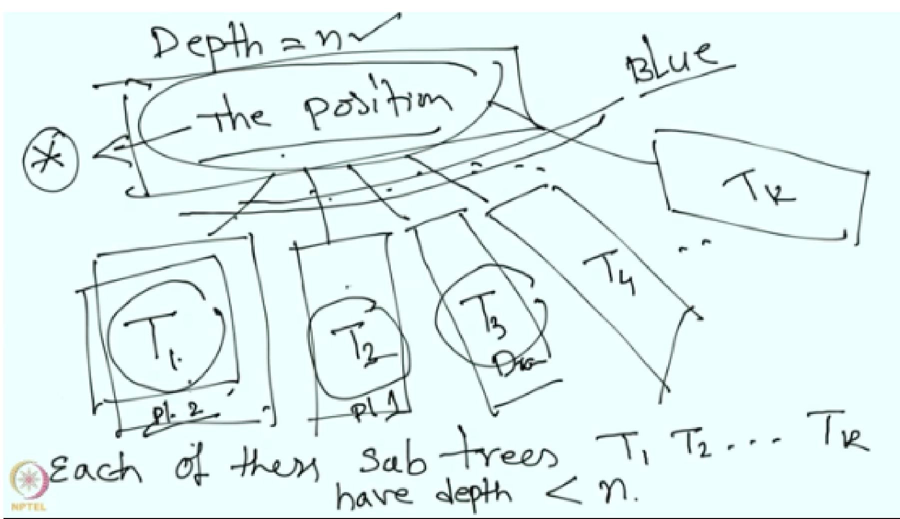
black player has a winning strategy or the game ends in a draw. This is the consequence of the Zermelo's theorem.

Proof: The proof is based on induction. First, it is based on the depth of the game tree. Recall that terminal positions are where a win, loss or draw is decided. *Depth* of a game tree is the length of the longest path from the initial position to a terminal position. We can easily see that a depth of zero means that we are at the terminal positions. A depth of one means that we are in a position just above a the terminal position. Now, we use induction on the depth of the game tree.



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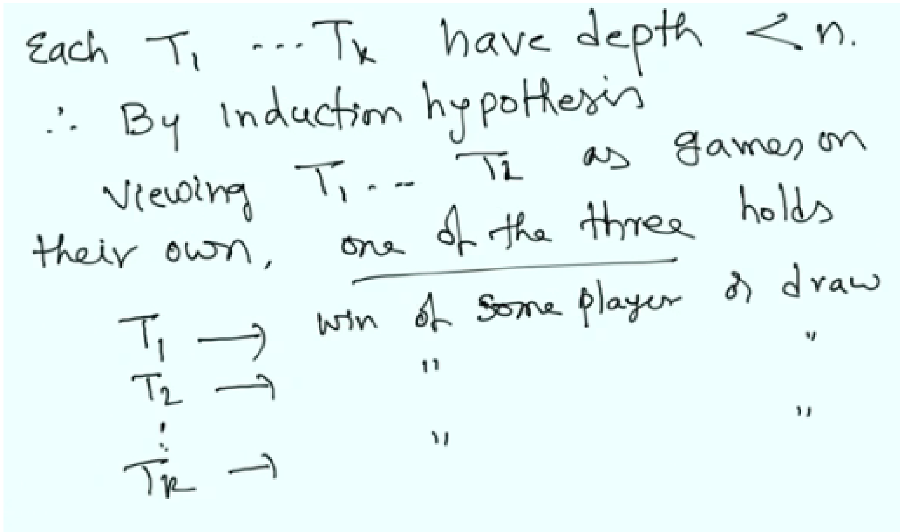
If the depth is zero, win, loss or draw has already been determined. Now, according to our induction hypothesis, we assume that the theorem is true for all game trees with depth less than n. In the next step, we need to prove the theorem for game trees with depth equal to n.

Let the depth of the game tree be n. Now, given a position, whoever the player, there are certain choices that (s)he is making. Call the sub-trees $T_1, T_2, .., T_k$. Here k is the number of moves available at the given position.



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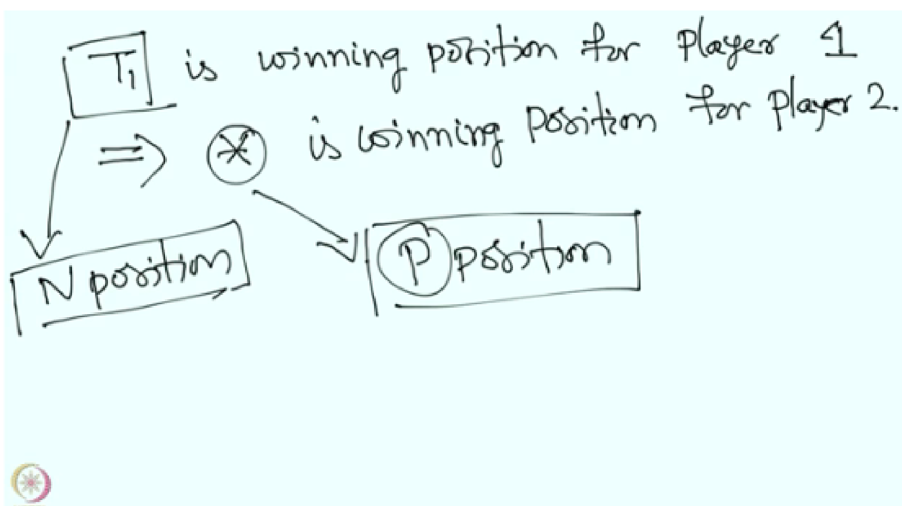
Now, each of these sub-trees $T_1, T_2, .., T_k$ have depth less than n. So, by induction hypothesis, viewing these $T_1, T_2, .., T_k$ as games on their own, one of the three outcomes holds. That is, either player 1 has a winning strategy or player 2 has a winning strategy or game ends in a draw. This means that each game sub-tree has either player 1 winning or player 2 winning or the game ending in a draw.

Remember, in all these games, it is not important who the player is, the important point is what position the player who is making the move is in.



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So, in the game tree T, if the sub-tree T_1 is a winning position for player 1, it means that the person who is going to make a move at the position T_1, he can force a win. And if T_1 is a winning position for Player 1, the starred position(initial position) is going to be a winning position for Player 2.

So, in a sense, by our induction hypothesis, this T_1 can be replaced by winning position for Player 2, say. Similarly, if T_1 is a Player 2 win, then we can replace the sub-tree T_1 with a winning position for Player 1. Or, it could also be a draw. Now, the player who is making a decision at the initial position is going to decide which of these choices are giving her the best outcome. There are finitely many choices with her. And whatever she chooses, the game ends in one of the three outcomes mentioned above. This completes the proof of the theorem.

So, to summarize this theorem, we have used a very simple mathematical principle of induction. If the depth is zero, then either it is a winning position or a losing position or a draw for the players and that is determined automatically. If the depth is less than n, we are assuming that the theorem is true and then we show that if the depth equals n then also, one of the above three outcomes holds.

This is the essence of this theorem. We will keep using this theorem again and again, in this combinatorial theory part as well as in the other parts of this course.