Management of Fixed Income Securities
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Lecture – 43
Estimation of Binomial Trees – III

Good morning and welcome back. So, in the previous class, we started the discussion on the estimation of the binomial trees. So, we have discussed about how to find out the values of u & d and there we have two approaches. One approach is your equilibrium model approach where we are trying to solve this value of u & d on the basis of certain assumptions and the other model is basically the calibration model.

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So, today we will be discussing about the calibration model and mostly we will see that what are those different conditions and different methods or different ways the pricing of or may be the estimation of the binomial trees they are carried out through this calibration model.

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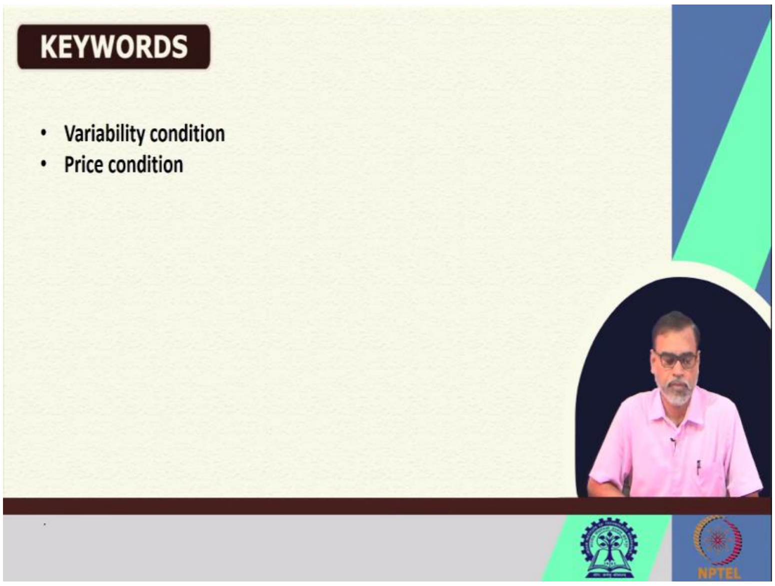

And in this particular context you will come across two different major keywords, what we will be using in this particular session, that is basically your variability conditions and the price conditions. So, these are the two conditions which basically come under the calibration model.

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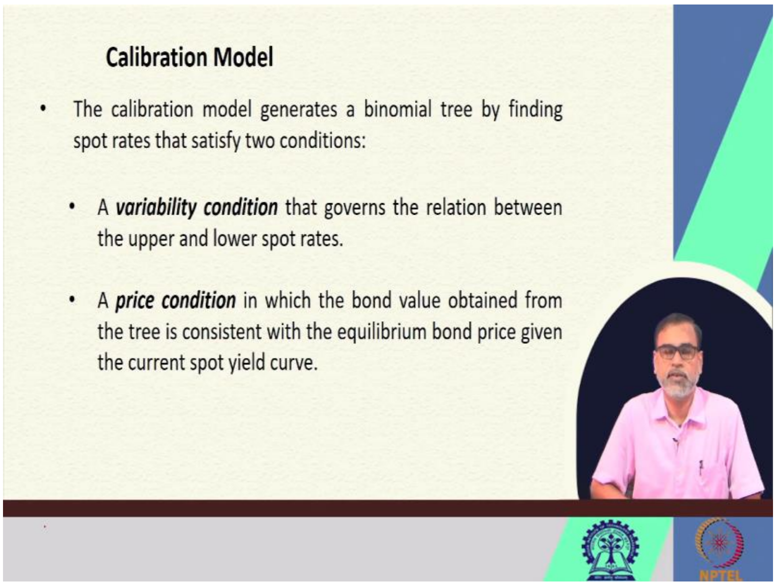

So, let us see that, what exactly the calibration model is all about. So, calibration model basically generates a binomial tree by finding the spot rates of the different periods or the different kind of notes. So, here whenever we are trying to use this calibration model to find out the spot rates two things you are always try to or you are trying to always keep in the mind or this particular model can work only in two conditions.

One is your variability condition, other one is the price conditions. So, what this variability condition is basically talks about that or basically says that or basically tries to explain the relationship between the upper and lower spot rates; the variability condition which basically governs the relation between the upper and lower spot rates and another one is the price condition.

So, here the price condition basically is what? Here, the value of the bond generally is found from the tree which is consistent with the equilibrium bond price given the current spot yield curve. If the current spot rate is known to you then and accordingly, you can find out that whether the price condition is satisfied or not. So, here the bond value generally is obtained from the tree which is consistent with the equilibrium bond price.

So, therefore the calibration model is based on the two conditions and the binomial tree basically it can be generated following these two conditions. One is variability condition, other one is the price condition. So, we will see that what exactly the variability condition is and what is this price condition?

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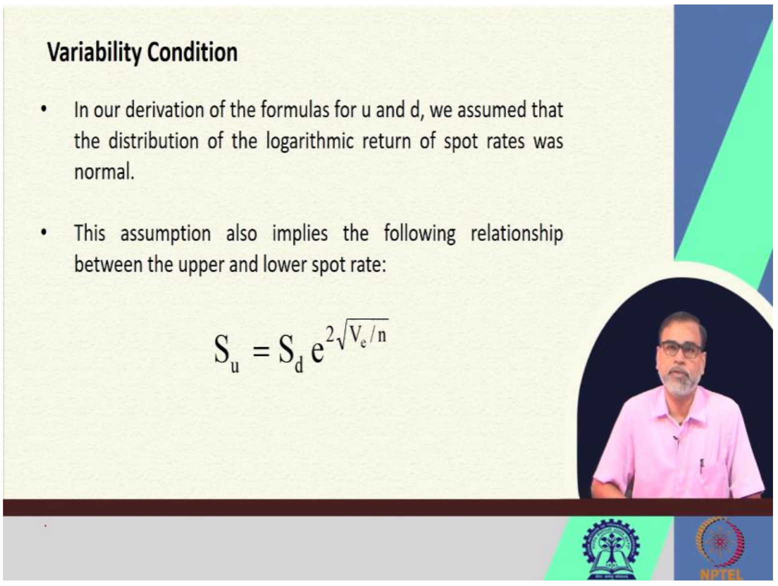

So, whenever you talk about the variability condition if you back in our previous session what basically we have discussed? that we are trying to find out the value of u and d and how we are trying to find out the value of u and d? We are basically trying to first calculate the log value of the returns or the logarithmic return of the spot rates and we are also assuming that this particular logarithmic returns are following the normal distribution.

So, following this normal distribution we have tried to solve the value of the u and d that is what basically what we have discussed in the previous class. So, whenever we are trying to use that particular kind of concept what basically we have used. So, here one assumption basically on the basis of that particular resolution we are basically trying to explain that the relationship between S u and S d.

So, if you are finding the value of S d how you are trying to find out the S u? So, here if you see that the value of S u = S d e to the power 2 into root of the V e by n.

$$S_u = S_d e^{2\sqrt{V_e/n}}$$

V e basically the variance of this particular returns. So, here how basically we get this? Why you are telling that your S u = S d e to the power 2 into root of V e by n, how basically we derive this?

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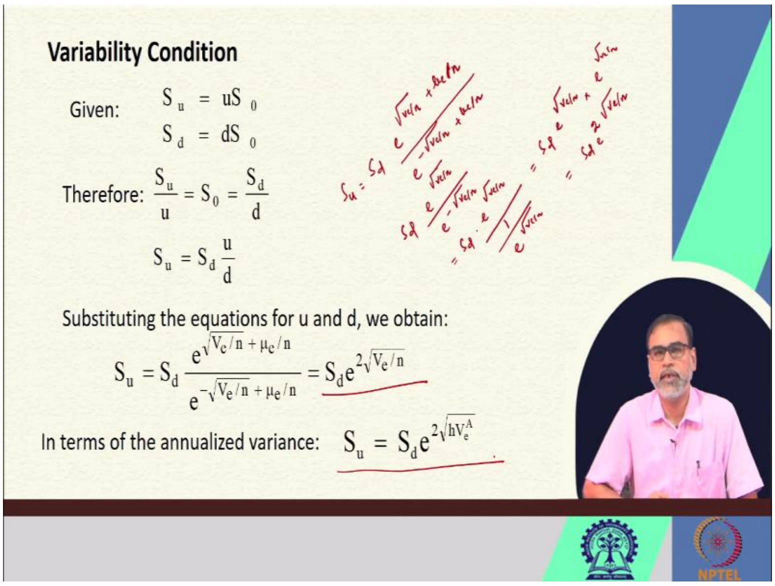

So, here if you assume what basically we have derived or what basically over the binomial tree concept we are trying to explain that your S u is nothing, but u into S 0. Your S u = u into S 0 and your S d = d into S 0. So, by default what basically you can get? S u by u = S 0 = S d by d then with this some kind of algebraic manipulation you will find S u = S d into u by d and already you know what is the value of u? u = e to the power root of V e by n + mu e divided by n that we have discussed in the previous class and your d = e to the power – root of V e by n + mu e by n. So, if you are assuming a large number of observations in the log normal distribution case our particular formula is coming to u = e to the power root of V e by n and d = e to the power – root of V e by n that basically we have discussed in the previous class.

So, from this what basically you have seen that your S d S u = S d e to the power root of your V e by n + mu e divided by n divided by e to the power – root of V e by n + your mu e divided by n. Now, let me e divided by n is basically we may not consider that part then automatically it will become S d e to the power root of V e by n divided by e to the power – root of V e by n.

Then if you say then it says nothing, but S d into e to the power root of V e by n and this is e to the power – root of V e by n means that is basically what we can say divided by 1 by e to the power root of V e by n. So, then automatically it will be S d into e to the power root of V e by n into e to the power root of V e by n. So, finally what basically you are getting S d 2 root V e by n e to the power 2 into root of V e by n.

So, here basically what we are trying to say that this is basically your what we get that this formula is basically derived that S u = S d e to the power 2 root of V e by n. So, now what basically we got? So, if you are trying to convert it in terms of the annualized variance then you will find your S u = S d into e to the power 2 into h V e A; V e basically annualized variance h into V e by n.

Variability Condition:

Given: $S_u = uS_0$

$S_d = dS_0$

Therefore: $\frac{S_u}{u} = S_0 = \frac{S_d}{d}$

$S_u = S_d \frac{u}{d}$

Substituting the equations for u and d, we obtain:

$$S_u = S_d \frac{e^{\sqrt{V_e/n+u_e/n}}}{e^{-\sqrt{V_e/n+u_e/n}}} = S_d e^{2\sqrt{V_e/n}}$$

In terms of the annualized Variance : $S_u = S_d e^{2\sqrt{h V_0{}^A}}$

And h and n all those kind of concepts that already we have discussed. So, that means here what basically we are trying to say? that we are trying to basically see that, how this S u and S d relationship can be established?

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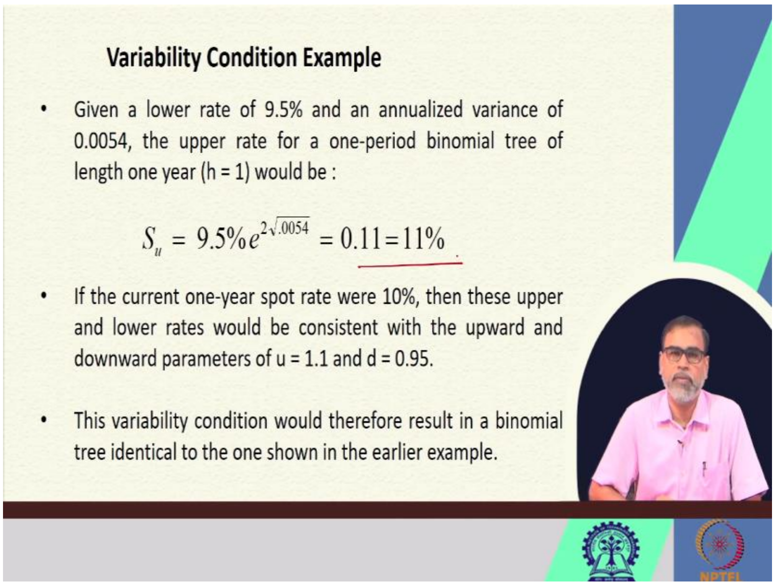

Then we will see that how this particular formula whatever we have derived that is going to help us to find out the actual values. So, let you take a kind of case in this particular context. Let the lower rate is 9.5% that means your S d = 9.5% and your annualized variance is let 0.0054. The annualized variance is 0.0054 and S d = 9.5% then what will happen let the upper rate you are trying to find out for one period and your h = 1.

$S_u = 9.5\% e^{2\sqrt{.0054}} = 0.11 = 11\%$

So, in this case if you are using that particular formula just now whatever we have just now discussed that S u = S d into e to the power 2 into root of 0.0054. So, now if you put those particular values here then we will find that your S u will become 11%.

So, here what basically we can say if the current one year spot rate is 10% then this upper and lower rate should be consistent with the upward and downward parameters of u and d.

The parameter value of u and d we have consider 1.1 and 0.95. if you go back, whenever you started the discussion about the binomial tree. Now, whatever we have seen that if the current one year spot rate is 10%, then using this particular formula we can find out your u = 1.1 and d = 0.95. This variability condition would result in a binomial tree identical to the example whatever we have explained before that is what basically what we are trying to see here.

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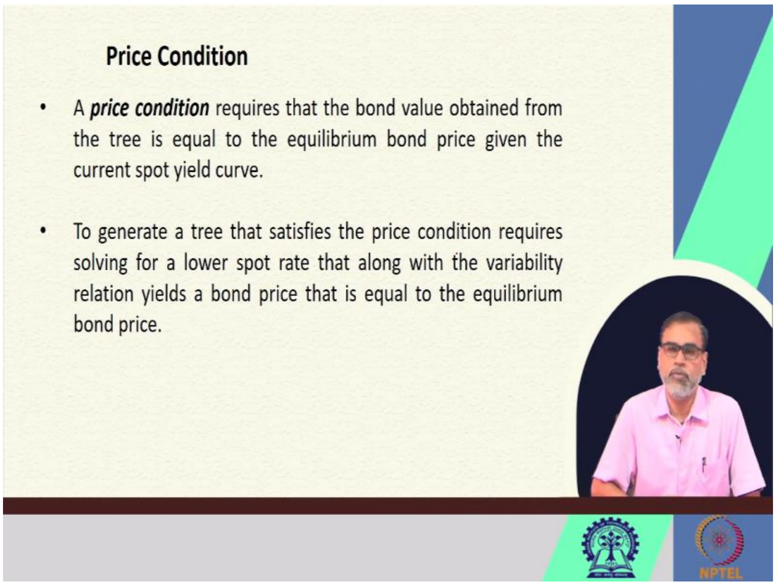

So, then let us see that what is the price condition? So, this is the variability condition and what is the price condition? The price condition says that the bond value what you are trying to obtain from the tree that is equal to equilibrium bond price given the current spot yield curve. So, then if you want to generate a tree which basically satisfies this price condition then what basically you have to do?

You have to basically solve for or you have to find out a lower spot rate which basically along with the variability relations yields a bond price that is equal to equilibrium bond price. So, whenever you are taking that condition that the value of a bond obtained from the tree is equal to the equilibrium bond price given the current spot yield curve, then what basically we have to do here to generate a tree which satisfies the price condition requires solving for the lower spot rate that along with the variability relation yields a bond price that is equal to the equilibrium bond price. that is what basically what you have to do, then how basically exactly we can do that? Let us see these things.

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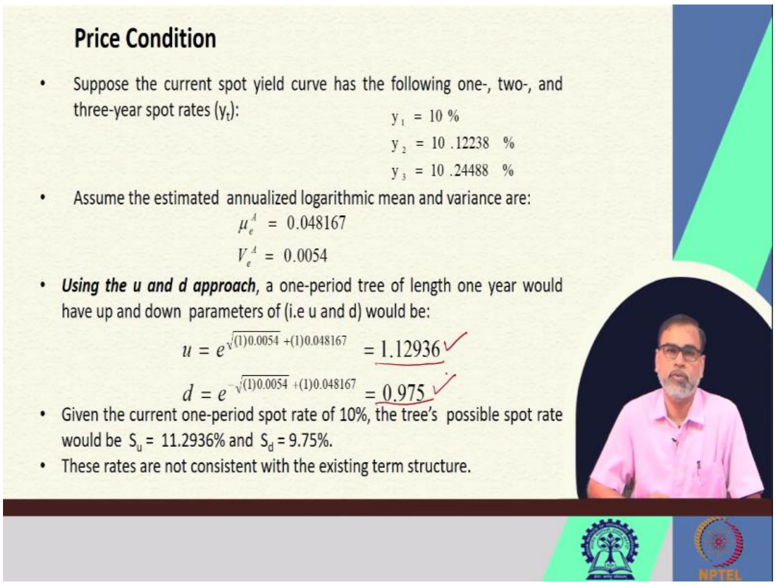

Suppose, this is a kind of example or hypothetical case you can see. Suppose, a current spot yield curve has the following one, two, three year spot rates this is given that your one year spot rate is 10%, second year spot rate is 10.12238% and third year spot rate is 10.24488% and you assume the estimated annualized logarithmic mean and variance the mean is 0.048167 and the variance is 0.0054.

So, whenever you are in the first case you are trying to use this u and d approach, then your one period tree of length one year would have up and down parameters then what will be the value of u and what will be the value of d? Just now, we are trying to explain that you know the formula for the u and d; u = e to the power your root of just now if you say that actual formula it will be your e to the power 2 into root of V e by n plus your mu e divided by n that is what basically what we have seen.

So, here in this case if you see period is 1. So, therefore you can say that your that will be your e to the power your h into the annualized volatility that is 0.0054 and your mu e is basically mu A = 0.048167 then using this formula what basically we find that is basically 1.12936 that u value = 1.12936. In the same way if you are trying to d you are finding 0.975. So, here what basically we observe? Give a one period spot rate that is 10% the binomial tree possible spot rate will be your S u = 11.2936% and your S d = 9.75%.

$$u = e^{\sqrt{(1)\,0.0054}\,+\,(1)\,0.048167} = 1.12936$$

$$d = e^{-\sqrt{(1)\,0.0054}\,+\,(1)\,0.048167} = 0.975$$

But these rates are not consistent with the existing term structure whatever we have assumed. In our example, whatever thing we have considered these rates are basically are not consistent with that because this particular u and d values are basically different from the u and d values whatever we have taken in the previous case.

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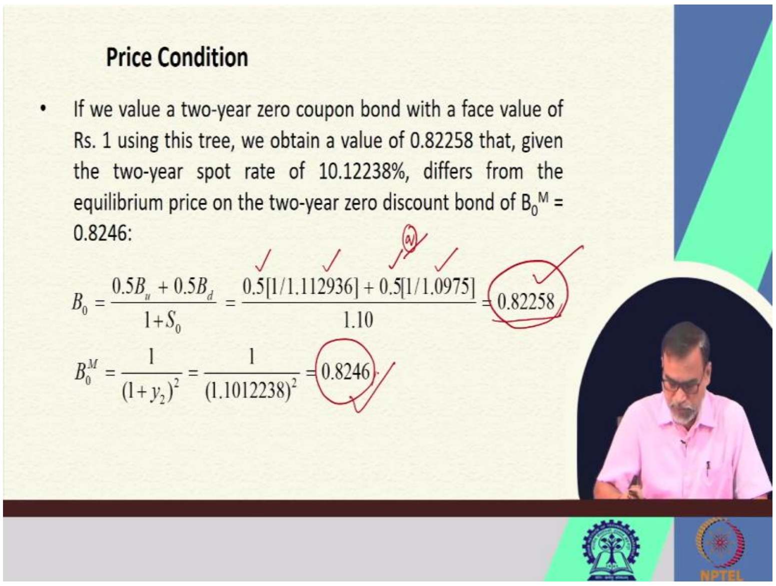

So, then what basically we have to do? Let us take one example in this case. let there is a zero coupon bond and we want to evaluate that zero coupon bond which term to maturity is two years and the face value of that zero coupon bond is Rs. 1 and we are trying to evaluate or trying to find out the value of the bond using the binomial tree, then how we can do this? You go back or you recall that particular thing whatever we have discussed using the valuation of zero coupon bond using the binomial tree.

Your B 0 will be 0.5 B u + 0.5 B d here we have taken the probability is 50%, 50% it can increase, 50% it can decline. So, that basically we have to discount it with respect to the spot rate which is prevailed in that particular period. So, what we have seen here using this u and d whatever we have considered in this case, we find that the yields of the particular kind of bond whatever we are considering that can increase up to 11.29% or it can decline up to 9.75% from the 10%.

And the current spot rate is one year spot rate is 10%. So, then it will be 0.5 into 1 by Rs. 1 is the face value of the bond that is 1 by 1.112936 + 0.5 into 1 by 0.0975 we are trying to find out the present value of the particular bond in these two different conditions then you multiply with that particular probability then we are discounting it with respect to 10% to find out the present value of that particular bond.

Then you find that the present rate so this is the rate whatever we have just now calculated. So, B 0.5, 0.5 is basically the value of q or the probability value. So, then what basically we got? We got 0.82258 that is basically the value of the zero coupon bond. So, now you have the two year spot rate already given to you. So, if you are going to that is basically 10.12238% that in the beginning we have taken that particular hypothesis or that example. Then your value in the market in that particular point of time that should be 1 by 1 + y 2 square and that will give you 1 by 1.1012238 whole square that value will become 8246.

$$B_0 = \frac{0.5B_u + 0.5B_d}{1+S_0} = \frac{0.5[\ 1/1.112936 + 0.5[\frac{1}{1.0975}]}{1.10} = 0.82258$$

$$B_0^M = \frac{1}{(1+y_2)^2} = \frac{1}{(1.1012238)^2} = 0.8246$$

Now, this value whatever you have calculated and this value you have calculated that is not same, that is basically not equal. So, then what basically we can say in that particular context.

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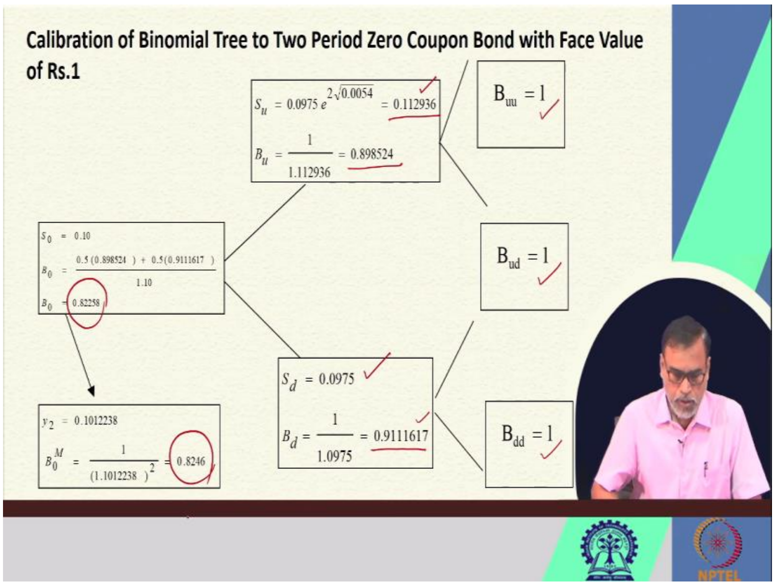

That if you are going for a calibration of the binomial tree to 2 period zero coupon bond with a face value of 1. What basically just now we have seen? the face value is Rs. 1 that means B uu = 1, B ud also 1, B dd also 1. Then. what we have done your S u? basically we have to find out your S u = your 0.0975 e to the power 2 into 0.0054 and period one basically one we are taking here.

So, that is why that does not matter to that particular value then we find this one. Then, if you discount it with respect to this rate then you are finding the value of this. Then, if you are going to do this S d = this then your value of the bond in the lower node will be 0.9111617. Now, again we are using these values here 0.5 into 0.898524 + 0.5 into 9111617 and you are discounting at a rate of 10% which is the one year spot rate then you find 0.82258.

But in the market the value of that zero coupon bond is 0.8246. So, this is what basically is not matching in this particular case.

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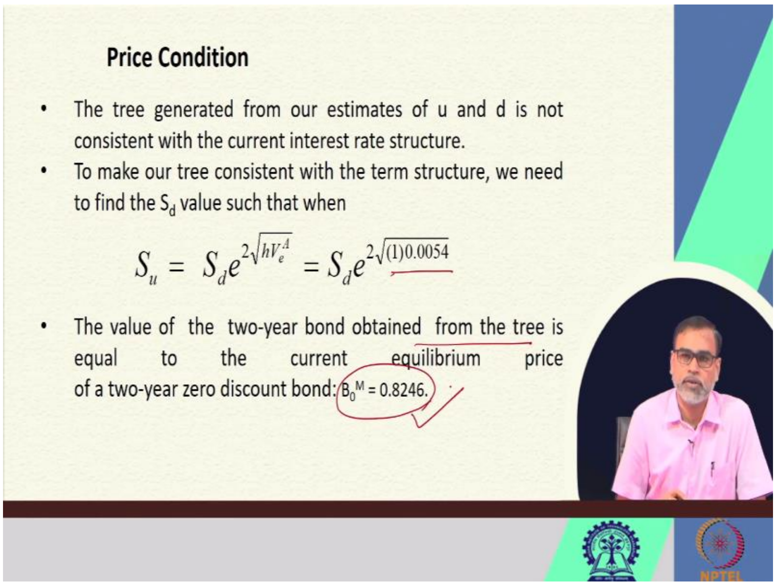

So, then what we can do the tree what basically generated from our estimates of u and d is not consistent with the current interest rate structure. So, to make our tree consistent with the term structure we need to find out the S d value which will satisfy this condition that means S u = S d e to the power 2 into root of h V e A; V e A is nothing, but the annualized volatility annualized variance. So, here that will become S d into e to the power 2 into 1 into 0.0054. So, if you use this particular formula then the value of this to your bond obtained from the tree these will be equal to exactly 0.8246.

$$S_u = S_d e^{\sqrt[2]{hV_0{}^A}} = S_d e^{\sqrt[2]{(1)\ 0.0054}}$$

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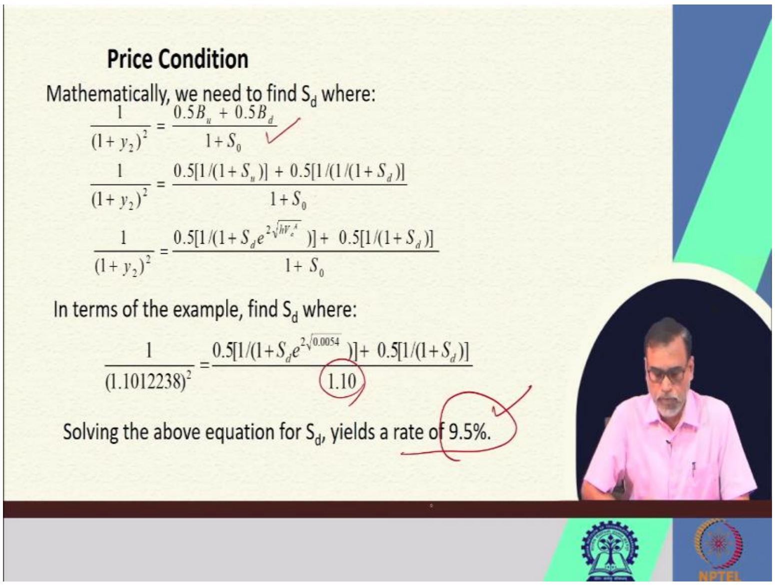

So, mathematically whenever we are trying to find out the value of S d that should satisfy this condition that 1 by 1 + y 2 whole square = 0.5 B u + 0.5 B d divided by 1 + S 0. How to find out S d that is what we are discussing then it is nothing, but 0.5 into 1 by 1 + S u + 0.5 into 1 by 1 + S d. So, then if you see that then what we got you are putting this particular values it is S u is nothing, but S d into e to the power 2 into h V e A.

$$\frac{1}{(1+y_2)^2} = \frac{0.5B_u + 0.5B_d}{1+S_0}$$

$$\frac{1}{(1+y_2)^2} = \frac{0.5\ [\frac{1}{1+S_u}] + 0.5[1/(\frac{1}{1+S_d}]}{1+S_0}$$

$$\frac{1}{(1+y_2)^2} = \frac{0.5[\frac{1}{1+S_d e^{2\sqrt{hV_0{}^A}}}] + 0.5[1/(\frac{1}{1+S_d}]}{1+S_0}$$

Then, if we put this particular values then you will find out in terms of our example what we can find that 1 by 1.1012238 whole square = 0.5 into 1 by 1 + S d e to the power 2 into square root of 0.0054 + 0.5 into 1 by 1 + S d. So, here basically what we got that you discount it with respect to 10% then if you solve this equation then you find out your S d = exactly 9.5% which is basically consistent with our term structure. Is it clear?

$$\frac{1}{(1.1012238)^2} = \frac{0.5[\frac{1}{1+S_d e^{2\sqrt{0.0054}}}] + 0.5[1/(\frac{1}{1+S_d}]}{1.10} = 9.5\%$$

So, that is what basically what we are trying to find out. So, once you find out your S d then it will be easy for you to find out the other values then finally the value of that particular bond can be calculated.

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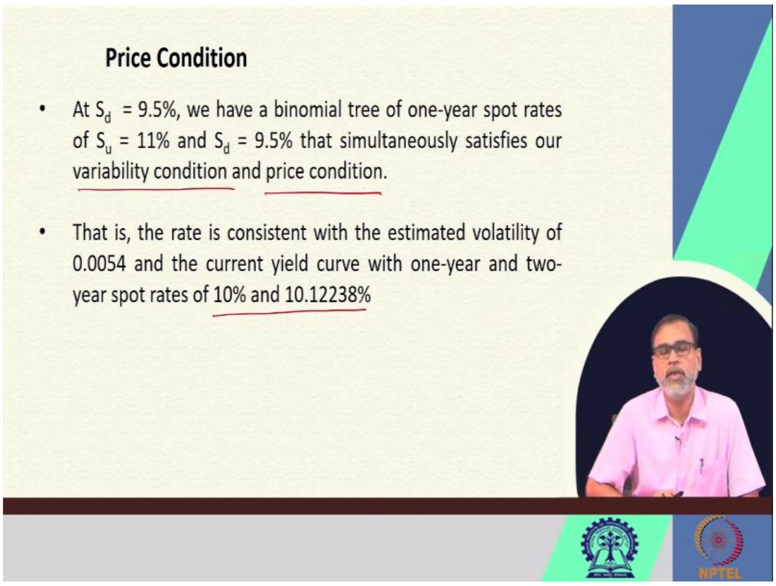

So, then what we find here our S d = 9.5% we have a binomial tree of one year spot rates of S u that should be 11% and S d will be 9.5% which basically satisfy our variability condition and the price condition both the condition will be satisfied in this case. So, the rate is consistent with the estimated volatility of 0.0054 and the current yield curve with one year and two year spot rate of 10% and 10.12238% that is basically what we call it the price condition.

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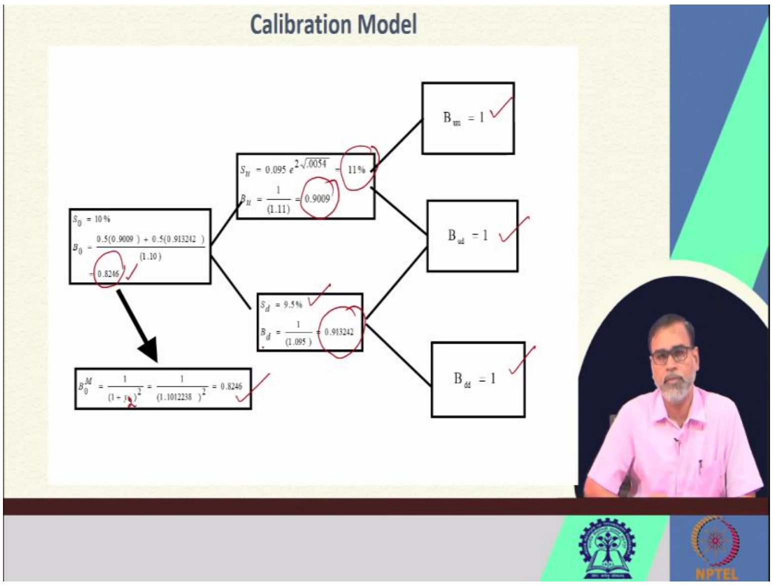

Now, we see that in the calibration model how this particular value is exactly matching. You start with 1 here B uu = 1 and B ud = 1, B dd = 1. So, now you find out your S d that is 9.5% and your S u = 0.095 e to the power 2 into square root of 0.0054 that you got 11%. So, automatically your B u = 0.9009 and your B d = 1 by 1.095 that will be 0.913242. So, now you go back to our formula 0.5 into 0.9009 + 0.5 into 913242 you will find the value 0.8246.

And your market value using the spot rate whatever we are considering that is basically 0.8246 that is what basically in the beginning we have achieved. So, that is the current spot rate in that particular period that is basically y 2 that is 8246. So, it is basically perfectly matching.

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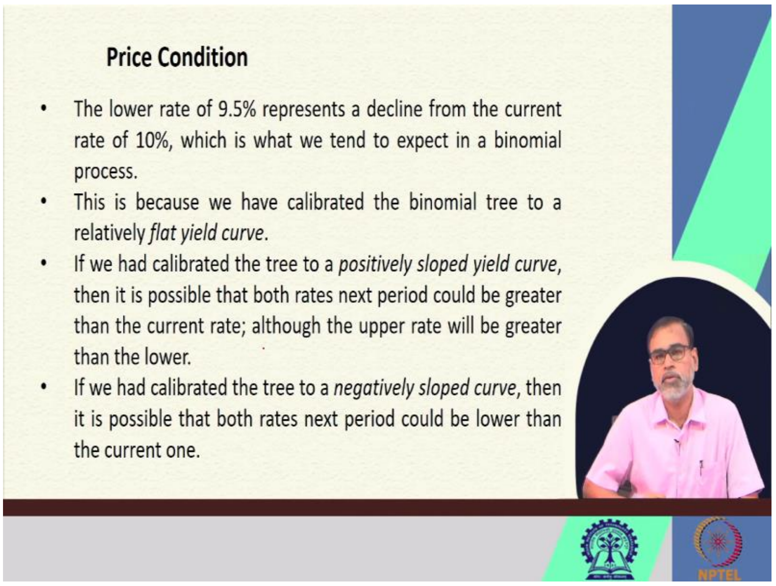

So, then the lower rate of 9.5% represents a decline from the current rate of 10% which basically tend to expect in a binomial process and this thing basically has happened because we have calibrated the binomial tree to a relatively flat yield curve. So, if we had calibrated the tree to a positively sloped yield curve, then it is possible that both rates in the next period could be greater than the current rate the 10% to 11% or it is 9.5% instead of that it can go up to more than 11% or it can go up to also more than 10% in the lower node.

So, both rates in the next period could be greater than the current rate although the upper rate will be upper than the lower tree. So, if you are calibrated this tree for a negatively sloped curve then it is possible that both the rates in the next period could be lower than the current that is also possible in that particular context.

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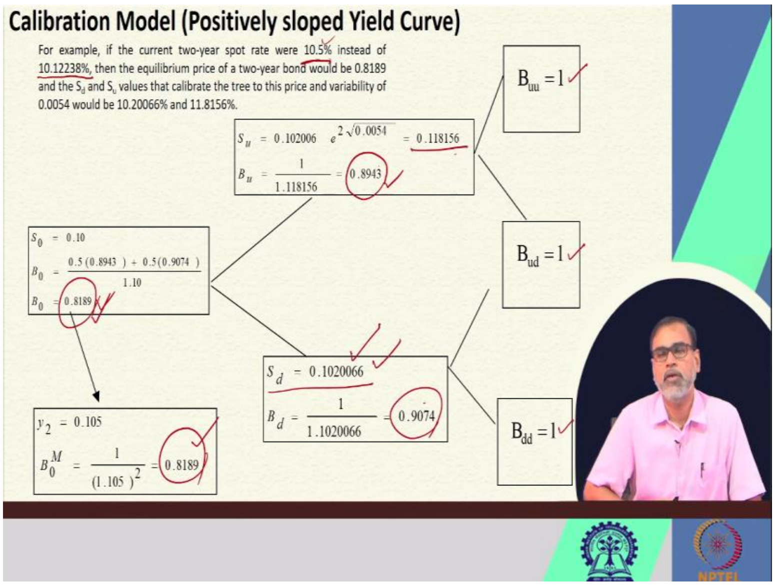

So, if you are let assuming a positively sloped curve. So, in the positively sloped curve in the same example if you see let your B uu = 1, ud = 1 dd = 1. So, in that case what basically we are trying to find out. Let we are finding that S d value we are assuming your 10.20066% so B d = this and S u you are trying to find out using this value then you got 0.118156 that 11.8156% then B u = this.

So, now if you put these values here then you will find that your B 0 = your 8189% because here what basically where we are assuming the two year spot rate is 10.5% instead of 10.12238%. So, then if you discount it with respect to 10.5% you will also find that 8189 is the value of the bond which is matching here and why it is basically matching because this particular rate is based on the rates whatever we have consider here that is 10.5% and how it is derived that already we have discussed.

Mathematically if you want to derive that S d by using the two year spot rate which is available in that particular point of time, then what basically you can find? you are using your binomial tree model the value whatever you are finding the price basically what you are finding that will be perfectly equal to the value what basically in the market you are going to calculate or you are going to find out using this that particular current spot yield available in that particular point of time. So, this 0.8189 will perfectly match.

So, this is basically can happen with respect to a positively sloped yield curve like that you can also assume a particular rate. Let two year spot rate will be lower than the current spot rate then accordingly these particular values can be changed then exactly you may find out the value of the bond using the tree the calibration model or the market yield curve what you are trying to use it both of them will be equal.

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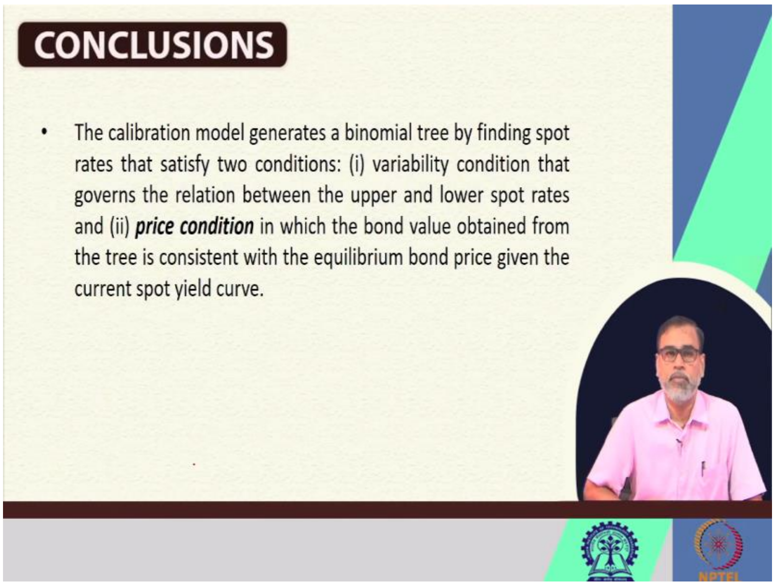

So, what basically we have discussed here. The calibration model basically generates the binomial tree by finding the spot rates which satisfy two conditions. One is variability condition and another one is the price condition. The variability condition basically governs the relationship between the upper and lower spot rate that means the lower spot rate is available how to find out the upper rate.

And the price condition basically tells the bond value what basically we find from the tree that should be consistent with the equilibrium bond price given in the current spot rate that is what just now we have seen both the cases the particular two rates are same.

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So, these are the references you can go through. Thank you.