Management of Fixed Income Securities
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Lecture – 44
Estimation of Binomial Trees – IV

Welcome back. So, in the previous class we started the discussion on the calibration model and there we have seen that how the variability condition and the price condition works and in that particular context the value of the bond what we are estimating using the binomial tree and the equilibrium price on the basis of the current spot yield curve that will be equal to same.

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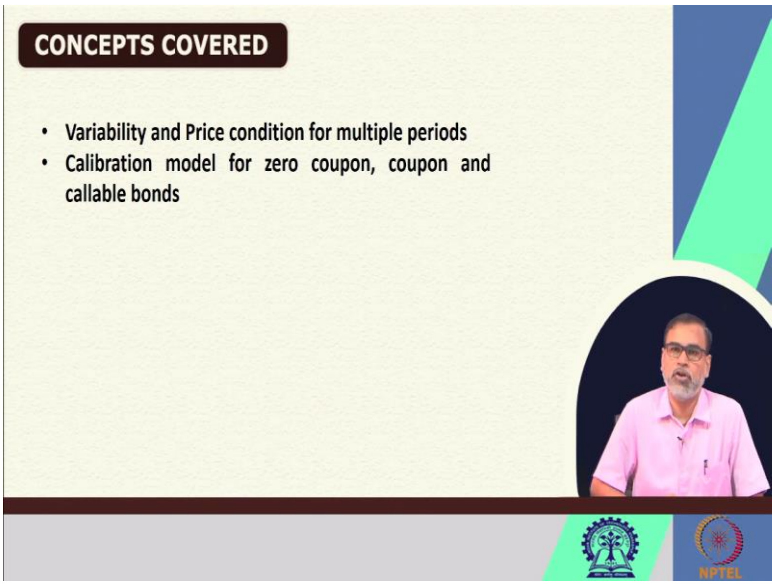

So, in today's class or in this particular session we will be discussing certain things that how the variability and price condition work for the multiple periods because the one period example we have taken in the previous case and how the calibration model can work for the zero coupon, coupon and the callable bonds. So, these are the things basically what we will be discussing in today's session.

Mostly you will see that how this thing will work for the bonds having the coupon features and also have the call options.

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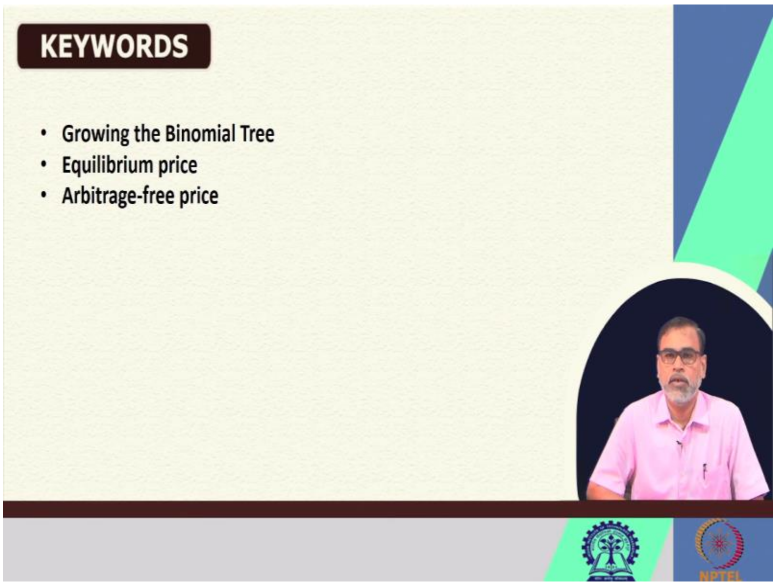

We will come across certain keywords that growing the binomial tree, equilibrium price, arbitrage free price. So, these are some of the keywords that we will come across while discussing about this particular topic in today's session.

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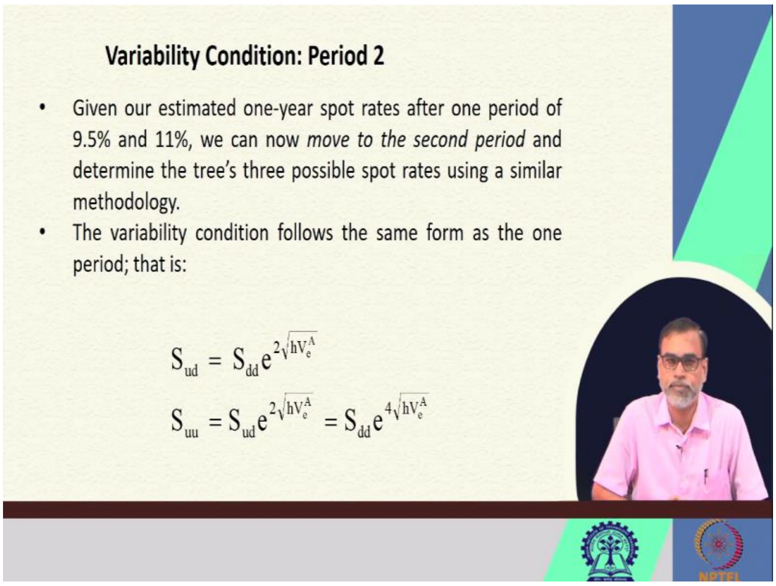

So, let us see that what will be the variability condition if the period is 2; if the period is 2 because one period case we have considered in the previous class and today's class if we see that let there is a 2 period model then how that particular condition will be and how we are trying to find out the value of that. So, given the estimated one year spot rates after one period of 9.5% and 11% we can now move to the second period and try to determine the trees three possible spot rates using the similar methodology.

So, the variability condition what basically we have discussed in the previous class follows the same for as the one period like S ud = S dd e to the power 2 into root of h into V e A V is nothing, but the annualized variance of that particular log returns then S uu is nothing, but S ud into e to the power 2 into root of h V e A and that basically is nothing, but S dd e to the power 4 into root of h V e A.

$$S_{ud} = S_{dd} e^{2\sqrt{hV_0{}^A}}$$

$$S_{uu} = S_{ud} e^{2\sqrt{hV_0{}^A}} = S_{dd} e^{4\sqrt{hV_0{}^A}}$$

That derivation that already we have discussed in the previous class. So, if you are increasing the number of periods accordingly this particular formula are going to be changed.

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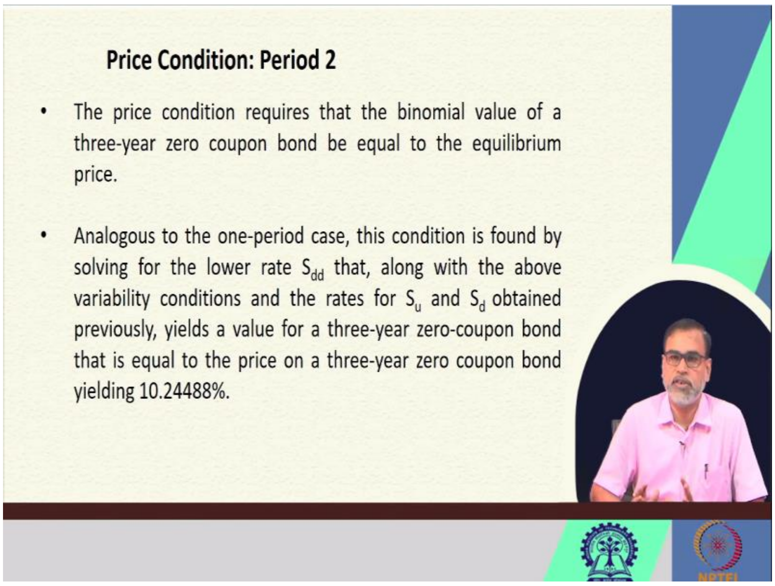

So, now let us see that how this price condition basically works here? Already you know that the price condition generally require that the binomial value of a three year zero coupon bond should be equal to the equilibrium price. In the previous case, we have taken the two year zero coupon bond and here we are talking about a three year zero coupon bond that will be equal to the equilibrium price.

So, analogous to the one period case how this condition can be found, how we can find out this condition? So, to find this condition and how we can find this condition? We can find it by solving the lower rates S dd previous case we are trying to find out S d. So, here we try to find out S dd along with the variability conditions and the rates for S u and S d what basically we have obtained previously.

So, now how basically with the same example if you are trying to see then if you go for a calibration model or using the calibration model if you are trying to try these particular trees then you will find out the value of a three year zero coupon bond will be exactly equal to the price on a three year zero coupon bond in the market which is yielding the 10.24488% that is the assumption we have taken from the beginning y 1 = 10% and y 2 = 10.12238% and y 3 = 10.24488% that is the example we have started and this is the continuation of that.

So, now if you see that then how basically the particular mechanism works, how that process basically works in this particular context.

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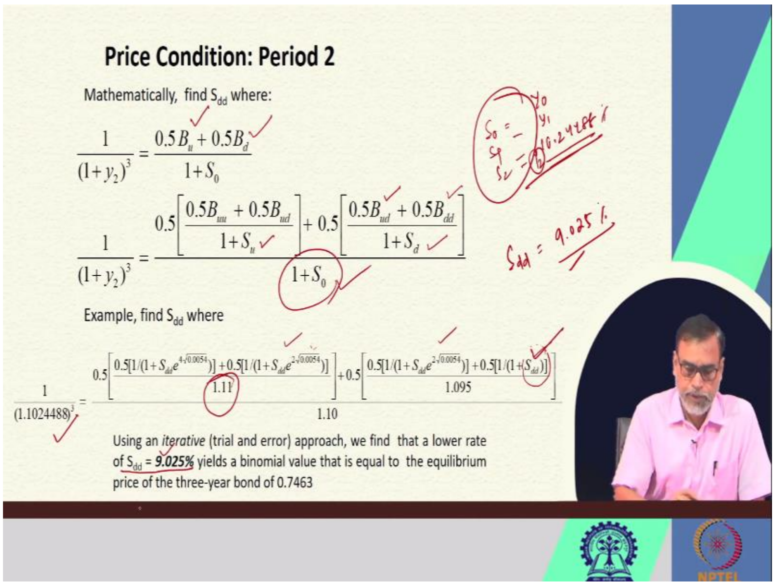

So, now let us see. First of all our objective is to find out the value of S dd that is the lowest node that basically first we have to find out then how basically we can find out the value of S dd. To find out the value of S dd we have to basically see this particular equation should be satisfied and what is this equation that this will be basically your 0.5 B u + 0.5 B d divided by 1 + S 0. That should be basically = 1 + y 2 to the power 3.

So, in this case what basically we have seen the three year spot rate is basically what in the beginning we have taken that is your three year zero coupon bond yield rate is 10.24488%. So, that means it is your 1 by 1.1024488 to the power 3. So, again if you observe here that this is basically you see that your 0.5 B u + 0.5 B d divided by 1 + S 0.

So, in this case what just now we have seen that 0.5 B u + 0.5 B d first of all you have to find out B u then the B d then your whenever you are trying to find out B u it is nothing, but your

0.5 B uu + 0.5 B ud divided by 1 + S u and here it is B d = 0.5 B ud + 0.5 B dd divided by 1 + S d. We are going basically backward and whole divided by you have to discount it with respect to the current rate that is basically S 0.

So, we have S 0, we have S 1, you have S 2. So, these three rates are available in the market. So, now what basically we are doing. So, whenever you are finding S d that these rate is giving you 10.24488%. So, that means your 1 by 1.1024488 to the power 3 = that means we are representing that y 0, this is y 1, this is let your y 2 for example. So, now we are writing 0.5 into 0.5 1 by 1 + S dd e to the power 4 into root of 0.0054 that already we have derived in the previous slide + 0.5 into 1 by 1 + S dd e to the power 2 into the square root of 0.0054.

Divided by your rate is 11% you have to discount it with respect to 11% then 1.11 + your 0.55 into B ud is nothing, but your S dd into e to the power 2 into square root of 0054 that already we have seen here and it is 0.5 1 by 1 + S dd that we have to find out. Now, you got all these values our objective is to find out the S dd which will basically satisfy these conditions.

So, now if you are going to solve this particular equation then we will find that your S dd will become 9.025%. So, there is a trial and error approach, iterative approach you can follow. You find that in this rate of S d this particular equation will be satisfied. Now, you find your S dd = 9.025%. So, if your S dd value you got now you got everything you will find out you S d, you will find out S ud, you will find out your S u, you will find out S uu.

$$\frac{1}{(1+y_2)^3} = \frac{0.5B_u + 0.5B_d}{1+S_0}$$

$$\frac{1}{(1+y_2)^3} = \frac{0.5\left[\frac{0.5B_{uu}+0.5B_{ud}}{1+S_u}\right]+0.5\left[\frac{0.5B_{ud}+0.5B_{dd}}{1+S_d}\right]}{1+S_0}$$

$$\frac{1}{(1.1024488)^3} = \frac{0.5\left[\frac{0.5[1/(1+s_{dd}e^{4\sqrt{0.0054}}] + 0.5[1/(1+s_{dd}e^{2\sqrt{0.0054}}]}{1.11} + \right] + 0.5\left[\frac{0.5[1/(1+s_{dd}e^{2\sqrt{0.0054}}] + 0.5[1/(1+s_{dd})]}{1.095}\right]}{1.10}$$

All kinds of things you can find out because that is calibrated on the basis of the S dd value which is the lowest value.

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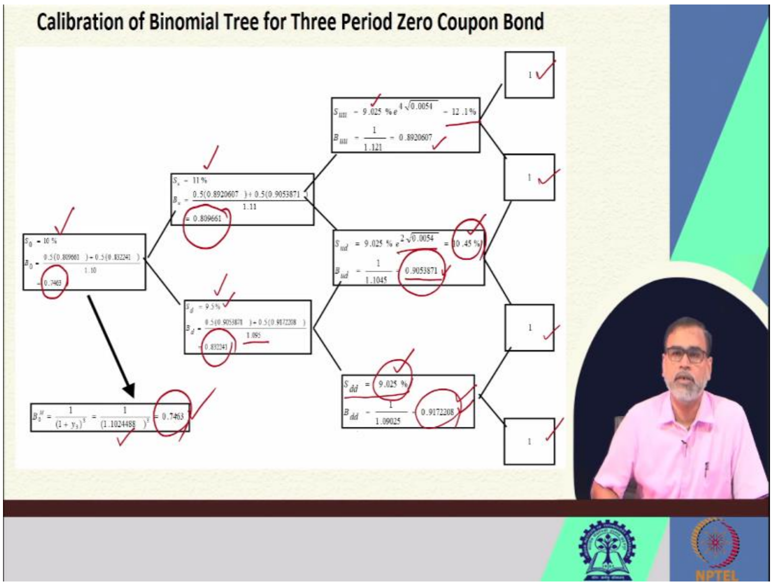

So, now let us see that how that particular model will look like it is for a three period zero coupon bond. We have taken the face value of 1. We are continuing with the previous example your face value is basically 1. So, now you got your S dd from that equation you got your S dd that is 9.025%. So, then you can find out your B dd where B dd will become 1 by 1.09025 that you got 0.9172208 you got your B dd.

If you got your B dd now if S dd you got you will find S ud. Your S ud is nothing, but 9.025% into e to the power 2 into square root of 0.0054. So, you got 10.45% you see it is exactly same whatever we have derived our binomial tree from the beginning. So, now what basically you can do? You can find out your B ud; your B ud has become 0.9053871. So, you got S ud, you got your B ud.

Then like that you can find out your S double u that is again based upon this 9.025 into e to the power 4 into square root of 0054 then you got 12.1% this is exactly also matching then your B uu become 0.1 by 1.121 that basically you get 0.8920607. Now, basically what you can do? Again roll back so use this particular values now you have three nodes from there you can find out the B u and B d.

So, your S u is basically 11% that already you know S d = 9.5% that basically we know. Now your 0.5 into 0.8920607 + 0.5 into 0.9053871 that you have to discount it with respect to 11% then that will be 1.11 then you got this particular value then again 0.5 into this value + 0.5 into this value like 0.9053871 + 0.5 into 9172208 divided by your discount rate is 9.5% that means 1.095. So, you got 0.832241.

So, now again these two values again roll back to the current period; current period spot rate we have assumed 10%. So, then it will be 0.5 into 0.809661 + your 0.5 into 0.832241 divided by 1.10 because that has to be discounted with the current spot rate that is 10% that one year spot rate that is 10% then you got 0.7463, but if you are going to use the current yield in that particular point of time then you get that value 0.7463.

That 10.24488% was the rate then it will be 0.7463. So, now what basically we have seen that it is also 0.7463 it is also 0.7463 that is the way the particular pricing our prices are matching. Now, we will see that how these particular models can be used or how the particular binomial tree can be obtained.

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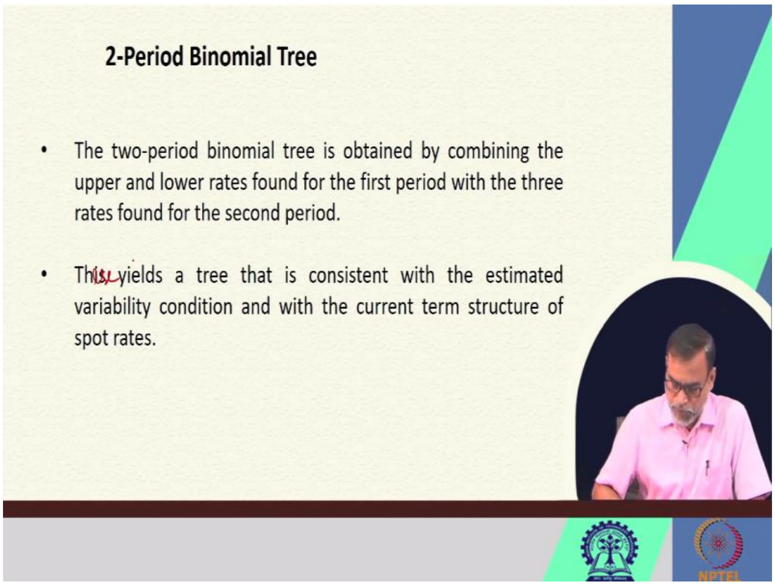

So, the 2 period binomial tree generally is obtained by the combining the upper and lower rates found for the first period with the three rates found for the second period and this yields whatever we are finding these yields this will be this these yields basically a tree which is basically trying to find out is consistent with the estimated variability condition and with the current term structure of the spot rates.

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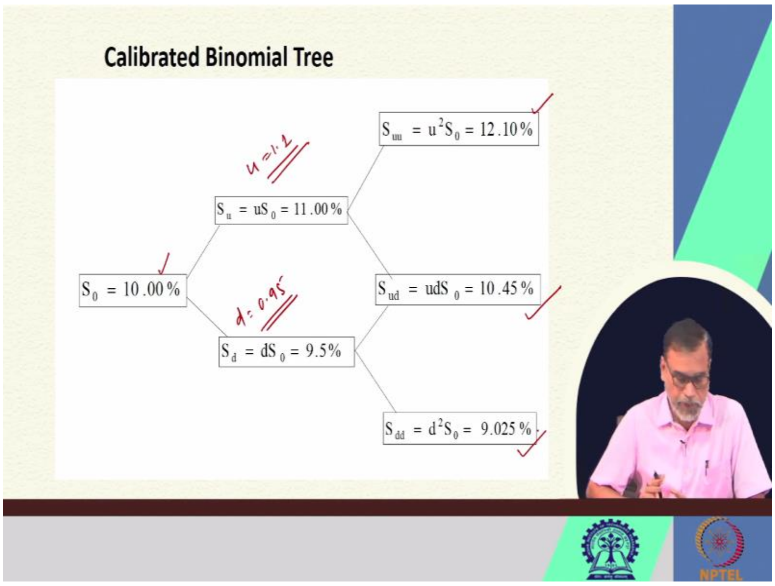

And if you see that we have started from here you recall in the beginning we have started from here that whenever the current spot rate was 10% it can up to 11% your u = we have taken 1.1 and here your d you have started with 0.95. So, S u become 11%, S d become 9.5% then your S uu become u square S 0 that is your 12.1% and your S ud that is u d S 0 that is 10.45% and S dd become 9.025%.

So, whenever this calibration model is imposed satisfying the variability and price conditions then the binomial tree whatever we have estimated that is basically consistent with this rates whatever we have assumed before that is what we have observed in this particular case.

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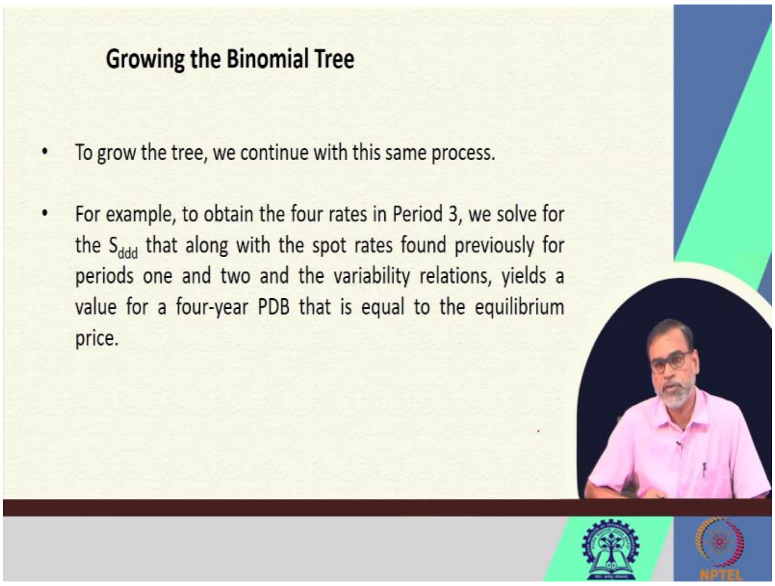

Now, let us see that you can expand this particular three all these trees for the next, next to next periods. So, to grow the tree we continue with the same process, for example, to obtain the four rates in period 3 we solve for the S ddd first and after that the other rates should be calculated on the basis of that formula whatever we can derive from there and along with the spot rates which is found previously for period 1 and 2.

And the variability relations which basically will yield a value for 4 year this pure discount bond or the zero coupon bond that is equal to the equilibrium price. So, you can also expand this particular period N number of periods and accordingly you can find out the lowest one and from the lowest one which will satisfy this particular condition then accordingly you can find out or you can estimate these particular trees.

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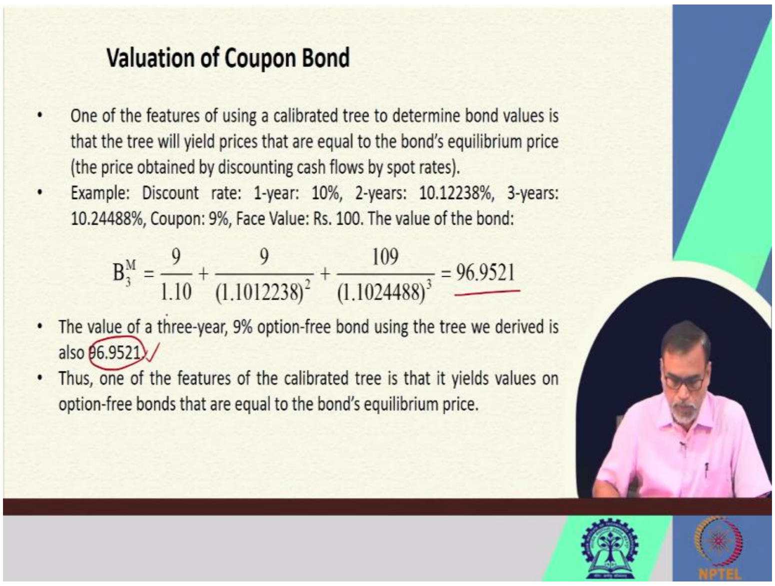

So, now we will see that how we can do these things for a coupon bond. We have discussed about the zero coupon bond now we will come back to the valuation of the coupon bond. So, one of the features of using the calibrated tree to determine the bond value is that the tree will basically yield the prices that equal to the bonds equilibrium price that already we have seen. So, let you have taken this example let the one year discount rate is 10%, two years 10.12238% the same example we are taking three years basically 10.24488%.

Another is the coupon bond and the coupon rate is 9% let the face value of the bond is Rs. 100 then what will be the value of the bond? So, Rs. 9 is the coupon in the first year so 9 by 1.1 again Rs. 9 in the second year then it will be 1.102238 square then in the end of the third year we will be getting your face value of the bond that is Rs. 100 and Rs. 9 is the coupon that means 109 divided by 1.1024488 to the power 3.

Then the value of the bond has become 96.9521. So, the value of a three year 9% option free bond using the tree if you want to derive then that should be equal to 9.9521.

$$B_3^M = \frac{9}{1.10} + \frac{9}{(1.1012238)^2} + \frac{109}{(1.1024488)^3} = 96.9521$$

So, let us see whether it is happening or not.

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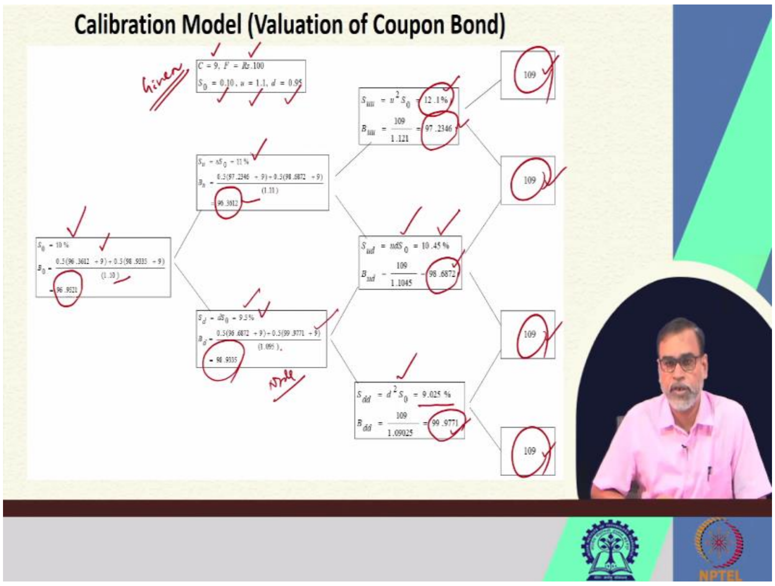

So, let us go for a calibration model for the valuation of the coupon bond. So, the price of the bond in the end the face value will be Rs. 100 and Rs. 9 is basically the coupon. So, here we have assumed your coupon is 9 per year, face value is 100, S 0 is 10%, u = 1.1, d = 0.95. These are the given. So, the cash flow what we will be getting at the end of the three periods that is 109, 109, 109, 109. So, these are the cash flows what you are getting in the end.

So, if you are getting this cash flow in the end then from there we can start it. First of all we have to find out the S dd. The S dd is basically we find 9.025% then your B dd has become 109 divided by 1.09025 that has become 99.9771 then you got your S dd and now you can find out your S ud; S ud is 10.45%. So, your B ud will become 98.6872. So, S uu that you got 12.1%, u square S 0 that is 12.1% then B uu become 97.2346.

Now, you use this particular value roll back to this period and whenever you rollback to this period we will find that 0.5 here the S u is 11%, S d = 9.5% that already we know. So, your B u become 0.5 into this + Rs. 9 is the coupon that what you will be getting then 0.5 into this + Rs. 9 is the coupon end of this period you will be getting Rs. 9 is the coupon then you discount it with respect to this 11%.

So, if you discount it with respect to that 11% you will be getting 96.3612. So, then in this node you have to use the values these two nodes this node and this node to find out the value of this node. So, then here this value is 98.6872, here the value is 99.9771 so then your 0.5 into 96.6872 + 9 + 0.5 into 99.9771 + 9 and you discount it with respect to 9.5% so you got your 98.9335 then you rollback to this previous period.

So, if you roll back to this previous period then what you will find S 0 = your 10% which was given in before we started from here then values whatever you are getting here end of the period one period you will get Rs. 9 with a coupon then price is cash flow here it is 96.3612 + 9 + 0.5 into 98.9335 + 9 and you discount it with respect to 10% then you got 96.9521 and just now whenever you use the current spot rates in this case we are finding that 9 by 1.1 + 9 by 1.1012238 square + 109 divide by 1.1024488 to the power cube you also find 96.9521.

So, using this particular tree also you will find that is 96.9521. So, now the rates or the price of the coupon bond whatever we have derived using this calibration model, using the binomial tree that is 96.9521 that is exactly equal to the equilibrium price in the model. So, that is what basically what we are trying to find out in this particular context.

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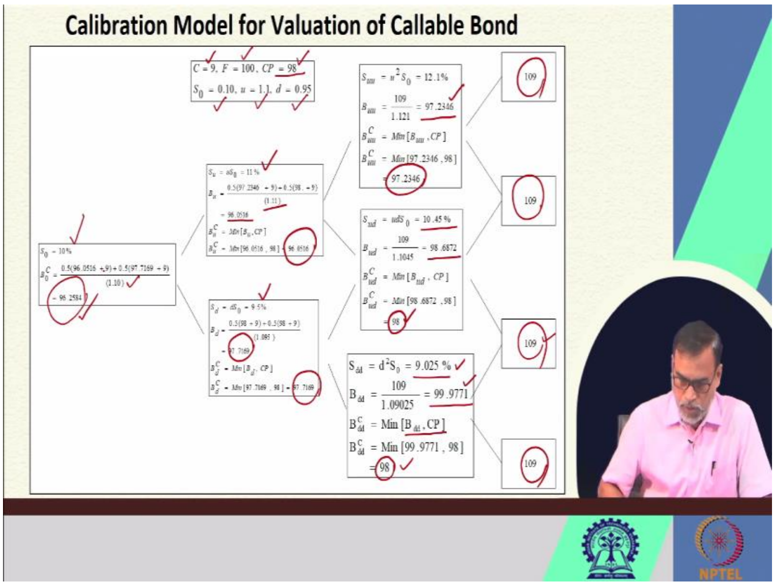

Then let us see what will happen if there is a call feature. If there is a call feature then what is going to happen. So, let the call price is 98 C = 9, F = 100, CP = 98 that means this is the call price S 0 = 10%, u = 1.1, d = 0.95. So, then what will happen these are the final cash flows

109, 109 and 109. Your S dd is 9.025% so if you discount it your B dd with respect to this cash flow this will be 97.9771.

And the call price is 98 and you know that your B dd c will be the minimum of these value or the call price value B dd or CP. So, now which is the minimum? The call price is the minimum so that is why the cash flow will be 98. Here, if you see the value whatever you calculate that is 98.6872 109 divided by 1.1045. Then you got 98.6872 the minimum of the call price and this is 98.

In this case you will find that the value is 97.2346, call price is 98 so that is why minimum is this one that is why the cash flow is this. So, now what you can do use this cash flows for discounting in this particular period S u is 11% that we find so 0.5 into 97.2346 + 9 + 0.5 into 98 + 9 and you will discount it up to 11% that is 1.11 then you got 96.0516. So, between this call price and this value whatever you have derived.

The 96.0516 is the minimum that is why we have considered this. Second case also whenever you find this value and this value you discount it with respect to the 9.5% that means 0.5 into 98 + 9 + 0.5 into 98 + 9 then you get this value of the 97.7169 then the minimum of these two will become 97.7169 then you rollback here discount rate is 10%, use these two values 96.0516 and 97.7169 + 9 + 9 discount it up to 10% you got 96.2584.

Now, the price has declined why it has declined because the cash flow has changed and wherever is the minimum the call price is the minimum or the price whatever you are calculating that node that is the minimum that we have to consider. So, because of this the value of the particular bond in the case in the end **(()) (29:00)**. So, this is the calibration model what we can use for the valuation of a callable bond.

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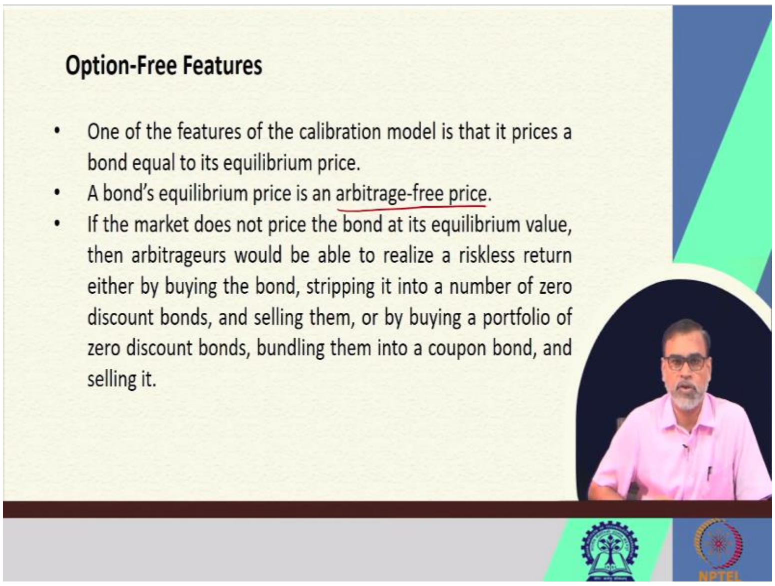

So, there are certain features one of the features of the calibration model is that it prices a bond equals to the equilibrium price and bonds equilibrium price is basically called as the arbitrage free price. So, if the market does not price the bond at its equilibrium value then the arbitrages should be able to always realize there is less return either by buying the bond, stripping into number of zero coupon bonds and selling them or buying a portfolio of the zero coupon or the zero discount bond bundling them into a coupon bond and selling it that can be possible.

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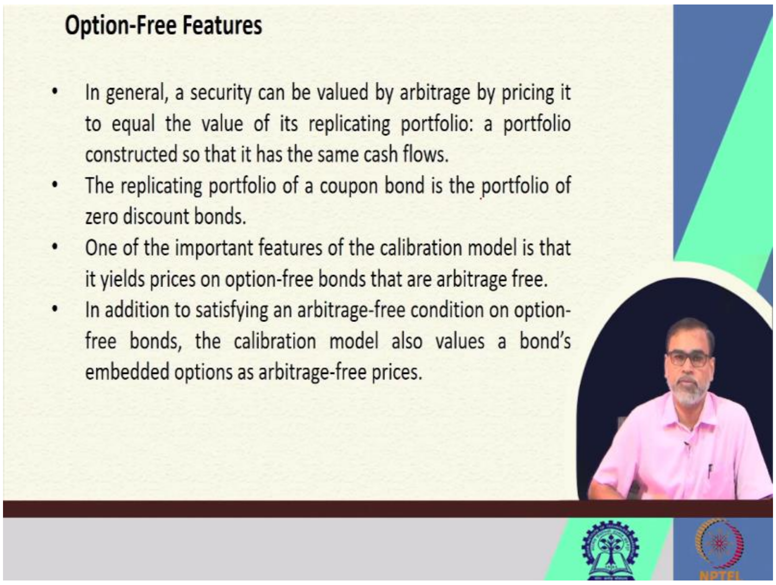

So, in general security can be valued by arbitrage by pricing it to equal to the value of its replicating portfolio that means a portfolio or constructed so that it has the same cash flows. So, the replicating portfolio of the coupon bond is the portfolio of the zero discount bond of zero coupon bond and another feature of the calibration model is that it yields prices on option free bonds that are arbitrage free.

And in addition to that satisfying an arbitrage free condition on option free bonds the calibration model also values the bonds embedded options as arbitrage free prices. So, these are the certain features what basically we can find out from this particular case.

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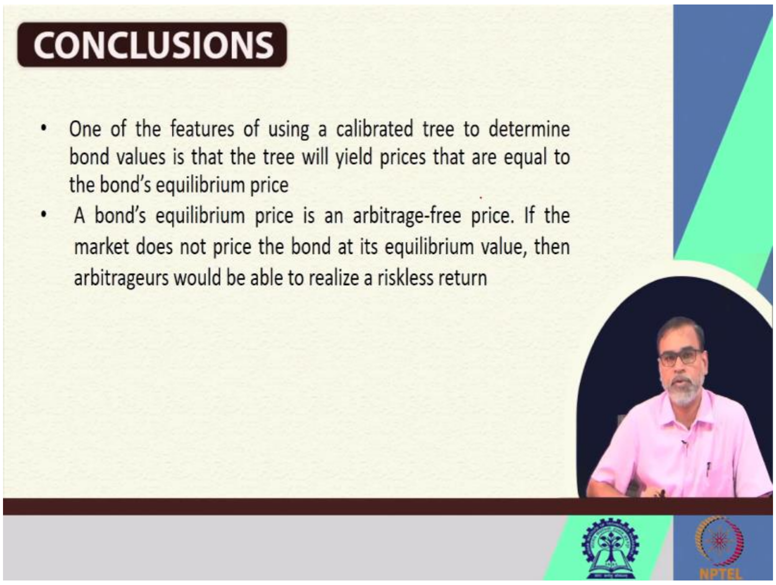

So, what we have discussed in this session that one of the features of using a calibrated tree to determine the bonds value is that tree will yield the prices which is equal to bonds equilibrium price and bonds equilibrium price is an arbitrage free price. So, if the market does not price the bond at its equilibrium value then the arbitrages always would be able to realize a risk less return or the risk free return.

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So, these are the references. Thank you.