

**Time value of money-Concepts and Calculations**  
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**Lecture - 10**  
**Annuities- 1 & 2**



Welcome to the lecture series on Time value of money-Concepts and Calculations. In this lecture we will cover Annuity; Annuity part 1 and Annuity part 2.

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An annuity is a series of regular, equally spaced, payments over a defined period of time (often called the term) at a constant rate of interest. The payments may occur weekly, fortnightly, monthly, quarterly or yearly. Annuities are classified into four categories. These are: (1) ordinary annuity, (2) annuity due, (3) deferred annuity, and (4) perpetuity. An Ordinary annuity is an annuity where the regular payment is made at the end of the successive time periods.

Examples of annuity are :

Regular payments to savings account	monthly home mortgage payments
Payment to superannuation fund	insurance premiums
Periodic payment to a person from retirement fund	straight bonds paying coupon payments

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An annuity is a series of regular equally spaced, payments over a define period of time often called the term, at a constant rate of interest. The payments may occur; weekly, quarterly, monthly quarterly or yearly. Annuities are classified in to 4 categories, these are one ordinary annuity, second annuity due, third deferred annuity and forth perpetuity.

An ordinary annuity is an annuity, where the regular payments are made at the end of the successive time periods. Example of annuities are regular payment to saving accounts, payment to superannuation fund, periodic payment to a person from retirement fund, monthly home mortgage payment, insurance premiums, straight bonds paying coupon payments.

Now let us see; what are the definitions of different annuity. Ordinary annuity, in ordinary annuity will equal payments are made at the end of each compounding period starting from the first compounding period.

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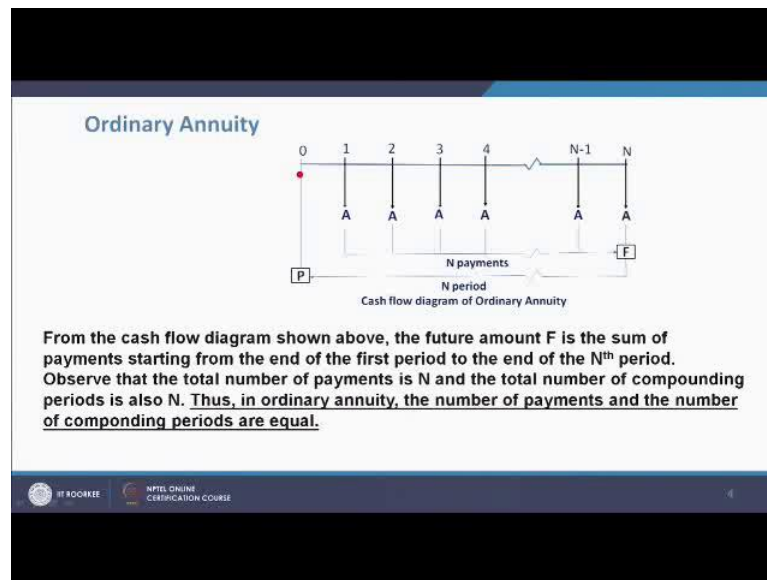
Types of annuity	Description
Ordinary Annuity	In ordinary annuity, the equal payments are made at the end of each compounding period starting from the first compounding period
Annuity Due	In annuity due, the equal payments are made at the beginning of each compounding period starting from the first period
Deferred Annuity	In deferred annuity the first payment is deferred a certain number of compounding periods.
Perpetuity	Perpetuity is an annuity where the payment period extends forever, which means that the periodic payments continue indefinitely

Note: In this lecture we will deal with ordinary annuity only.

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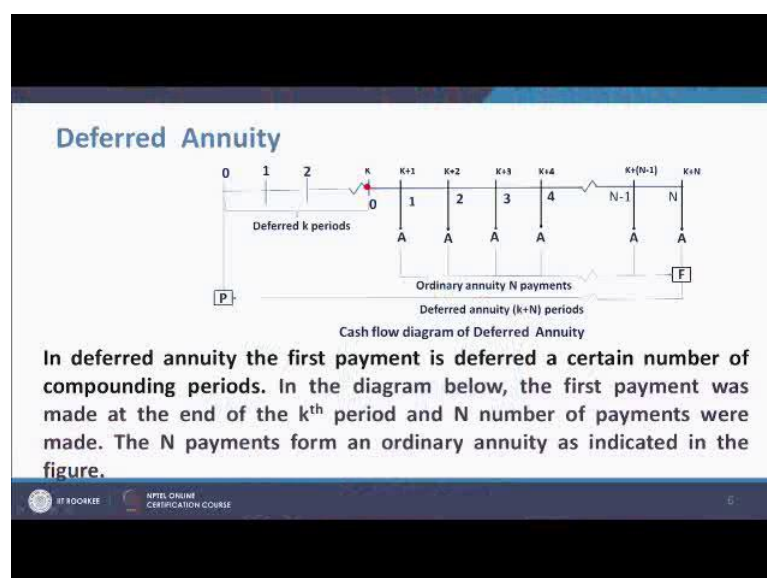
Annuity due; in annuity due the equal payments are made at the beginning of each compounding period starting from the first period deferred annuity. In deferred annuity the first payment is deferred a certain number of compounding periods. Perpetuity; perpetuity is an annuity where the payment period extends for ever which means that the periodic payments continue indefinitely. In this lectures will deal with only ordinary annuity.

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Let us see pictorially value what is an ordinary annuity. In this picture, here we see that the principle case put here at  $P$  is equal to 0 and then we get the annuities at 1 2 3 4 up to  $N$ . Where, at the  $N$  we can find out the future value of the sum of all the annuities. So, here  $N$  payments are made and there are  $N$  periods. From the cash flow diagram shown above the future amount  $F$  is the sum of payments starting from the end of the first period to the end of the  $N^{\text{th}}$  period. Observe that the total number of payments is  $N$  and the total number of compounding periods is also  $N$ . Thus, in ordinary annuity, the number of payments and the number of compounding periods are equal.

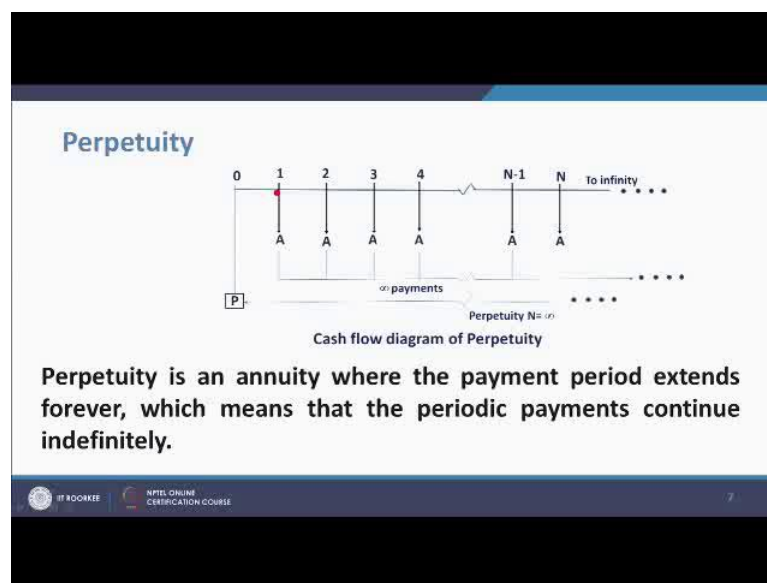
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Now, let us take the Annuity due. In annuity due we see that the first payment is made at  $t$  is equal to 0 time and then up to  $N$  minus 1 periods the payments is made. And the end of the  $N$ th period the final value is calculated. So, as indicated in the figure above  $F_1$  is the sum of ordinary annuity of  $N$  payments. The future amount  $F$  of annuity due at the end of  $N$ th period is 1 compounding period away from  $F_1$  in symbol  $F$  is equal to  $F_1$  in to  $i$  plus  $i$ , future amount of annuity due is  $F$  deferred annuity.

Here we will see that the payment is deferred to the end of the period of  $K$ th, in deferred annuity the first payment is deferred a certain number of compounding periods. In the diagram below the first payment was made at the end of the  $K$ th period and  $N$  number of payments was made. The  $N$  payments from an ordinary annuity as indicated in the figure.

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Now Perpetuity; perpetuity is an annuity where the payment period extends forever. Which means that the periodic payments continue in definitely here we will see that at the end of the first period the payments start and it continues every year or every time term up to the infinite. Now let see the derivation of an annuity for annually compounding.

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**Derivation of annuity for annually compounding**

**Amount of annuity (A) and the sum of all the annuity payments; is therefore**

$$S = A(1+i)^{N-1} + A(1+i)^{N-2} + A(1+i)^{N-3} + \dots + A(1+i) + A \quad (1)$$

$$S(1+i) = A(1+i)^N + A(1+i)^{N-1} + A(1+i)^{N-2} + \dots + A(1+i)^2 + A(1+i) \quad (2)$$

**Subtract equation (1) from equation(2)**

$$S + Si - S = A(1+i)^N - A$$

$$Si = A(1+i)^N - A$$

$$S = A \left[ \frac{(1+i)^N - 1}{i} \right]$$

**"Annuity" For annually compounding**

$$P = A \left[ \frac{(1+i)^N - 1}{i(1+i)^N} \right]$$

**"present worth" of annuity for annually compounding**

**We know**

$$S = P(1+i)^N = A \left[ \frac{(1+i)^N - 1}{i} \right]$$

**A = Amount of annuity per year**  
**S = future value of sum of all annuities**  
**P = present value of sum of all annuities**  
**i = interest rate per year**  
**N = no. of years the annuities are paid**  
**No. of payments are equal to no. of compounding years**

Now you can write down the S that is sum future value of the all annuities payments made for ordinary annuity is equal to the Annuity A into i plus to the power N minus 1 plus A into 1 plus i to the power of N minus 2 and so on and so forth. A 1 plus i plus A; so if i multiply i plus 1 with the left hand side as well as right hand side this becomes S into i plus i is equal to A 1 plus i to the power N plus A 1 plus i to the power N minus 1 so and so forth and my last term will be A 1 plus i.

Now, we subtract these 2 equations that is the equation number 1 from equation number 2. Then it becomes S plus Si minus S is equal to A 1 plus i to the power N minus A. So, this S has canceled out. So, Si is equal to A, 1 plus i to the power N minus A, if i take A common then it becomes 1 plus i to the power N minus 1 and this i is divided. So, S is equal to A, in brackets 1 plus i to the power N minus 1 whole divided by I. This gives as the formula to find out annuity for annually compounding.

Now, if we place the value of S with P. So, it is P into 1 plus i to the power N is equal to A in brackets 1 plus i to the power N minus 1 whole divided by I, then P becomes A into 1 plus i to the power N minus 1 whole divided by i into 1 plus i to the power N. So, it is shows present worth of annuity for annual compounding. Where A is the amount of annuity per year, S is the future value of sum of all annuities, P present value of sum of all annuities, i interest rate per year, N number of years the annuities are paid. Here the numbers of payments are equal to number of compounding years.

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**Derivation of annuity for discretely compounding**

Amount of annuity (A) and the sum of all the annuity payments for discretely compounding is :

$$S = A \left(1 + \frac{r}{m}\right)^{mN-1} + A \left(1 + \frac{r}{m}\right)^{mN-2} + \dots + A \left(1 + \frac{r}{m}\right)^{mN-m} + A \quad (3)$$

$$S \left(1 + \frac{r}{m}\right) = A \left(1 + \frac{r}{m}\right)^{mN} + A \left(1 + \frac{r}{m}\right)^{mN-1} + \dots + A \left(1 + \frac{r}{m}\right)^{mN-2} + A \left(1 + \frac{r}{m}\right)^{mN-1} \quad (4)$$

Subtract equation (3) from equation (4)

$$S \left[1 + \frac{r}{m} - 1\right] = A \left(1 + \frac{r}{m}\right)^{mN} - A$$

$$S = \frac{A \left[ \left(1 + \frac{r}{m}\right)^{mN} - 1 \right]}{\frac{r}{m}} \quad \text{"Annuity" For discretely compounding}$$

$$S = P \left(1 + \frac{r}{m}\right)^{mN} = \frac{A \left[ \left(1 + \frac{r}{m}\right)^{mN} - 1 \right]}{\frac{r}{m}} \quad \text{or} \quad P = \frac{A \left[ \left(1 + \frac{r}{m}\right)^{mN} - 1 \right]}{\left(\frac{r}{m}\right) \left(1 + \frac{r}{m}\right)^{mN}}$$

A = Amount of annuity per period  
S = Future value of sum of all annuities  
P = Present value of sum of all annuities  
r = nominal interest rate per year; r/m = interest rate per period  
N = no. of years the annuities are paid  
m = no. of periods per year  
No. of payments are equal to no. of compounding periods

Now, if we go for the derivation of annuity for discretely compounding. The amount of annuities A and the sum of all the annuity payments for discretely compounding is S is equal to  $A \left(1 + \frac{r}{m}\right)^{mN-1} + A \left(1 + \frac{r}{m}\right)^{mN-2} + \dots + A \left(1 + \frac{r}{m}\right)^{mN-m} + A$ . In similar manner we multiply S with  $1 + \frac{r}{m}$  left hand side and right hand side and then equation 3 and 4 is formed. Then we add up the equation 3 and 4, it becomes  $S \left(1 + \frac{r}{m} - 1\right) = A \left(1 + \frac{r}{m}\right)^{mN} - A$  and S is equal to this value which is  $A \left[ \left(1 + \frac{r}{m}\right)^{mN} - 1 \right] / \frac{r}{m}$  and if i replace the value of S with  $P \left(1 + \frac{r}{m}\right)^{mN}$  then P becomes this.

So, this is the present value of the annuities when compound is discretely compounding. A is equal to amount of A annuity per period, S is equal to future value of some of all annuities, P is equal to present value of sum of all annuities, r is equal to nominal interest rate per year and  $r/m$  is equal to interest rate per period N is number of years the annuities are paid and m is the number of periods per year.

Here also number of payments is equal to the number of compounding periods. Sometimes only the effective, rate per period that is per month may be per day is known however, compounding for m periods at an effective interest rate per period is not affected by the definition or length of the period for example, compounding for 360 fine

periods that is days at an interest rate of 0.03808 percent and for this if I find out the effective this will be 1 plus 0.000808 to the power 365 minus 1. Which comes out to be 0.149 and which 14.9 percent is.

So, this is  $i$  effective, that is effective interest rate is 14.9 percent. Is the same as compounding for 12 periods, that is month per month compounding at an interest rate of 1.164 percent and for this percent if I find out  $i$  effective that is effective interest rate, this is 1 plus 0.164 to the power 12 minus 1 which comes out to be around 14.9 percent all 1S at an effective annual interest rate of 14.9 percent.

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Sometimes, only the effective rate per period (e.g., per month) is known. However, compounding for  $m$  periods at an effective interest rate per period is not affected by the definition or length of the period. For example, compounding for 365 periods (days) at an interest rate of 0.03808%  $((1+0.0003808)^{365}-1=0.149=14.9\%)$  is the same as compounding for 12 periods(months) at an interest rate of 1.164%  $((1+0.0164)^{12}-1=0.1489=14.9\%)$ , or once at an effective annual interest rate of 14.9%. In each case, the interest rate per period is different, but the effective annual interest rate is the same. If only the daily effective rate were given, the discount factor formulas could be used with  $i = 0.03808\%$  and  $n = 365$  to represent each yearly cash flow. Hence Eq. derived for annuity compounded annually can be converted into annuity compounded per period by simply replacing  $i = r/m$  and  $N$  by  $mN$

$i = \frac{r}{m}$

Annuity for annually compounding  $S = P(1+i)^N = A \left[ \frac{(1+i)^N - 1}{i} \right]$   $i_e = (1+i)^n - 1$

Annuity for discretely compounding  $S = P(1+r/m)^{mN} = A \left[ \frac{(1+r/m)^{mN} - 1}{r/m} \right]$   $= \left(1 + \frac{r}{m}\right)^n - 1$

10

In each case the interest rate per period is different effective annual interest rate is the same if only the daily effective rate where given the discount factor formula could be used with  $i$  is equal to 0.03808 percent and  $N$  is equal to 265 to represent each yearly cash flow hence the equation derived for annuity compounded annually can be converted into annuity compounded per period by simply replacing  $i$  is equal to  $r$  divided by  $m$  and  $N$  by  $m$  into  $N$ . So, annuity for annually compounding is  $S$  is equal to  $P$  equal to 1 plus  $i$  to the power  $N$  is equal to  $A$  in brackets 1 plus  $i$  to the power  $N$  minus 1 whole divided by  $i$ .

Now, if  $i$  replace  $i$  by  $r$  by  $m$  and this  $N$  by  $mN$ , then I get the annuity for discretely compounding as this  $P$  is equal to 1 plus  $i$  is replaced by  $r$  by  $m$  and  $N$  is replaced by  $m$



into N here also i is replaced by r by m N is replaced by m into N minus 1 and i is replaced by r by m.

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**Derivation of annuity for continuously compounding**

Amount of annuity (A) and the sum of all the annuity payments for continuous compounding is:

$$S = A + Ae^r + Ae^{2r} + Ae^{3r} + \dots + Ae^{(N-1)r} \quad (1)$$

$$Se^r = Ae^r + Ae^{2r} + Ae^{3r} + \dots + Ae^{Nr} \quad (2)$$

Subtract equation (2) from equation (1)

$$S - Se^r = S(1 - e^r) = A - Ae^{Nr} = A(1 - e^{Nr})$$

$$S = A \left[ \frac{(1 - e^{Nr})}{(1 - e^r)} \right] = A \left[ \frac{(e^{Nr} - 1)}{(e^r - 1)} \right] \quad \text{"Annuity" For continuously compounding}$$

We know

$$S = Pe^{rN} = A \left[ \frac{(1 - e^{Nr})}{(1 - e^r)} \right] = A \left[ \frac{(e^{Nr} - 1)}{(e^r - 1)} \right]$$

**Diagram:** A timeline from t=0 to t=N. At t=0, there is a present value P. At regular intervals, there are annuity payments A. At t=N, there is a future value S.

**Formulas:**

$$\left( \frac{A}{P}, r, N \right) = A \left[ \frac{e^{rN}(e^r - 1)}{(e^{Nr} - 1)} \right]$$

$$\left( \frac{A}{F}, r, N \right) = A \left[ \frac{(e^r - 1)}{(e^{Nr} - 1)} \right]$$

**Present Worth of annuity for continuous compounding:**

$$P = A \left[ \frac{(e^{Nr} - 1)}{e^{rN}(e^r - 1)} \right]$$

**Legend:**

- A = Amount of annuity per year
- S = future value of sum of all annuities
- P = present value of sum of all annuities
- i = interest rate per year
- N = no. of years the annuities are paid No. of payments are equal to no. of compounding years

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Now, derivation of annuity for continuously compounding amount of annuity A and the sum of all the annuity payments for continuous compounding is this A plus A into e to the power r plus A into e to the power two r plus A into e to the power 3 r and so and so forth. A e to the power N minus 1 r.

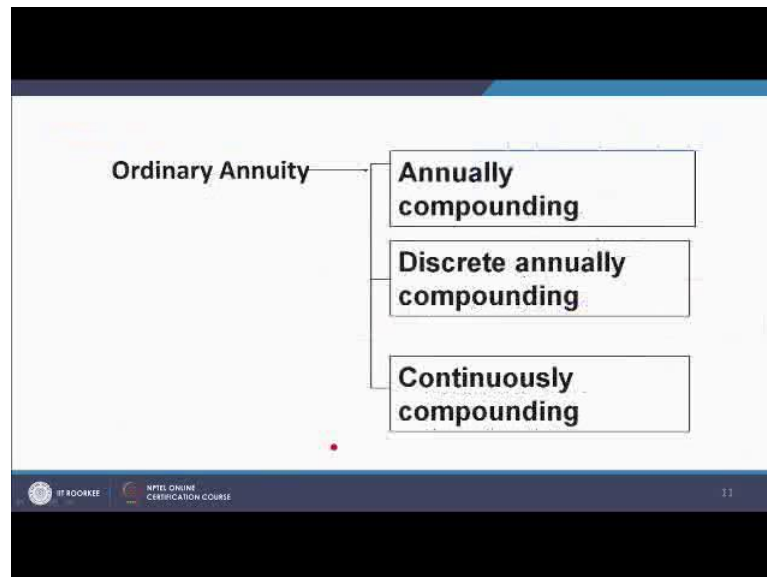
Now, if i multiply this S into e to the power r this becomes A into e to the power r plus A into e to the power two r. So, and. So, forth and A e to the power Nr. Now if I subtract equation 2 from equation 1. So this is S minus S e to the power r is equal to S in brackets minus e to the power r is equal to A minus A e to the power Nr is equal to A minus A e to the power Nr, which is A in brackets 1 minus e to the power Nr. So, I can write down here S is equal to A in brackets 1 minus e to the power Nr divided by 1 minus e to the power r bracket close or I can take negative command.

This becomes A e to the power Nr minus 1 divided by e to the power r minus 1 and this gives you annuity for continuously compounding. Now annuity factor can be define like this A by P r N is equal to A in brackets e to the power rN in brackets e to the power r minus 1 divided by e to the power Nr minus 1 and this factor A by F r N is this factor which is A into in brackets e to the power r minus 1 whole divided by e to the power Nr minus 1.



Now, the present worth is given with this formula  $P$  is equal to  $A$  in brackets  $e$  to the power  $Nr$  minus 1 whole divided by  $e$  to the power  $Nr$  in brackets  $e$  to the power  $r$  minus 1 and this has been derived from here, where  $S$  has been put as  $P$  into  $e$  to the power  $rN$  the value of  $A$  is taken from here say value of  $S$  has been taken from here.

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Now, if you see the ordinary annuity, it can be divided into 3 parts annually compounding, discrete annually compounding and continuously compounding. So we have derived the formula for annually compounding, derived formula for discrete annually compounding and for continuously compounding. Now this was the ANSI functional forms for annuity.



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ANSI Functional forms for Annuity			
Factor Name	Symbol	Formula	Converts
Capital recovery factor: Multiplies a single amount at time zero to give the annual rate of a series of N equal amounts	$(A/P, i, N)$	$\frac{i(1+i)^N}{(1+i)^N - 1}$	$P \rightarrow A$ $A = P(A/P, i, N)$
Present-worth factor: Multiplies the annual rate of a series of N equal amounts to give a single amount at time zero	$(P/A, i, N)$	$\frac{[(1+i)^N - 1]}{i(1+i)^N}$	$A \rightarrow P$ $P = A(P/A, i, N)$
Sinking Fund factor: Multiplies a single amount at N to give the annual rate of a series of N equal amounts	$(A/F, i, N)$	$\frac{i}{(1+i)^N - 1}$	$F \rightarrow A$ $A = F(A/F, i, N)$
<b>Note:</b> In the ANSI form, the distinction between discrete and continuous compounding or discounting is the replacement of i in the discrete compounding form with r for the continuous compounding.			

Now, capital recovery factor this multiplies a single amount at a time 0 to give the annual rate of a series of N equal amounts and this is called A by P, I, N and this given by this  $\frac{i(1+i)^N}{(1+i)^N - 1}$ . Now this converts P to A; that means, present worth to annuity and it is written like this A is equal to P into this factor A by P, i, N then there is A present worth factor which multiplies the annual rate of A series of N equal amounts to give A single amount at time 0. So this is represented by in brackets P by A, i, N and is given by this, in brackets  $\frac{[(1+i)^N - 1]}{i(1+i)^N}$ .

This converts the annuity to present worth. So, written as P is equal to A into P by A, i, N. Third is this sinking fund factor it multiplies a single amount at N to give the annual rate of A series of N equal amounts, it is given by A by F, i, N this is i divided by  $(1+i)^N - 1$ . This converts F the final worth or value to A that is annuity and given as A is equal to F in the brackets A by F, I, N the same ANSI functional forms of annuity for other factors, future worth factor, this multiplies the annual rate of a series of N equal amounts to give A single amount at N and S is called F by A, i, N and this is given as  $\frac{i}{(1+i)^N - 1}$  this converts annuity to the future value.

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ANSI Functional forms for Annuity			
Future worth factor: Multiplies the annual rate of a series of N equal amounts to give a single amount at N years	$(F/A, i, N)$	$\frac{(1+i)^N - 1}{i}$	$A \rightarrow F$ $F = A(F/A, i, N)$
Capital recovery Factor: Multiplies a single amount at time zero to give the annual rate of a series of N equal amounts	$(A/P, r, N)$	$\frac{e^{rN}(e^r - 1)}{e^{rN} - 1}$	$P \rightarrow A$ $A = P(A/P, r, N)$
Sinking Fund Factor: Multiplies a single amount at N to give the annual rate of a series of equal amounts	$(A/F, r, N)$	$\frac{(e^r - 1)}{e^{rN} - 1}$	$F \rightarrow A$ $A = F(A/F, r, N)$
<b>Note:</b> In the ANSI form, the distinction between discrete and continuous compounding or discounting is the replacement of i in the discrete compounding form with r for the continuous compounding.			
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		14	

So is written as F is equal to A into this factor F by A, i, N. Then capacity recover factor, this is for continuous compounding, multiplies A single amount at time 0 to give the annual rate of A series of N equal amounts given by A by P r and N. So, it is equal to e to the power rN in brackets e to the power r minus N divided by e to the power rN minus 1. This converts P which is the present value to annuity. So annuity can be calculated as A is equal to P into A by P r and N.

You will see that the i has been replaced by r. So, this in the note you will see that in the ANSI form the distinction between discrete and continuous compounding or discounting is the replacement of i in the discrete compounding form with r for the continuous compounding form and that is why i has be replaced by r and the last factor is sinking fund factor multiplies A single amount at N to give the annual rate of A series of equal amounts and this is called A by F, r, N.

This factor is shown like this and the value is e to the power r minus 1 divided by e to the power r N minus 1. It converts the future worth to annuities and annuities can be calculated with this A is equal to F and this factor multiplied by this factor, means this quantity.

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**Table 1: Annually Compounding problem matrix**

$$A = P \frac{i(1+i)^N}{(1+i)^N - 1} \quad \dots \text{Eq. 1}$$

The above equations which contains four variable can be solved to find out the value of a single unknown variable only when other three variables are known. Thus four types of problem can be generated out of these equations. The problem matrix for these equations are shown below:  $Pi(1+i)^N = A[(1+i)^N - 1]$

Given	Find	Formula	Remarks	Problem Type
P, i, N	A	$A = P \frac{i(1+i)^N}{(1+i)^N - 1}$	Find annuity(A), when present value(P), interest rate(i) and time in years N are known	<b>A</b>
A, i, N	P	$P = \frac{A[(1+i)^N - 1]}{i(1+i)^N}$	Find the present value(P), when annuity(A), interest rate i and time in years N are known	<b>B</b>
A, N, P	i	$Pi(1+i)^N = A[(1+i)^N - 1]$	Find the interest rate(i), when annuity(A), present value(P) and time in years N are known	<b>C</b>
A, P, i	N	$N = \frac{\ln\left[\frac{A}{A-Pi}\right]}{\ln(1+i)}$	Find time in years(N), when annuity(A), present value(P) and interest rate(i) know	<b>D</b>

**Note: Problem type B have been solved in lecture: Present and Future value**



15

Now if you see the annually compounding problem matrix. So, we have this equation and here if you see the above equation which contains four variables can be solved to find out the value of A single unknown variable only when other 3 variables are known; that means, 4 type of problems can be created out of it. Thus 4 type of problem can be generated out of this equation the problem matrix of this equation are shown bellow. Now the first is that P is given i is given and N is given, we have to find out A and the equation is A is equal to P i 1 plus i to the power N divided by 1 plus i to the power N minus 1. What it does? It finds annuity when present value P interest rate i and the time in years N are known and such type of problem is called problem A.

The second type of problem give r A i and N we have to find out the present worth or present value P and the equation is P is equal to A in brackets 1 plus i to the power N minus 1 and whole divided by i into 1 i to the power N. It finds the present value of the annuity when annuity A interest rate i and time in years N are known such type of problem is called problem type B. Then problem type C, A is known, N is known, P is known, we have to find out the value of i and the equation is P i 1 plus i to the power N is equal to A in brackets 1 plus i to the power N minus 1 and this finds the interest rate when annuity A present value P and the time in years N are known as such type of problem is called problem type C. Here we see that in the l h S and r h S left hand side and right hand side i exist and that is why in such type of problems are solved by trial and error.

The fourth type of problem is A is given P is given and i is given, we have to find out the value of N and N is given as N is equal to  $\frac{1}{N} \ln \left( \frac{A}{A - P} \right)$  divided by  $\ln(1 + i)$  and this problem find the time of N years N. Where annuity A present value P and the interest rate are now such type of problem is called problem t.

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Problem Type- A

**Example-3:** What annual end of the year payment for 15 years is necessary to repay the present loan of Rs.20,000 at annual interest rate of 10% ?

**Solution:**  
 $F = P(1+i)^N$  ;  $A = S \cdot i / [(1+i)^N - 1]$  ; Hence;  $A = P \cdot i(1+i)^N / [(1+i)^N - 1]$

Where;

<p>A = Amount invested at the end of each year</p> <p>F = Total amount due at the end of N years</p> <p>P = The principal or present value of investment</p> <p>N = Number of interest periods</p> <p>i = annual interest rate in %</p> <p><math>i = i/100</math></p>	<p><b>For the present problem:</b></p> <p>A = ?</p> <p>P = Rs.20,000</p> <p>N = 15</p> <p><math>i = 10/100 = 0.1</math></p>
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$$A = P \cdot i(1+i)^N / [(1+i)^N - 1] = 20000(0.1(1+0.1)^{15} / [(1+0.1)^{15} - 1])$$

$$= 20000(0.4177248 / [4.177248 - 1]) = \text{Rs.}2629.47$$

16

Now, let us take problems now first we take the problem type A, this is the example 3 what annual end of the year payment for 15 years is necessary to repay the present loan of 20000 at annual interest rate of 10 percent. Obey this solution, yes, the formula is F is equal to  $P(1+i)^N$  and A is equal to  $S \cdot i / [(1+i)^N - 1]$ . Enhance from this two equations I can find out the relationship between A and P A is the annuity A is equal to  $P \cdot i(1+i)^N / [(1+i)^N - 1]$  and this is the value where A is the amount invested at the end of each year, F total amount due at the end of N years P the principle or present value of investment, N number of interest periods, i annual interest rate in percentage enhance i when i is used I have already told you that it is used as a fraction. So, i divided by 100.

Now, A is not known P is known it is 20000 N is 15 years i is 0.1. So, in this equation we put the value and find out that A is rupees 2629.47. Let see a different problem, which is problem type B. Next example five if I agree to repay A loan by paying rupees 5000 a year at the end of every year for 5 years and the annual interest rate is eight percent then how much one could borrow today.

(Refer Slide Time: 25:58)

Problem Type: B

**Example-5:** If one agrees to repay a loan by paying Rs.5,000 a year at the end of every year for 5 years and the annual interest rate is 8% then how much one could borrow today ?

**Solution:**  
**Given ;** A= 5000; i=0.08; N=5; P = ?  
**Method-1:**

$$A = P \cdot i(1+i)^N / [(1+i)^N - 1] = P \cdot (0.08(1.08)^5 / [(1.08)^5 - 1]) = P \cdot 0.25045645$$

$$P = 5000 / 0.25045645 = \text{Rs. } 19963.55$$

**Method-2:** Find out present value of all investments done at the end of each year

$$\text{Present value} = 5000/(1+0.08) + 5000/(1.08)^2 + 5000/(1.08)^3 + 5000/(1.08)^4 + 5000/(1.08)^5$$

$$= 4629.63 + 4286.69 + 3969.16 + 3675.15 + 3402.92 = \text{Rs. } 19963.55$$

=Rs.19963.55

0                      1                      2                      3                      4                      5  
 Present Value      5000                      5000                      5000                      5000                      5000  
 19963.55

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Now, if you analyze this we find that, what is given, the value of A is given that is annuity is given which is 5000, i the interest rate is given which is 0.08 that is 8 percent and the N the value of years is given which is N is equal to 5 and we have to calculate what is the value of P. So, we will use a formula which relates A to P and this is the formula which relates A to P and if we substitute the values then we find that this is equal to P into 0.25045645 and from here i can find out what is the value of P, which is equal to 5000 which is the value of a 5000 is the value of A divided by this quantity and this come out to be 19963.55. So, this is what we have done through the equations which have developed, but this can be solved by a second method from the first principles method.

So, let us see the second method in the second method find out the present value of all investments done at the end of each year, we are investing at the end of each year and the first year end of the first year we have investing 5000 end of the second year another 5000 end of the third year another 5000 and end of the forth year another 5000 and end of the 5th year another 5000. If I find out the present value of all these sums here bring it to t is equal to 0 and then sum it. So I will find out the results. So, let us see when I find out the present value of this sum 5000 which is invested at the end of the 5th year, this is 5000 divided by 1 plus 1.08 to the power 5 and this 1.08 is equal to 1 plus i divided by 100. So, this will generate a present value of 3402.92.

Similarly, this value that is which I have been invested in the end of fourth year will generate a present value which should be equal to 5000 divided by 1.08 to the power 4 and the present value is this. Similarly the present value of the third year these values are computed and written here and when I sum them up it becomes 19963.55. So, we saw two methods, one quick method and the other method which works on first principle, but gives you A very clear view of participating done is before you.

(Refer Slide Time: 28:49)



Problem Type- B

**Example-6:** Suppose you will retire in exactly one year and want an account that will pay you Rs.30,000 a year for the next 20 years. (The fund will be depleted at the end of the twentieth year.) Assuming a 8% annual interest rate, what will be the amount you would need to deposit now?

**Solution:**  
 Given: A= Rs.30,000; i =8%; N= 20 yr., P= ?

$$P = A \left( \frac{P}{A}, 8\%, 20 \right) = \text{Rs.} 30,000 * \frac{[(1+i)^N - 1]}{i(1+i)^N} = 30,000 * \{4.660957 - 1\} / [0.08 * 4.660957]$$

$$= \text{Rs.} 30,000 * 9.8181477 = \text{Rs.} 294544.43$$

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18

Now, let us take a problem. Problem type B, this is example 6, suppose you will retire in exactly 1 year and what an account that will pay you rupees 30000 a year for the next 20 years. The fund will be depleted at the end of the 20th year. Assuming 8 percent annual interest rate, what will be the amount you need to deposit. Now when you say that what amount you want to deposit now; that means, I want the present value or the principle value. So if you analyze this problem we find that A is given, that is annuity is given which is rupees 30000, the interest rate is given i is equal to 8 percent, the value of N is given which is 20 and what is required is P. So, I take a equation which relates P to A this is the equation. So, I know the value of A then I multiply with this factor and my value of P becomes now 294544.43.

Another problem which is problem type c if 1 agrees to repay A loan of rupees 19963.55 by paying rupees 5000 a year at the end of every year for 5 years find out under what interest rate he has taken the loan.



(Refer Slide Time: 30:30)

Problem Type - C

**Example-5:** If one agrees to repay a loan of Rs.19963.55 by paying Rs.5,000 a year at the end of every year for 5 years. Find out under what interest rate he has taken the loan.

**Solution:**  
Given ;  $A = 5000$ ;  $i = ?$ ;  $N = 5$ ;  $P = \text{Rs.}19963.55$   
It is known that  $Pi(1+i)^N = A[(1+i)^N - 1]$ . From this expression the value of  $i$  can be computed using trial and error as given below  
 $LHS = Pi(1+i)^N$     $RHS = A[(1+i)^N - 1]$

$i$	LHS	RHS
0.06	1602.944	1691.128
0.07	1959.994	2012.759
0.08	2346.64	2346.64
0.09	2764.476	2693.12

As at  $i=0.08$ ,  $LHS=RHS$   
The required result is  $i = 8\%$

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So, if you analyze we find that what is given is  $A$  is 5000,  $i$  is unknown,  $N$  is equal to 5 and  $P$  is rupees 19963.55. So, you use this equation which relates  $P$  to  $A$  through  $i$  and  $N$  and this has left hand side and this has right hand side and in both the side  $A$  exist enhance such type of problems are solved through trial and error from this expression the value of  $i$  can be computed using trial and error as given bellow my l h S is  $Pi$  in bracket 1 plus  $i$  to the power  $N$  and r h S is right hand side is  $A$  into 1 plus  $i$  to the power  $N$  minus 1.

So, i take different values of  $i$  and calculate l h S is equal to r h s. So, we see at  $i$  is equal to 0.08 that is 8 percent l h S is equal to r h S enhance the required result is 8 percent now we see problem type d example 5.

(Refer Slide Time: 31:36)

Problem Type - D

**Example-5:** If one agrees to repay a loan of Rs.19963.55 by paying Rs.5,000 a year at the end of every year for certain amount of years. If the interest rate is 8% find out the number of year when the loan will be repaid.

**Solution:**  
 Given ; A= 5000; i= 0.08; N= ? ; P = Rs.19963.55

$$N = \frac{\ln\left(\frac{A}{A-Pi}\right)}{\ln(1+i)}$$

Or  $N = \ln(5000/(5000-19963.55*0.08))/\ln(1+0.08) = \ln(1.469328)/\ln(1.08)=0.384805/0.076961$   
 $= 4.99999 = 5$  years

20

If he agreed to repay a loan of 19963.55 by paying rupees 5000 a year at the end of A every year for certain amount of years, if the interest rate is 8 percent find out the number of the year, when the loan will be repaid. So if you analyze A, we find that the A is given that is 5000, i is given which is 0.08, N is unknown and P is 19963.55. So, this is the equation which we have used to find out N and once we put all the values that is value of A P i then we calculate it comes out to be N is equal to 4.99999 which is equal in to 5 years.

(Refer Slide Time: 32:27)

**Table-2 Annually Compounding problem matrix**

$$A = \frac{Fi}{(1+i)^N - 1} \quad \dots \text{Eq. 2}$$

The above equations which contains four variable can be solved to find out the value of a single unknown variable only when other three variables are known. Thus four types of problem can be generated out of these equations. The problem matrix for these equations are shown below:

Given	Find	Formula	Remarks	Problem Type
F, i, N	A	$A = \frac{Fi}{(1+i)^N - 1}$	Find annuity(A), when Future value(F), interest rate(i) and time in years (N) are known	E
A, i, N	F	$F = \frac{A[(1+i)^N - 1]}{i}$	Find the Future value(F), when annuity(A), interest rate(i) and time in years (N) are known	F
A, N, F	i	$Fi = A[(1+i)^N - 1]$	Find the interest rate(i), when annuity(A), Future value(F) and time in years (N) are known	G
A, F, i	N	$N = \frac{\ln\left(\frac{F}{F-A}\right)}{\ln(1+i)}$	Find time in years(N), when annuity(A), Future value(F) and interest rate(i) know	H

**Note: Problem type F have been solved in lecture: Present and Future value**

11

Now, we have seen how A is related to t and this set of problems we will see how the A is related to FV that is final value. There we have seen that A is related to PV that is present value. Here we see how the A is related to the final value and based on this equations what type of different questions can be created.

So, this is the equation which is taken into account A is equal to  $F_i$  divided by  $1 + i$  to the power  $N$  minus 1, the above equation which contains four variable can be solved to find out the value of a single unknown variable, only when other three variables are known. Thus 4 types of problem can be generated out of these equations. The problem matrixes for these equations are shown bellow. So, if you see here if  $F, i, N$  are given then A has to be find out and the equation is A is equal to  $F_i$  divided by  $1 + i$  to the power  $N$  minus 1.

So, here we have to find out the value of A that is annuity, when future value F interest rate i and the time in years N are known this type of problem will call it problem e, then given A i N we have to calculate F. So F is equal to A in brackets  $1 + i$  to the power  $N$  minus 1 divided by i. So, this formula will be used and here find out the future value F when annuity A interest rate i and time in years N are known we call such type of problem type F.

The third type of problem is that A is given N is given F is given, we have to calculate i and this is written as  $F_i$  is equal to  $A (1 + i)^{N-1}$ , here we see that in both the side i exists enhance such type of problem will be solved by trail and error by equating left hand side to right hand side. So, here find the interest rate i when annuity A future value F and time in years N are known such type of problem will be called problem G.

The last type of problem were A is given F is given and i is given, we have to calculate the value of N, where N is given by this equation. When N is equal to  $\frac{\ln F_i}{\ln A + 1}$  divided by  $i$   $\frac{1}{1 + i}$ . Here find the time in years N, when annuity A future value F and the interest rate i are known and this type of problem will be called problem H. So, we see start with the problem e, how much 1 should invest at the end of each year for 10 years to get rupees 20000 at the end of 10 years with the annual interest rate of 10 percent.

(Refer Slide Time: 35:30)

Problem Type- E

**Example-1:** How much one should invest at the end of each year for 10 years to get Rs.20,000 at the end of 10 years with the annual interest rate of 10% ?

**Solution:**

$$A = F * (i / ((1+i)^N - 1))$$

Where; A= Amount invested at the end of each year  
F = Total amount due at the end of N years  
N = Number of interest periods  
i = annual interest rate in %  
 $i = i/100$

For the present problem:

A = ?  
F = Rs.20,000  
N = 10  
 $i = 10/100 = 0.1$

$$A = F * (i / ((1+i)^N - 1)) = 20000 * (0.1 / ((1+0.1)^{10} - 1)) = 20000 * (0.1 / [2.59374 - 1]) = \text{Rs.}1254.91$$

22

The solution is we have to pick up A equation which correlates A and F. So, this is the equation where A is the amount invested at the end of each year F is A total amount due at the end of N years, N is the number of interest periods, i is the annual interest rate in percentage, i is equal to i by 100 and in the for the present problem, A is unknown, F is known, N is known and i is known. So they we put up in this equation. So, the A comes out to be 1254.91.

We take a next problem, problem e, next type. If a 40 year old student wants to start saving for retirement and saves rupees 5 a day in her PG bank at the end of the year she invests the accumulate savings which is 1825 in a bank and expects an annual return of 9 percent.

(Refer Slide Time: 36:33)

Problem Type- E-mixed

**Example-2:** If a 40-year-old student wants to start saving for retirement and saves Rs 5 a day in her piggy bank. At the end of the year, she invests the accumulated savings (Rs 1825) in a bank and expects an annual return of 9%. She will earn Rs.93367.21 at the 60 years. However, if some other student starts his saving at 20 years under the same conditions gets Rs. 616635.46 at the 60 years of age, how much must the 40-year old deposit annually to catch the 20-year old ?

**Solution:**  
Given:  
 $F = \text{Rs.} 616635.46$ ;  $N=20$ ;  $i=9\%$ ;  $A = ?$   
 $A = S * (i / [(1+i)^N - 1]) = 616635.46 (0.09 / [(1+0.09)^{20} - 1]) = \text{Rs.} 12053.04$   
The 40 year old has to deposit Rs.12053.04 at the end of the year for 20 years to catch the 20-year old.

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She will earn about 93367.21 at the 60 years; however, if some other students start saving at 20 years, under the same condition gets rupees this is 616635.46 at the 60 year of age.

How much must the 40 year old deposit annually to catch the twenty year old, this is the question and what are given and what are absent. We see F is given 616635.46, N is given 20 years, i is given 9 percent and A is missing that we have to calculate the value of A. So, we use the equation A is equal to S into in the brackets i divided by 1 plus i to the power N minus 1 and find out the value of A which is 12053.04. So the 40 year old has to deposit 12053.04 at the end of the year for 20 years to catch the 20 year old student. Now we take the problem; problem type F, A 20 year old student want to start saving for retirement and saves rupees 5 a day in her PG bank. At the end of the year she invest the accumulated saving of 1825 in a bank, expects an annual return of 9 percent. How much money she will have when she will be 60 years old?

(Refer Slide Time: 38:24)

Problem Type- F

**Example-4:** A 20-year-old student wants to start saving for retirement and saves Rs 5 a day in her piggy bank. At the end of the year, she invests the accumulated savings (Rs 1825) in a bank and expects an annual return of 9%. How much money she will have when she will be 60 years old ?

**Solution:**

**Given:**  
Time period for investment = 40 years  
 $A = \text{Rs.}1825$   
 $i = 9\%$   
 $A = F \cdot (i / [(1+i)^N - 1])$  or  $1825 = F \cdot (0.09 / [(1+0.09)^{40} - 1])$   
or  $F = \text{Rs. } 616635.46$

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What is given here A is given, which is 1825, i is given 9 percent. So, we have to find out F. So, we take pick up equation which relates A to F as given bellow and we put the values and then we find out the F is equal to 616635.46. We will see that i am using one type of problems and by changing A little bit I am finding out the different parameters this is to give you the clarity.

(Refer Slide Time: 38:57)

Problem Type- G

**Example-1:** One in depositing Rs.1255 at the end of each year for 10 years to get Rs.20,000 at the end of 10 years . What is the annual interest rate?

**Solution:**

Given:  $F = \text{Rs.}20,000$ ;  $A=1255$ ;  $N= 10$  ;  $i=?$   
For this problem(from Table-2)  $F i = A [(1 + i)^N - 1]$   
From the above equation the value of i can be estimated by trial and error as shown below:

LHS=  $F i$   
RHS=  $A [(1+i)^N - 1]$   
At  $i = 10\%$  LHS is almost equal to RHS  
Hence  $i = 10\%$

i	LHS	RHS
0.06	1200	992.5139
0.07	1400	1213.775
0.08	1600	1454.451
0.09	1800	1716.041
0.1	2000	2000.147
0.12	2400	2642.84

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Now, problem G. 1 is depositing 1235 at the end of the each year for 10 years to get rupees 20000 at the end of 10 years. What is the annual interest rate? So I already told

you that now the interest rate there will be required trial and error method. So, here F is given 20000, A is 1255, N is 10 and i value is missing, that miss we have to find out what will be the value of I, because this problem F i is equal to A in brackets 1 plus i to the power N minus 1. So, LHS is F I, RHS is A into in brackets 1 plus i to the power N minus 1.

So, we see that if we take i equal to 10 percent LHS is almost equal to RHS and if you see this i is equal to 0.1 LHS is two thousand and RHS is 2000.147 there almost equal enhance answer is i is equal to 10 percent.

(Refer Slide Time: 39:58)

Problem Type- H

**Example-1:** One is depositing Rs.1255 at the end of each year for N years to get Rs.20,000 at the end of N years. If the interest rate  $i = 10\%$  then find the value of N.

**Solution:**

Given:  $F = \text{Rs.}20,000$ ;  $A = 1255$ ;  $N = ?$ ;  $i = 10\%$

For this problem (from Table-2)  $N = \frac{\ln\left[\frac{F}{A} + 1\right]}{\ln(1+i)}$

$N = \ln(20000 \cdot 0.1 / 1255 + 1) / \ln(1 + 0.1) = \ln(2.593625) / \ln(1.1)$   
 $= 0.9530567 / 0.09531 = 9.9995 = 10 \text{ yr}$

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20

Now, we take the problem H. One is depositing rupees 1255 at the end of each year for N years, to get 20000 at the end of N years, if the interest rate is 10 percent then find out the value of N. Here F is given 2000, A is given 1255, i is 10 percent and N is unknown. So, from the table two we can pick up equation which defines N with other parameters this is the equation and we put the values and we get N is equal to 10 years. Now take some other problems of continuous compounding.

What annual end of the year payment for 15 years is necessary to repay the present loan of rupees 20000 at annual interest rate of 10 percent, when continuous compounding is used. So what is given to us in the present problem is A.P is equal to rupees 20000, N is 15, i is 0.1.



(Refer Slide Time: 41:04)

**Example-3:** What annual end of the year payment for 15 years is necessary to repay the present loan of Rs.20,000 at annual interest rate of 10% when continuous compounding is used ?

**Solution:**

Where;  
A= Amount invested at the end of each year  
P = The principal or present value of investment  
N = Number of interest periods  
i = annual interest rate in %  
 $i = i/100$

For the present problem:  
A = ?  
P = Rs.20,000  
N = 15  
 $i = 10/100 = 0.1$

$$A = P(A/P, 10\%, 15) = P \cdot \frac{e^{rN}(e^r - 1)}{e^{rN} - 1} = 20000 \cdot 4.48169 \cdot (1.10517 - 1) / (4.48169 - 1)$$
$$= 9426.87 / 3.48169 = \text{Rs.}2707.56$$

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So, we use the equation here A is equal to P into e to the power rN in brackets e to the power r minus 1 divided by e to the power rN minus 1. So, this is use for continuous compounding we have already derive this. So, if we put the values in this equation the value of A comes out to be 2707.56.

This is again another question on continuous compounding how much 1 should invest at the end of each year for 10 years to get rupees 20000, at the end of 10 years with the annual interest rate of 10 percent when continuous compounding is used.

(Refer Slide Time: 41:52)

**Example-1:** How much one should invest at the end of each year for 10 years to get Rs.20,000 at the end of 10 years with the annual interest rate of 10% when continuous compounding is used?

**Solution:**

$$A = F(A/F, r, N) = F(A/F, 10\%, 1) = F \cdot \frac{(e^r - 1)}{e^{rN} - 1}$$
$$A = 20000(1.105171 - 1) / (2.718282 - 1)$$
$$A = 20000 \cdot 0.105171 \cdot 0.5819767 = \text{Rs.}1224.14$$

Where; A= Amount invested at the end of each year  
F = Total amount due at the end of N years  
N = Number of interest periods  
i = annual interest rate in %  
 $i = i/100$

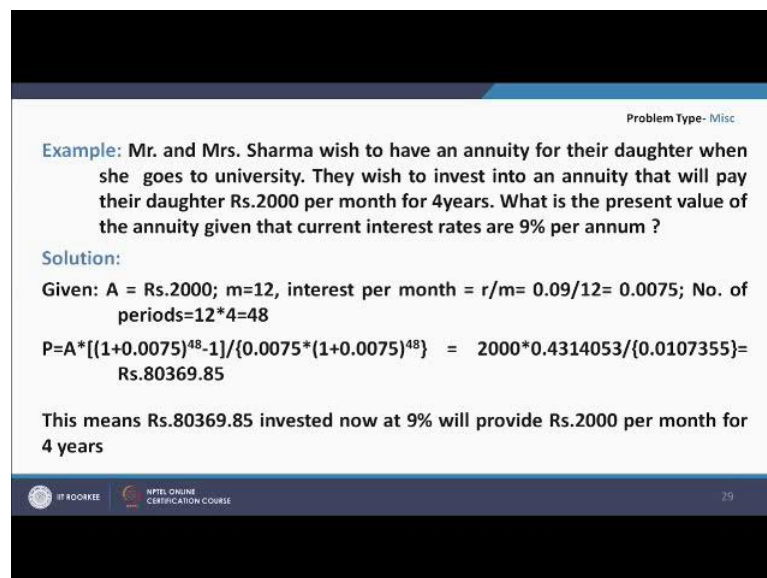
For the present problem:  
A = ?  
F = Rs.20,000  
N = 10  
 $i = 10/100 = 0.1$

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So, here what we want is A and this factor A by AF, r, N is known to us. So, we use this. So, F into e to the power r minus 1 divided by e to the power rN minus 1, we use this equation and put the values. So A comes out to be 1224.14. Now this is again we take a mix type of problem MR. and Mrs. Sharma wish to have an annuity for their daughter, when she goes to university.

They wish to invest into an annuity that will pay their daughter rupees 20000 per month for 4 years, what is the present value of the annuity given, that current interest rate is 9 percent per annum.

(Refer Slide Time: 42:32)



**Problem Type- Misc.**

**Example:** Mr. and Mrs. Sharma wish to have an annuity for their daughter when she goes to university. They wish to invest into an annuity that will pay their daughter Rs.2000 per month for 4 years. What is the present value of the annuity given that current interest rates are 9% per annum ?

**Solution:**

Given: A = Rs.2000; m=12, interest per month =  $r/m = 0.09/12 = 0.0075$ ; No. of periods =  $12 \times 4 = 48$

$$P = A \left[ \frac{1 - (1 + 0.0075)^{-48}}{0.0075} \right] = 2000 \times 0.4314053 / 0.0107355 = \text{Rs.}80369.85$$

This means Rs.80369.85 invested now at 9% will provide Rs.2000 per month for 4 years

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So, we have rupees 2000 m is 12 interest rate per month is r by m is 0.09 by 12 that is 0.0075 number of periods is 2 into 4 because she studying a course of 4 years. So we have to pay per month up to 4 years. So, the number of periods is 48. So, we use this equation which relates P to A and put the values and P comes out to be 80369.85. This means if rupees 80369.85 is invested now at 9 percent, will provide rupees 2000 per month for the next 4 years.

(Refer Slide Time: 43:57)

Problem Type- Misc

**Example:** Mr. Mishra borrows Rs.5,00,000 to buy a car. He wishes to make monthly payments for 5 years. The interest rate he is charged is 10.5% p.a. What is the size of each monthly payment?

**Solution:**

Given:  $A = ?$  ;  $P = \text{Rs.}5,00,000$  ; Frequency of payment per year( $m$ )= $12$ , interest per month  $= r/m = 0.105/12 = 0.00875$ ; No. of periods= $12 \times 5 = 60$

$$A = P \{0.00875 * (1 + 0.00875)^{60} / [(1 + 0.00875)^{60} - 1]\} = P * 0.01475778 / 0.0686603 = \text{Rs.}10746.95$$

This means to borrow Rs.5,00,000 at 10.5% from Bank one has to pay Rs.10746.95 per month for 5 years.

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Another mix type of problem we take, Mr. Mishra borrows rupees 500000 to buy A car. He wishes to make monthly payments for 5 years, the interest rate he is charged is 10.5 percent per annum. What is the size of each monthly payment?

So obviously, we need the value of A that is annuity, P is 500000 frequency of payment per year, m is 12 that is per month we have to pay interest per month is r divided by m is comes out to be 0.105 divided by 12, it comes out to be 0.00875 and number of periods is 12 in to 5 that is 60 because he has to pay for 5 years.

So, we take an equation which relates A to P and put the values and find out that the value of A is 10746.95, this means to borrow rupees 500000 at 10.5 percent from bank. One has to pay rupees 10746.95 per month for 5 years.

Thank you gentlemen.