Game Theory

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Lecture 22
Non-Zero-Sum Games: Lemke-Howson Algorithm - I

In this lecture we start understanding the Lemke-Howson algorithm. This requires some preliminary results. First, we go through the notation.

- (A,B) is the bimatrix game with order $m \times n$.
- $S_1 = \{1,2,..,m\}$
- $S_2 = \{m+1, m+2, .., m+n\}$
- $\Delta_1 = \{x \in \mathbb{R}^m_{\geq 0},\ \sum_i x_i = 1\}$ where $x^T \nVdash = 1$.
- $\Delta_2 = \{y \in \mathbb{R}^n_{\geq 0},\ \sum_j y_j = 1\}$ where $y^T \nVdash = 1$.
- $u_1(x,y) = x^T Ay = < x, Ay >$
- $u_2(x,y) = x^T By = < x, By >$
- $\nVdash$ is the column vector of order $n \times 1$ with 1 as each entry.
- For $x \in \Delta_1$, $Supp(x) = \{i \in \{1,2,..m\} | x_i > 0\}$, and for $y \in \Delta_2$, $Supp(y) = \{j \in \{m+1, m+2, ..m+n\} | y_j > 0\}$ denote the *support* of the mixed strategy profile (x,y).

Next, we look at an interesting result known as *Best Response Conditions.*
Theorem. Let x and y be mixed strategies of players 1 and 2 respectively. Then, x is the best response to y iff $\forall i \in S_1$,

$$x_i > 0 \Rightarrow \big(Ay\big)_i = u = \max\{\big(Ay\big)_k : k \in S_1\}$$

Proof. $x \in argmax_{x \in S_1} < x', Ay >$ where $x' = \sum_{i=1}^m x'_i (Ay)_i$. Now, $\sum_i x_i (Ay)_i$ is a convex combination of $(Ay)_i$'s. If x is the best response, that means x is going to put largest probability on the i's where this $(Ay)_i$ is maximized. Hence, x_i has to be positive if and only if $(Ay)_i$ is the maximum of $(Ay)_j$'s. This ends the proof.

Next, we introduce what is called a *non-degenerate game*. A two player game is called a non-degenerate game if no mixed strategy of support size k has more than k pure best responses.

Proposition. In any non-degenerate game every Nash equilibrium (x^*, y^*), x^* and y^* have supports of equal size.

Now, if a game is not non-degenerate, we call that as a *degenerate game*. But, we will mostly concentrate on non-degenerate games. This above proposition can be used as an algorithm. This is known as *Equilibrium by support enumeration.*
Algorithm:

- Input : non-degenerate game
- Output : Nash Equilibria
- Method : $\forall k = \{1, 2, .., \min\{m, n\}\}$ and each pair (I, J) of k-sized subsets of S_1 and S_2 respectively, solve the following equations:

$$\sum_{i \in I} x_i b_{ij} = v \text{ for } j \in J, \sum_i x_i = 1$$
$$\sum_{j \in J} a_{ij} y_j = u \text{ for } i \in I, \sum_j y_j = 1$$

with $x \geq 0$ and $y \geq 0$.

We can easily see the correctness of this algorithm using the Best Response Condition theorem that we have seen earlier. The detailed proof is left for the reader as an exercise.

Next, we introduce the notation of a *polytope*. First, let us look at an *affine combination*.

Given $Z_1, Z_2, ..Z_k \in \mathbb{R}^d$, an affine combination is given by

$$\sum_{i=1}^{k} \lambda_i Z_i,\ \lambda_1 \in \mathbb{R}, \sum_i \lambda_i = 1$$

When $\lambda_i \geq 0$, we call this a convex combination. Moreover, a convex set is a set in which all the convex combinations of the elements are in the set.

Next, we define what is called *affinely independent*. $Z_1, Z_2, .., Z_k \in \mathbb{R}^d$ are affinely independent if no Z_i is an affine combination of others. A convex set has dimension d' if and only if it has $d' + 1$ affine independent points and no more.

Next, we define a *Polyhedron*. A Polyhedron $P \subseteq \mathbb{R}^d$ is a set $\{x \in \mathbb{R}^d | Cz \leq q\}$ for some matrix C and vector q. We say that it has full dimension if it has dimension d. If this is bounded, then we call this as polytope.
Next we define the *Face* of P. The face of P is given by

$$\{z \in P | c^T z = q_0\}$$

for some $c \in \mathbb{R}^d, q_0 \in \mathbb{R}$. Note that, $c^T z = q_0$ is a hyperplane. A *vertex* of P is a unique element of a zero-dimensional face of P. When you take a hyper plane, if the hyperplane intersects this polyhedron exactly at one point and the polyhedron lies entirely on one side, then that unique point is known as a vertex. In fact, in convex analysis there is a very important theorem, which is known as a Krein-Milman theorem which says that any bounded convex set has an extreme point which is, in a sense, the vertex here, in this case.

An *edge* is a one-dimensional face of P. A *facet* is a face of dimension $d - 1$. These are some of the notations that we will require further in this section.