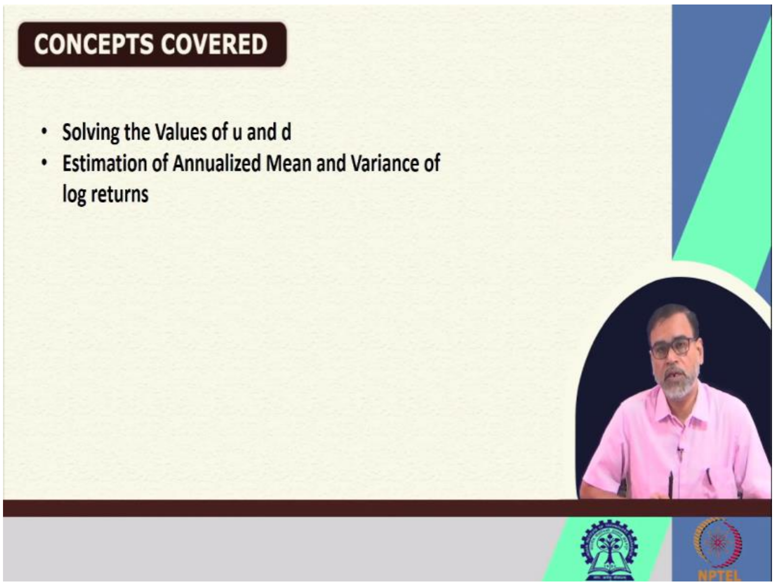
Management of Fixed Income Securities
Prof. Jitendra Mahakud
Department of Humanities and Social Sciences
Indian Institute of Technology – Kharagpur

Lecture – 42
Estimation of Binomial Trees – II

So, in the previous class we discussed about the subdivision of the binomial tree which looks more realistic in nature and also we started the discussion on the different approaches which are used for the valuation of the u and d. We will continue with that particular discussion.

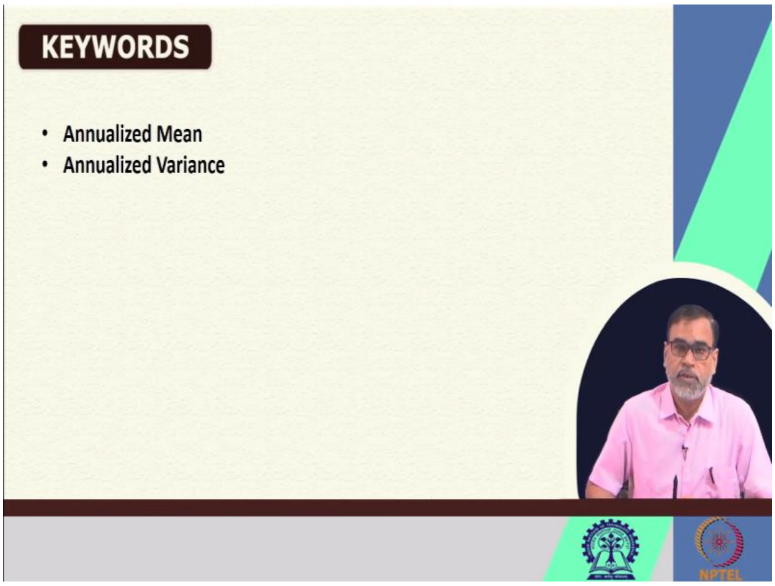
(Refer Slide Time: 00:39)



So, today we will be discussing how the u and d value we can solve, we can get and the another concept is basically the estimation of the annualized mean and variance of the log returns that what we have introduced because we have introduced the concept of the normal distributions of this particular data and we talked about this binomial process and the binomial distribution.

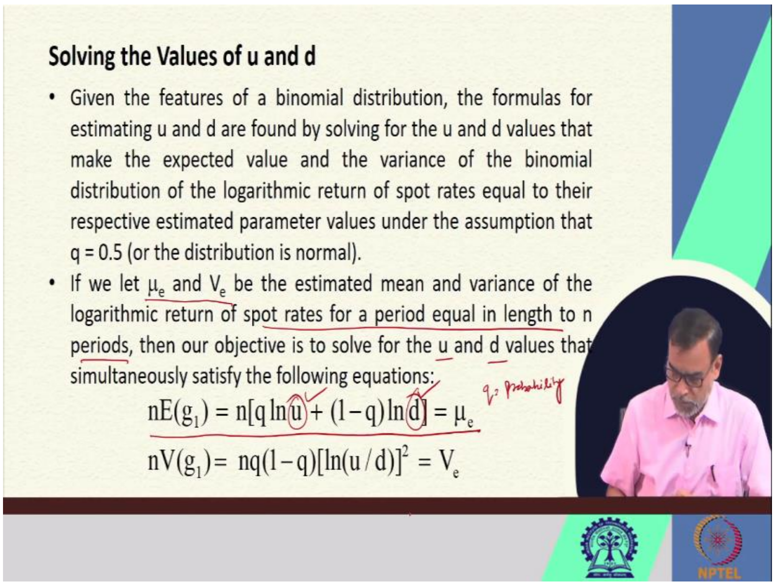
Then, accordingly we will see that on the basis of the equilibrium model how this u and d is calculated and as well as how the estimation of the annualized mean and variance of the returns also calculated particularly the log returns can be calculated.

(Refer Slide Time: 01:26)



So, annualized mean and annualized variance these are the major keywords what we are going to in particular session.

(Refer Slide Time: 01:36)



So, first of all come back to our discussion which is continued in this particular session; that is basically solving the values of u and d and already, we have seen that we have assumed a binomial distribution and the formula for estimating the u and d generally we can find by solving for u and d values, that makes the expected value and variance of the binomial distribution of the logarithmic return of the spot rates equal to their respective estimated parameters under the assumptions that the distribution is normal.

Very minutely you observe this. We can solve these values of u and d. how? We can only formulate the formula or we can find out the formula for the value of the u and d by solving the particular concept or particular thing where the expected or the values of u and d which make the expected value and variance of the binomial distribution of the logarithmic returns of the spot rates equal to their respective estimated parameters values.

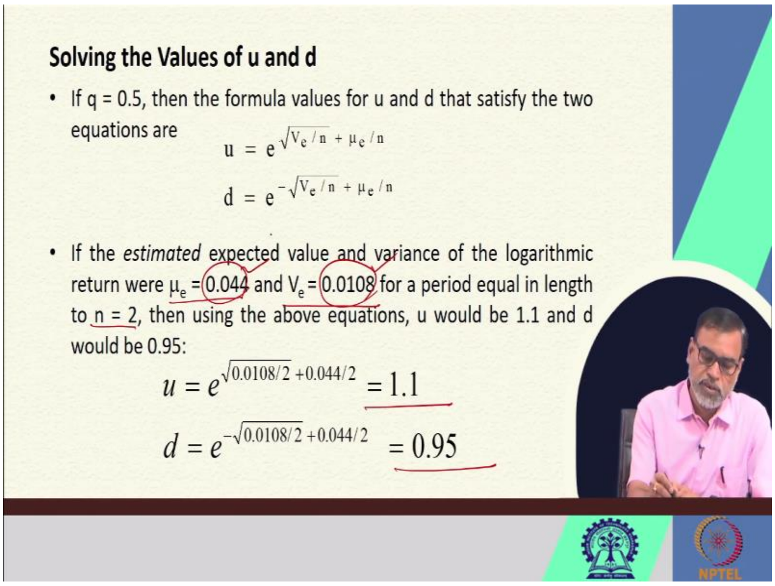
And here our assumption is there is a normal distribution. So, if you are assuming the mu e and the V e be the estimated mean and variance of the logarithmic return of the spot rates for the periodic fall in length to n periods then our objective is to solve for the u and d values which satisfy the following equations. What are those equations. So, here your n E g 1 here your n E g 1 = n q ln u + 1 – q ln d.

$$nE(g_1) = n[\, q \ln u + (1-q) \ln d] = \mu_e$$

$$nV(g_1) = nq(1-q)\, [\ln (u/d)]^2 = V_e$$

So, q is basically the probability and g 1 is the log returns that already you know and 1 – q = the 1 – probability and u already you know the d also you know. So, if you assume that the mu e and V e is the estimated mean and variance of the logarithmic return of the spot rates for a period equal in length to n periods then we can find out the value of u and d which basically satisfy these conditions. So, if you have these values with us then from there the u and d values can be calculated.

(Refer Slide Time: 04:55)



So, let us see that let q = 0.5, then the formula values for u and d that satisfy the two equations will be u = e to the power root of V e by n + mu e divided by n.

$u = e\sqrt{V_e/n} + \mu_e/ n$

$d = e\sqrt{V_e/n} + \mu_e/ n$

What do you mean by the mu e? The mu is basically the mean and V is basically the variance and d = e – the root of V e by n the square root of the V e by n + your mu e by n. V is nothing, but the variance of the returns and your mu is basically the mean of the returns.

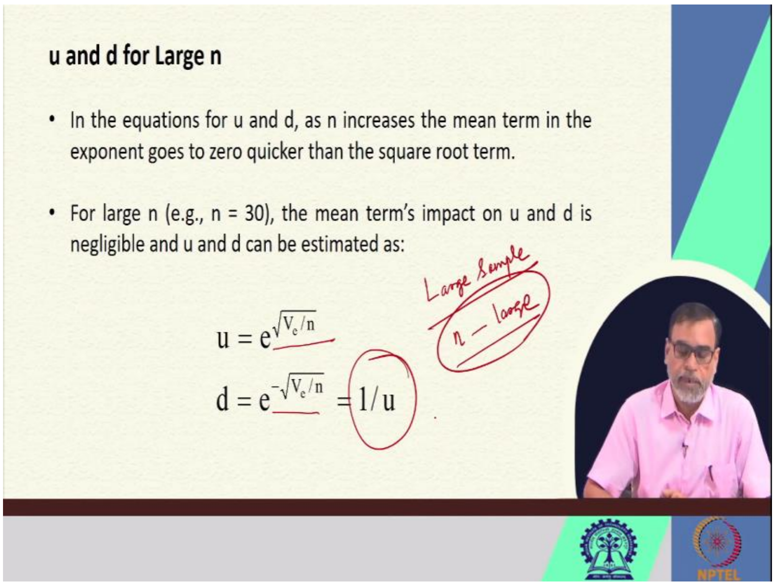
So, if from the previous example we have calculated this mu e and V e. Let, if you find that the expected value and variance of the logarithmic returns are mu = this and V = this for a period equal in length and n = 2 because these values we get whenever the two period analysis we are doing then you will find that u = e to the power your V = 0.0108 then it is square root of 0.0108 divided by 2 + your 0.044 by 2 that will be 1.1. Here it is e to the power – of this that will give you 0.95.

$u = e\sqrt{0.0108/2} + 0.044/2 = 1.1$

$d = e^{-}\sqrt{0.0108/2} + 0.044/2 = 0.95$

So, now the question is that we have derived this particular formula with the assumptions that q = 0.5 and mu = this and V = this.

(Refer Slide Time: 06:58)



If you are assuming that your n the number of observations are quite large, if you are assuming or if you feel that the number of observations are quite large, then as the n increases by statistical properties, when the n increases the mean term in the exponent goes to 0 quicker than the square root term. As the n increasing, the mean value in the exponent goes to 0 and it is reaching 0 quicker than the square root term.

So, in statistics or in a mathematical sense if you look at for large n, for example, n is greater than or equal to 30 generally we consider, but generally it should be quite large in number. The mean term's impact on u and d is negligible. The mean term on u and d is negligible. So, that is why this is your mu values what just now we have taken that your mu e divided by n and your mu e divided by n these two terms the role of these terms becomes negligible in that particular case.

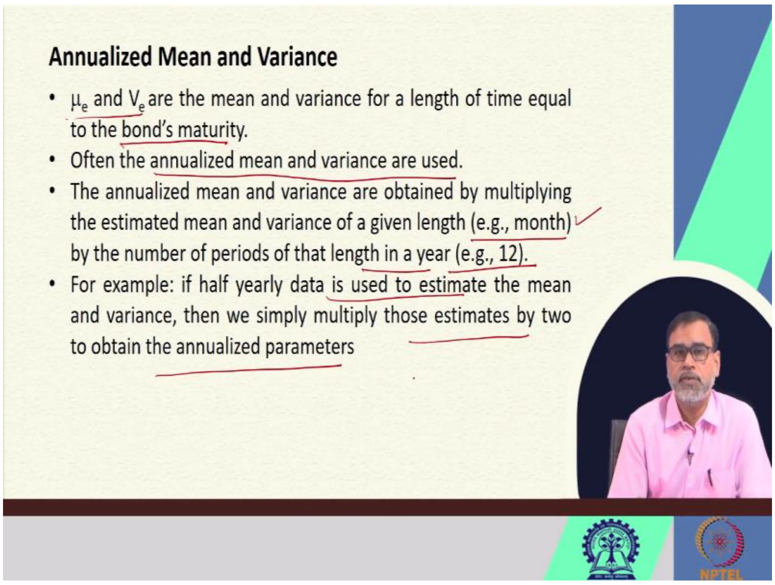
And this formula becomes u = e to the power square root of V e by n and d = e to the power – the square root of the V e by n that is nothing, but the 1 by n that generally happens for the large samples. So, if your n is quite large then your mean term becomes 0 the mean term in the exponent basically goes to 0 and only this variance term will remain so that is why the formula becomes this e to the power square root of V e by n and e to the power – this is square root of V e by n that is nothing, but the 1 by u.

u and d can be estimated as:

$u = e\sqrt{V_e/n} + \mu_e/ n$

$d = e^{-}\sqrt{V_e/n} + \mu_e/ n$

(Refer Slide Time: 09:23)

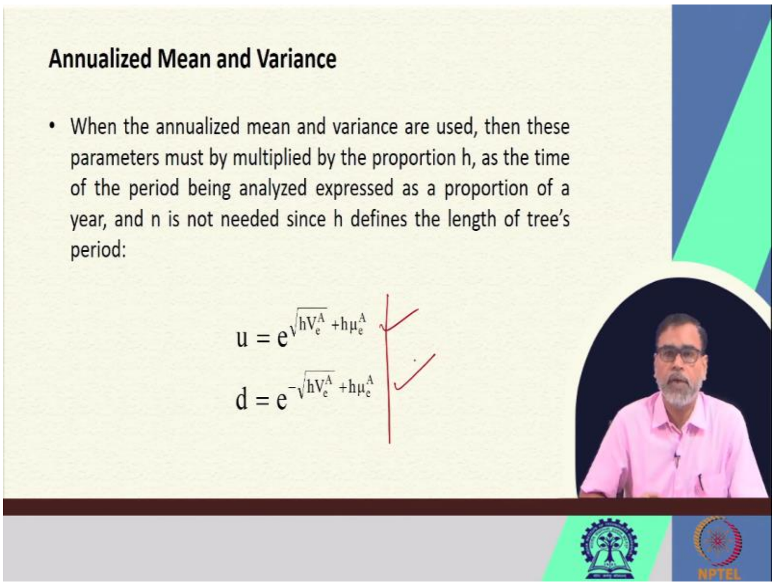


So, then we have a concept of the annualized mean and variance in this case. So, the mu e and V e; these two terms mu e and V e these are the mean and variance for the length of time = bonds maturity's. Mean and variance for a length of time equals to the bonds maturity. Generally, whenever we deal with this bond markets for the fixed income securities market, the annualized mean and variance generally are used.

And how the annualized mean and variance are found or are obtained that is generally always we find by multiplying the estimated mean and variance of a given length. Let, for example, you have taken month by the number of periods of that length in a year that means if you are talking about the month is the frequency then you have to multiply with a 12. For example, if half yearly data is used, then we simply multiply those estimates by 2 to obtain the annualized parameters.

If it is let quarterly then you multiply it by 4. So, like that the annualized mean and variance can be calculated.

(Refer Slide Time: 11:04)

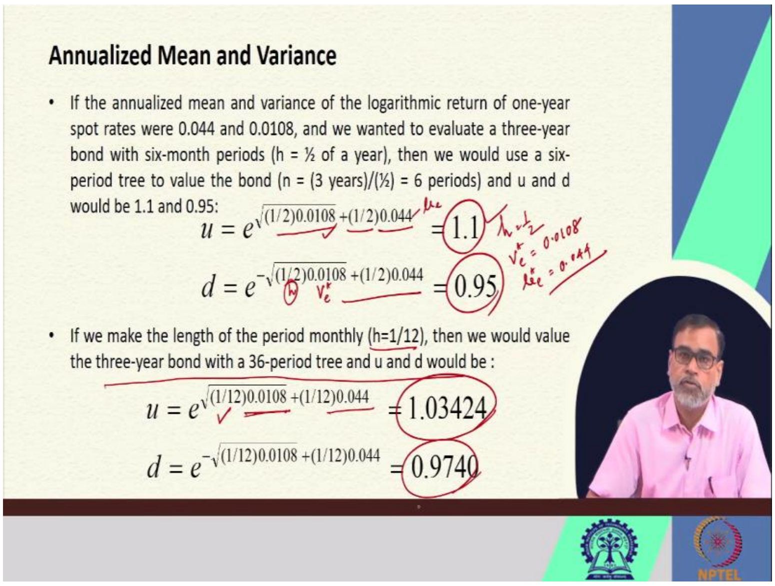


So, when the annualized mean and variance are used, then these parameters must be multiplied by the proportion h, as the time of the period being analyzed expressed as a proportion of a year, and n is not needed since h defines the length defines the length of tree's period. So, in that case you can write u = e to the power h into V e A + h into mu e A and d = e to the power – the square root of h V e A + h mu e A. U = e to the power the square root of h V e A + h mu e A and d = e to the power – the square root of h V e A + h mu e A that is the thing basically what you have to keep in the mind.

$u = e\sqrt{hV_e{}^{A}/n} + h\mu_e{}^{A}$

$d = e^{-}\sqrt{hV_e{}^{A}/n} + h\mu_e{}^{A}$

(Refer Slide Time: 12:17)



So, let us see that if the annualized mean and variance of a logarithmic return of one year spot rates that are 0.044 and 0.0108 and we wanted to evaluate a three-year bond with 6 months period. Let h = 1 by 2 of the year then what we should do? Then we would use basically a 6-period tree to value the bond that means n = your 3 divided by 1 by 2 that is the 6 period and u and d basically you can calculate from that case then, what will be the value of the d and u?

The value of u will be e to the power square root of h = here h = 1 by 2 then your V is basically 0.0108 and your mu e is basically your 0.044. So, now you are putting that values in this formula that is 1 by 2 multiplied by your 0.0108 the square root of that plus h into this is basically your mu e 044. So, then we get 1.1 then if you are going for d; d = e to the power – of the square root of h this is h this is basically your V e A. So, this is basically your V e A this is mu e A. So, then it is 1 by 2 into 0.044 that you get 0.95.

$u = e\sqrt{0.0108/2} + 0.044/2 = 1.1$

$d = e^{-}\sqrt{0.0108/2} + 0.044/2 = 0.95$

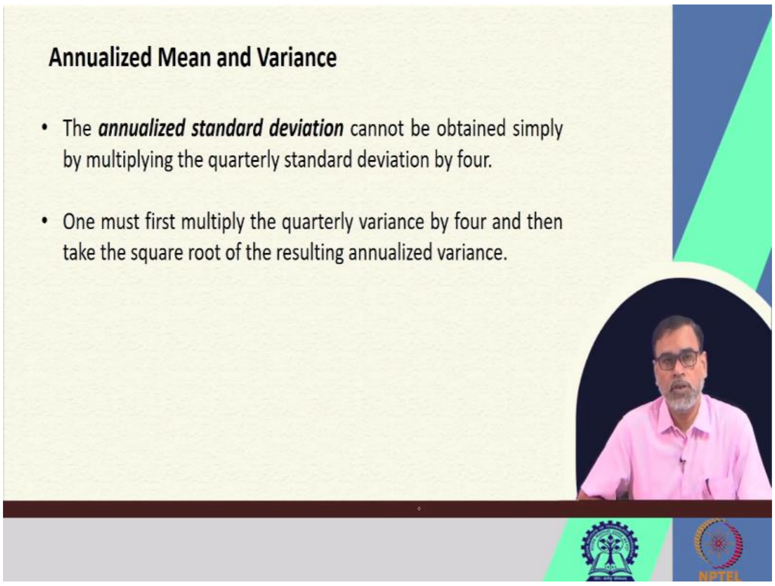
So, like that if you make the length of the period let h = 1 by 12 then we would value the three year bond with a 36 period tree and your u will be h = 1 by 12 into 0.0108 plus your 1 by 12 into 0.044 that will become this and d = will become this.

$u = e\sqrt{0.0108/12} + 0.044/12 = 1.03424$

$d = e^{-}\sqrt{0.0108/12} + 0.044/12 = 0.9740$

So, this is the way basically the annualized mean and variance can be calculated that actually you have to keep in the mind.

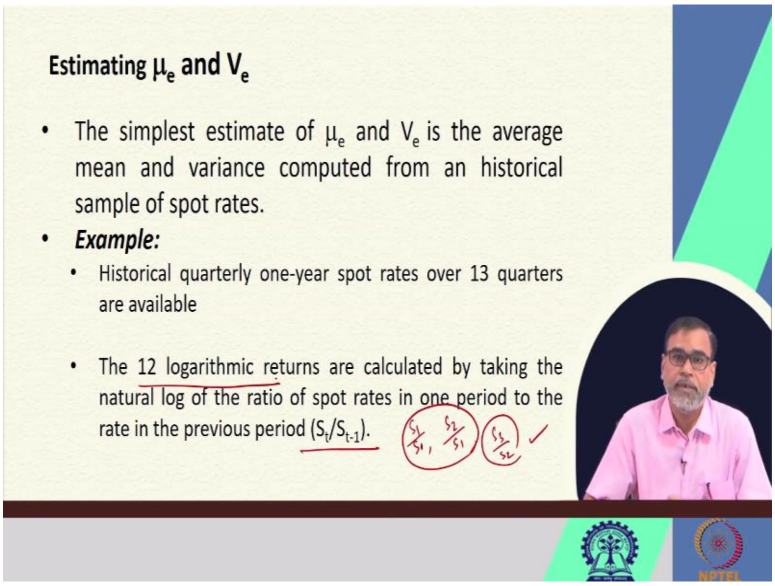
(Refer Slide Time: 15:11)



The annualized standard deviation if you are considering, that cannot be obtained simply by multiplying the quarterly standard deviation by four. If you are dealing with the standard deviation, then by multiplying only the number with that particular value will not give you that. We must first multiply the quarterly variance by 4 and then the square root of the resulting annualized variance has to be considered that actually after picking the standard deviation then you multiply by 4 that will basically will be incorrect.

What basically you have to do first of all you have to find out the quarterly variance and then first multiply the quarterly variance by 4 to make it annualized and then take the square root of the resulting annualized variance to find out the annualized standard deviation. that is why there is some technicality involved in that with respect to the calculations the standard deviation or annualized standard deviation that actually you have to keep in the mind.

(Refer Slide Time: 16:22)

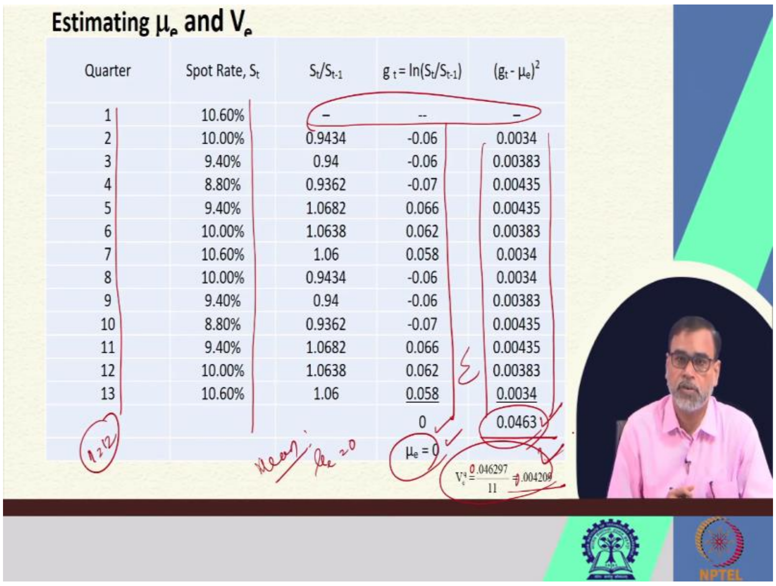


So, the simplest estimate of the mu e and V e is nothing, but the average mean and variance computed from the historical sample of the spot rates. First of all how you calculate the mu e and V? The simplest way is basically to calculate the average mean and variance from the sample of the spot rates wherever you have. You take the sample of the spot rates from there you find out the mean and variance of that.

From there you can use it for calculation of your u and d. So, let, for example, how basically this can be used or this generally is used by the analyst or by investor to find out the value of u and d in that particular consideration. Let us see that what basically happens in that particular case? So, let we have a data for historical quarterly one year spot rates over 13 quarters are available.

Then from there you can calculate the 12 logarithmic returns by taking the natural log of the ratio of the spot rates in one period to the rate in the previous period that is basically your S t by S t – 1 that means your S 1 by S 0 or S 2 by S 1 and so on that S 3 by S 2 like that you can calculate these 12 logarithmic returns value from this 13-quarter pattern. After that whenever you have this data available with you, you can calculate this average mean and variance of that particular series which can be used for calculations of your u and d.

(Refer Slide Time: 18:21)



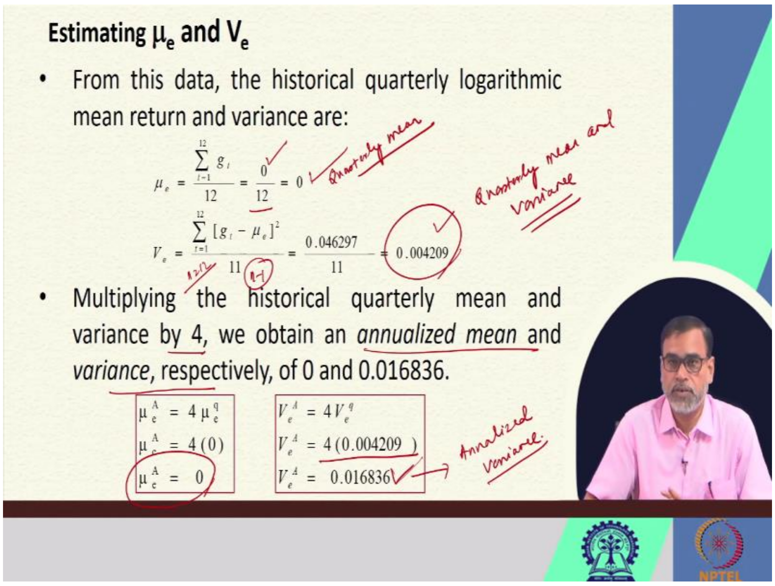
So, let this is your data this is your different quarters 1 to 13. Let these are the spot rates which are given in the percentage term. So, you have the S t by S t – 1 so obviously first period you will not have any data because previous data is not available that is why we lose one observation in that particular case, that means the 12 logarithmic returns can be consider for our particular analysis.

So, this is basically S t by S t – 1, then after that, what basically we are doing? that we can go for calculating the log of this S t by S t – 1. then from there you find out your mean here your mu e = 0 that means the mean = 0, the mu = 0 and then the deviation from the mean you have to consider in this case and after that you take the square of that. So, after you calculate the square of that you take the summation of that; that is basically the summation of all.

So, from rounding wise whenever we do the calculation in X n issue just ignore that you just find out the process into that; that is why some deviation you may find because we have just calculated using the excel. that is why some kind of deviations you will find in terms of the numbers. So, the total value you got 0.0463 so then your value will be your exactly the value 0.046297 we have rounded up to 0.0463.

So, then your variance of that particular series will be your 0.046297 divided by n – 1 where n = 12 whatever we have considered and that is 11 then you got the value is 0.004209. The variance basically in this case was basically 0.004209 that is what basically what you got. So, now what has happened you got the mean value, you got the variance value and from there you can use that particular value for the further calculations.

(Refer Slide Time: 21:11)



So, just now we have just given this how it is calculated the historical quarterly logarithmic mean and return mean return and variance is nothing, but this is the summation of t = 1 to 12 g t divided by 12 that total summation has gone 0, 0 divided by 12 that has become 0 then V is t = 1 to 12 into g t – mu e the square of that summation of the square of this divided by this is basically n – 1 then n = 12. So, n = 11 then you got 0.004209.

$$\mu_e = \frac{\sum_{t=1}^{12} g_t}{12} = \frac{0}{12} = 0$$

$$V_e = = \frac{\sum_{t=1}^{12} [g_t - \mu_t]^2}{11} = \frac{0.046297}{11} = 0.004209$$

So, that is basically the V e or the variance then you multiply the historical quarterly mean and variance by 4. So, these are the quarterly mean and variance then once you calculate your quarterly mean and variance then what basically you can do? You multiply this with 4 then you get this annualized mean and variance from this. So, obviously your quarterly mean was 0.

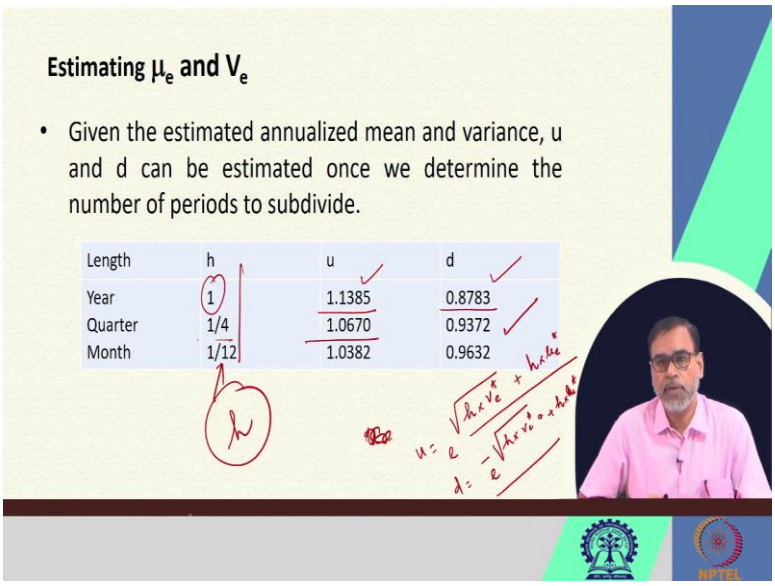
So, this is your quarterly mean so this is already 0. So, obviously your annualized mean also will be equal to 0, but quarterly variance is 0.004209. So, the annualized variance shall be 4 into this so this you will get 0.016836. This is basically your annualized variance. So, now what you can do using your quarterly data whenever you calculate this particular values? these are the quarterly mean and variance.

And from the quarterly mean and variance we are basically getting this annualized mean and variance by multiplying by 4. So, now these particular values can be utilized for the calculations of the u and d. So, this is the simplest approach for the calculations of that. So, if you are in general case if you are calculating this mean and variance and you have the probability values then you have the probability of the spot rates in the different periods.

There also you can calculate your mean and variance or annualized mean and variance, but in case of other data is not available then the simplest way of calculating this thing is by considering the historical data for the particular spot rates then take the log of that and from this whatever frequency data is available you can utilize it to calculate this mean and variance then you multiply with the respective numbers on the basis of quarterly data if you are using then you multiply with 4.

If you are using monthly data then you multiply with 12 like that then finally your annualized mean and variance can be calculated from that.

(Refer Slide Time: 25:03)



So, from that this estimated annualized mean and variance you can find out your u and d. Once we determine the number of periods to subdivide, that is why the concept of subdivision of the periods is very much important in this particular case. So, if you have the h is let 1 or one period in this case then this is your u, this is your d using that particular formula we are finding it out that what is the formula you are using that u = what is the formula you are using?

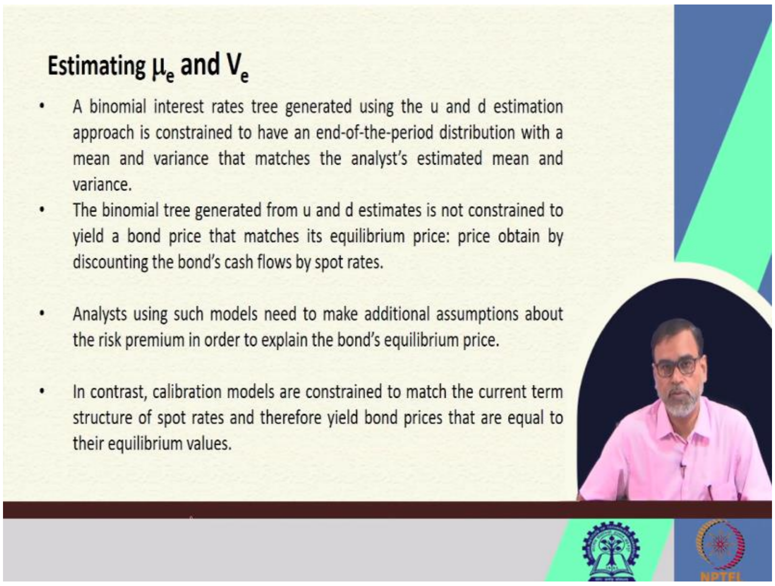
This is the formula what basically you are using, but if you are going by a large number then the different formula, but this is h into your variance what basically you are getting annualized variance plus h into this one already we have discussed that. So, from this generally we can use this particular formula here then we can calculate this u value and the d value.

So, if your h = 1 by 4 then accordingly this will be because in the square root this is h into your V e + h into your mu e. This is what basically what we are considering this is u e to the power this. If you are calculating d = e to the power – of this into V e A square root of this + your h into mu e A. Just now I have shown you this is the formula what basically you are using it for calculating this u and d.

And on the basis of the h your u and d values are basically changing. So, given this estimated annualized mean and variance u and d can be estimated once we determine the number of periods to subdivide that means you are finding out your h which is basically nothing about this. So, then this is the way the u and d estimation can be that then further we will see that how the calibration model is the extension of that or what are the different other approaches have been considered in that particular case.

And in the different scenario, how this particular value of the bond or the kind of binomial process and distributions are going to be utilized for finding out or estimating this binomial tree?

(Refer Slide Time: 28:17)

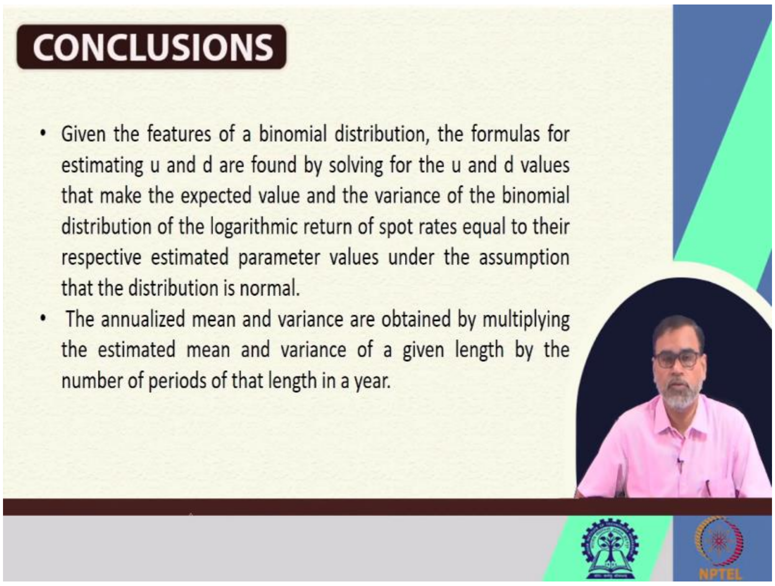


So, binomial interest rate tree generally generated using u and d estimation approach is constrained to have an end of the period distribution with a mean and variance that matches the analyst's estimated mean and variance. So, the binomial tree generated from the u and d estimate is not constrained to yield a bond price that matches its equilibrium price that means what price we obtained by discounting the bond's cash flow by the spot rates.

And generally the investors or the analysts using such models to make the additional assumptions to about the risk premium in order to explain the bonds equilibrium price because of the risk involved in that case. but in contrast to this model, the calibration models are constrained to match the current term structure of the spot rates which generally yield the bond prices that are equal to the equilibrium values.

That we will see in detail that what exactly happens in terms of the calibration model and how basically it is different from the equilibrium model for the estimation of the binomial tree interest rate fluctuations particularly?

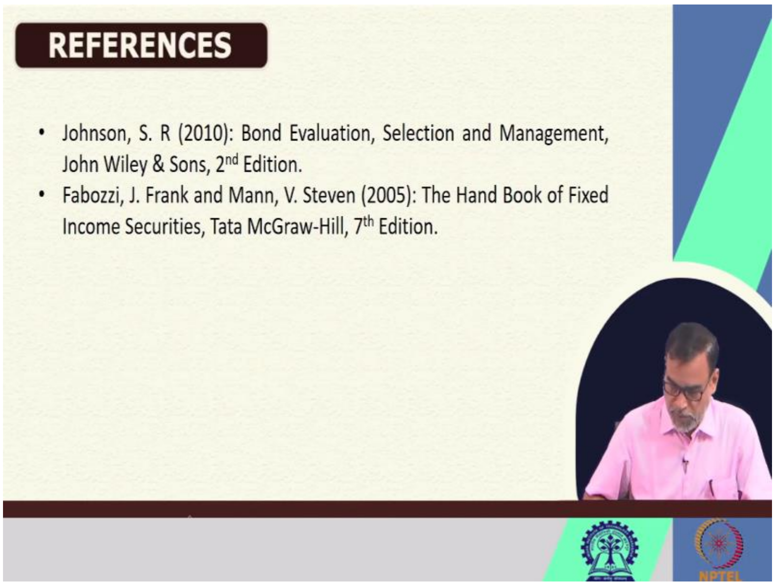
(Refer Slide Time: 29:35)



So, what basically we discussed in this session that the formula for u and d can be find out by solving the expected which make the expected value and variance of the binomial distribution of logarithmic return of the spot rates equal to their respective estimated parameters with the assumptions of the normal distribution and the annualized mean and variance generally are obtained by multiplying the estimated mean and variance for a given length by the number of periods of that length in a year.

That means it is monthly multiply 12, if it is quarterly multiply 4, if it is 6 monthly then you multiply 2 that is the way the annualized mean and variance can be calculated.

(Refer Slide Time: 30:24)



These are the references you can see. Thank you.