Management of Fixed Income Securities

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Lecture – 41
Estimation of Binomial Trees – I

Welcome back. So, in the previous sessions we discussed about how the binomial models or binomial interest rate tree models are going to help us for the valuation of the bonds and as well as the valuation of the bonds having the different kind of embedded options like the call feature, sinking fund provisions, convertible bonds having the call feature and all these things.

But the question here is, there are two issues generally comes to our mind. there we have taken that the interest rate can change in one year that either in one period it can go up to a particular amount or it can go down to a particular amount, but if you change this interest rate in a short horizon or frequently, we want to see that how the interest rate movements are happening?

And how they are going to affect the value of the bond that will be more realistic in nature that is number one and number two basically you have to keep in the mind we have considered that u value is 1.1, d value is something 0.95 or 0.90. So, how this u and d are calculated? what is the basic understanding for the estimation of that u and d. So, these are the two important questions always we should keep in the mind.

And what are those methods are generally used to calculate this value of u and d. So, in today's class basically we will start this estimation of the binomial trees.

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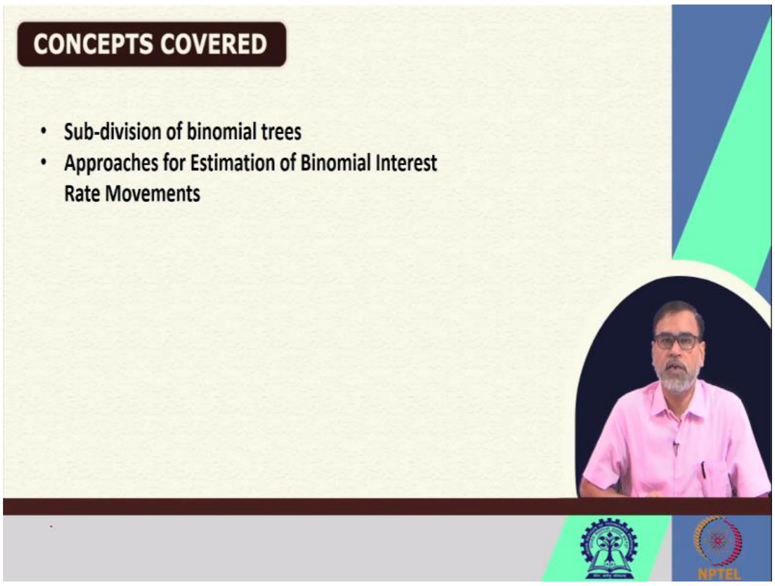

So, here first thing we will discuss the subdivision of the binomial trees, that means we are frequently changing this interest rate number one. Number two, we have to also see what are those different approaches or different methods are used to estimate the binomial interest rate movements? Mostly our botheration is our objective is to find out the values of u and d. So, this is what the basic objectives of the two densities.

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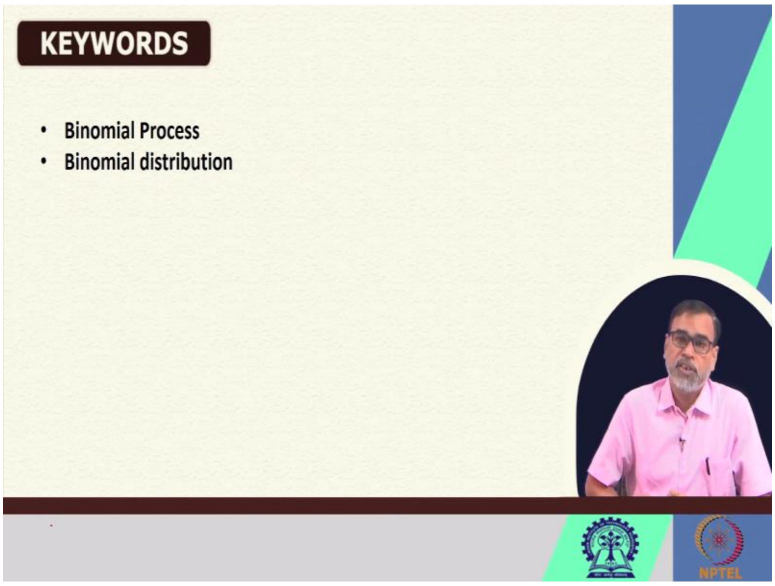

So, in this process you will understand or you will come across certain keywords like binomial process, binomial distribution and all these things.

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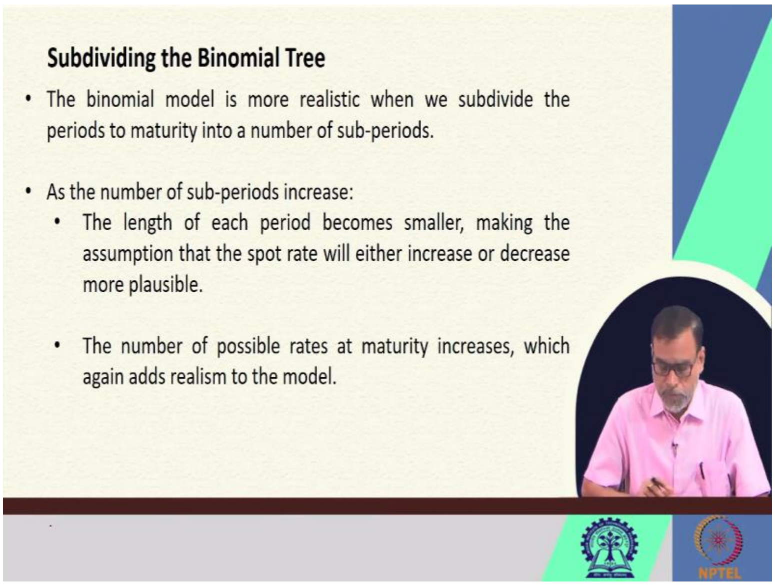

So, let us first discuss about the subdivision of the binomial tree. So, if you look at the binomial model whatever we have used or we have formulated for the valuation of these particular bonds it will look more realistic, when we subdivide the periods to maturity into a number of sub periods that means we are increasing the number of periods and by that your nodes are going to be changed. in the different periods the nodes are basically going to be changed.

So, when the number of sub periods increase, the length of each period become smaller making the assumptions that the spot rate will either increase or decrease more logically. that means we are assuming that the spot rates are going to be changed frequently and the number of possible rates at maturity also is going to increase which again add some kind of realism to the model.

It will be more realistic in that particular sense because the interest rate can change frequently. So, that also bring certain kind of realism into our analysis. Now, let us see that how basically it will look?

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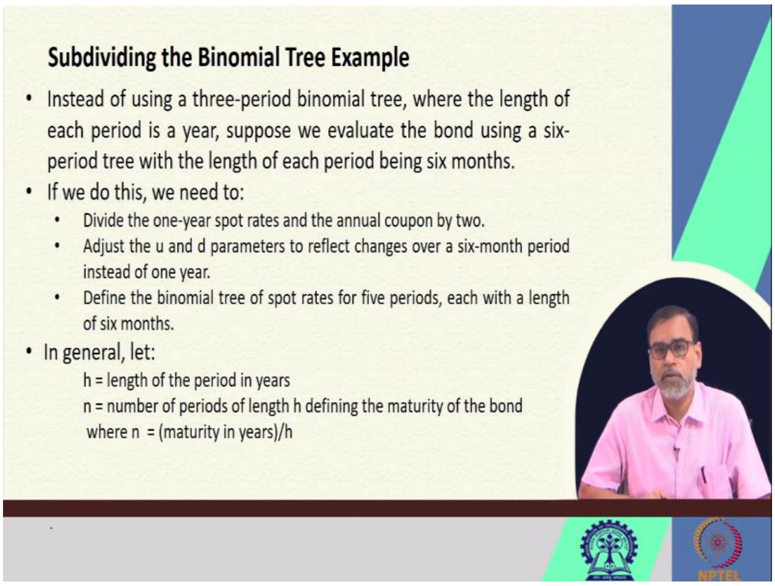

So, whenever we are using a three-period binomial tree what basically we have discussed in the previous sessions? There we have seen that the length of each period is one year. Suppose, we want to do a valuation of a bond using a six-period tree with the length of each period being six months. Let we are assuming the length of each period is six months then it will be automatically a six-period tree.

Then if you are going to do that then, what are those changes or the adjustments we are going to make? how we are going to make the adjustments in terms of the inputs which are used for this particular analysis? First of all divide this one year spot rates in the annual coupon by two because we have assumed that the length of each period is six months. So, we have to divide the one period spot rates and the annual coupon by two.

Accordingly, adjust the u and d parameters to reflect the changes over the six-month period instead of one year. U and d parameters whatever you are considering to find out the value of that particular price of the asset or the value of that particular asset either in the increasing scenario or in the decreasing scenario that also we are going to adjust it. To define this binomial tree of spot rates for five periods each with a length of six months.

So, in general if you want to see let h = the length of the period in years and n = the number of periods of length h defining the maturity of the bond, where your n = maturity in years divided by h. So, let us see how basically let us take some examples how this thing is basically going to be changed or how the numbers are going to be basically differ?

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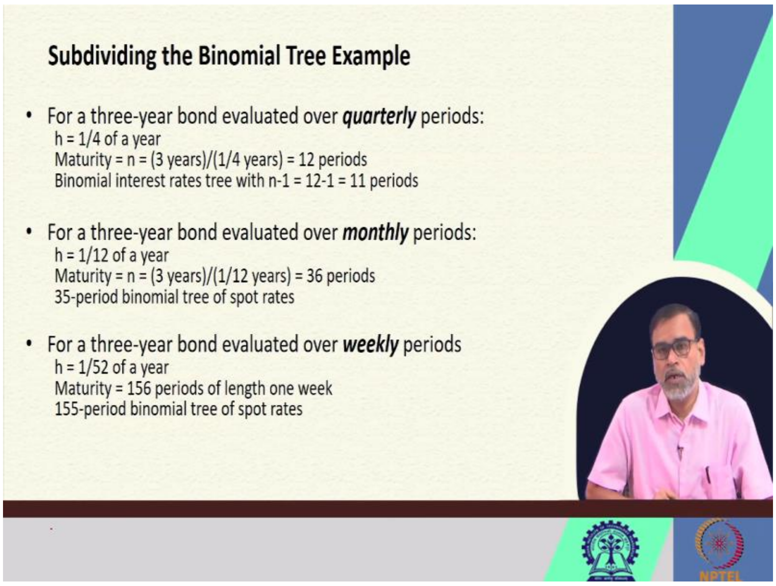

So, here if you see for a three year bond let you are going to evaluate that bond over quarterly basis. If you are going to do the evaluation on the basis of the quarterly basis then h = your 1 by 4 of the year then maturity will be n that is your h what basically we have considered in years maturity is 3 then it will be 3 divided by 1 by 4 that means it will be 12 periods the n = 12.Then how many binomial interest rate tree will be there that already you know that is n – 1 that means 12 – 1 that will be 11 periods.

For a three-year bond evaluated over quarterly periods:

h = ¼ of a year

maturity=n= (3 years)/ (1/4 years) = 12 periods

Binomial interest rates tree with n-1 = 12-1=11 periods

So, if you are going for monthly basis like that h = 1 by 12 then your n = 3 by 1 by 12 then it will be 36 then 36 – 1= 35 period binomial tree of the spot rates have to be formulated.

For a three-year bond evaluated over monthly periods:

h = 1/12 of a year

maturity=n= (3 years)/ (1/12 years) = 36 periods

Binomial interest rates tree with n-1 = 36-1=35 periods

So, if you are going for weekly like that the figures will change h = 1 by 52.Then the maturity will be 3 by 1 by 52 that means it is 1 by 56 periods then you have to formulate 155 period binomial tree of the spot rates.

For a three-year bond evaluated over weekly periods:

h = 1/52 of a year

maturity=n= (3 years)/ (1/52 years) = 156 periods

Binomial interest rates tree with n-1 = 156-1=155 periods

So, like that once your period analysis will change or your h will change automatically your number of trees of the spot rates also going to be changed.

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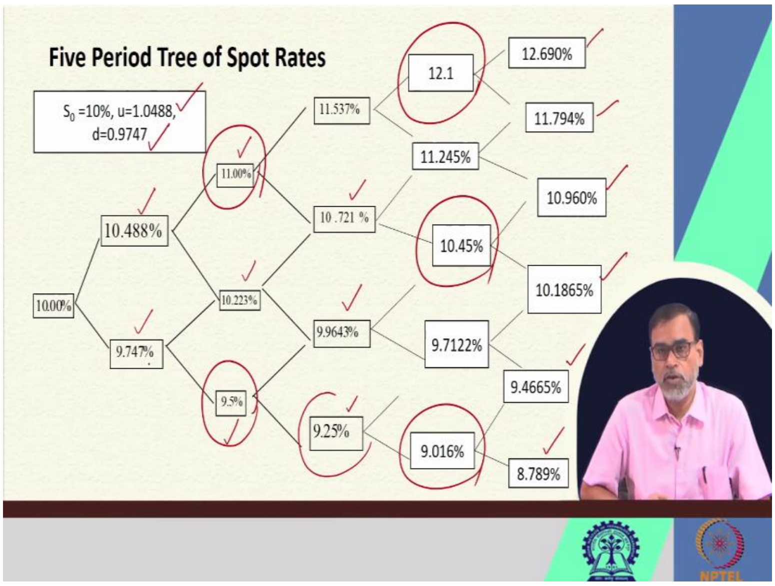

So, then let us see how this particular thing is going to be changed. Let you are going to have a 5 period tree of the spot rates with our previous example we have started from the 10%. So, it is 10% let u = 1.0488 d = 0.9747 then like that it will reach in six months it will reach because u and d also has to be adjusted because in one year it will be 1.1 and it will be 0.95 accordingly it will reduce in the first case and second case it will increase.

So, therefore your u become 1.0488 and here your u has become 1.0488 and d become 0.9747. So, accordingly your interest rate movements figure also going to be changed. It will become 10.488 in the first 6 months or it has come down to 9.747 further it can increase up to 11 or decline up to 10.223 or again it is 9.5. These are the numbers what we get whenever we have consider one period.

Further, it can increase up to these and decline up to 10.721 or from here it can decline up to 9.964 or it can further decline up to 9.25, but if you look at these figures the 12.1, 10.45, 9.016, these are the figures what we are getting whenever we are considering the one year period. Like that further it can go up to this or go down to this or it can reach here, it can reach here, reach here, reach here, reach here.

Like that the number of nodes are going to be changed and the number of periods are going to be changed because these things looks more realistic in nature the reason is that the spot rates or the interest rate can fluctuate frequently. So, as you go on increasing by your number of periods, number of nodes then this validity of the model or the applicability of the model is going to be better, going to be more realistic in nature.

So, this is what the 5-period tree of the spot rates will look like if you are considering the six months h = 1 by 2 years. So, that means you will have the 1, 2, 3, 4, 5 periods will be there it will be generally n – 1. n = 6 then your number of periods will be n – 1, that will be 5.

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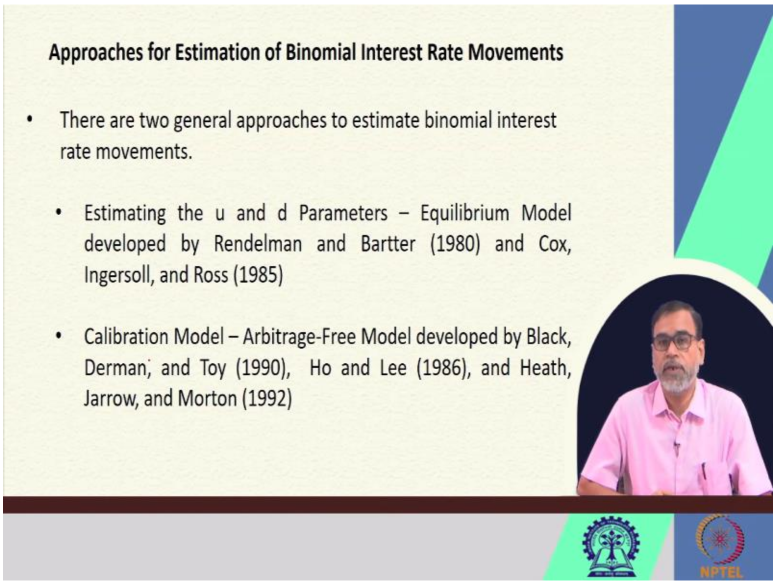

But in the beginning, I discussed with you the most important thing is how this u and d is formulated. We have assumed certain u and d in the one-year period analysis. we have taken this is 1.1 and 0.95, but how that u and d is calculated? So, there are generally two approaches we use to estimate this binomial interest rate movement to find out this u and d. One is u we call it the equilibrium model, through the equilibrium model we find out the u and d.

And further we can use the calibration model which basically helps us to exactly find out this or to estimate this binomial interest rate tree movement. So, this equilibrium model generally developed by Rendelman and Bartter (1980). further again, Cox and Ingersoll and Ross (1985) and the calibration model generally is also called the arbitrage free model. It is developed by Black, Derman and Toy (1990), Ho and Lee (1986), Heath, Jarrow and Morton (1992).

So, these are the people who basically try to develop this model to estimate this binomial interest rate movements or more specifically to find out the value of u and d.

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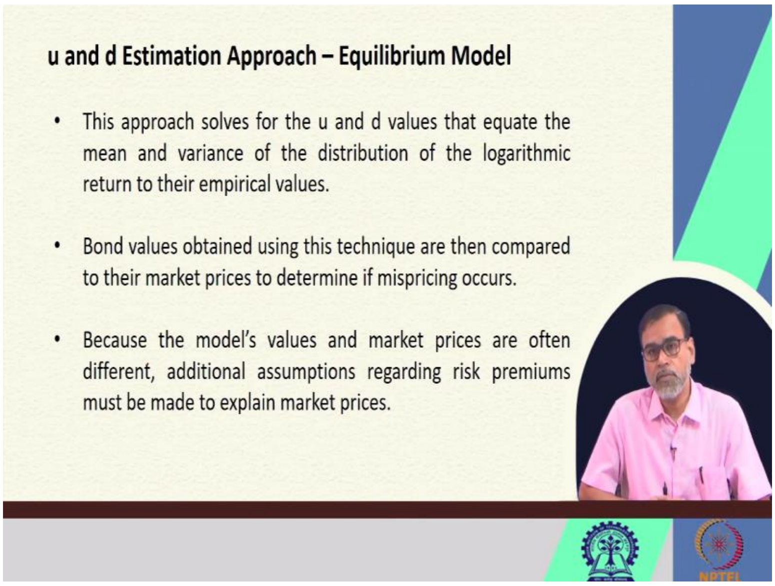

Let us first before we go for this, we can start with this u and d estimation approach; that is the equilibrium model. What this model basically does? This particular approach try to solve the value of u and d which equate the mean and variance of the distribution of the logarithmic return to their empirical values. Why it is considered the log of the returns? Generally, whenever we go for analyzing certain things in statistical way in the financial market we assume that the distribution should follow a normal distribution.

But whenever we go for the calculation of the returns or trying to decide this interest rate trees that may not follow the normal distribution. So, the investors or the analyst basically try to find out the value of the u and d which is trying to equate the mean and variance of the distribution of the logarithmic return to their empirical values. So, the bond values which are obtained using this technique are then compared to their market prices.

To determine if there is any mispricing that basically they want to determine after estimating this value of a u and d. So, in general, if you see the models value and the market prices generally differ. So, if it is different than the additional assumptions regarding the risk premium must be made to explain the market prices, why it is different because of the risk concept?

So, that is why the additional risk premium has to be added to that particular interest rate that is what basically what we call it the equilibrium model. How that particular model basically looks like? what basically it explains let us see?

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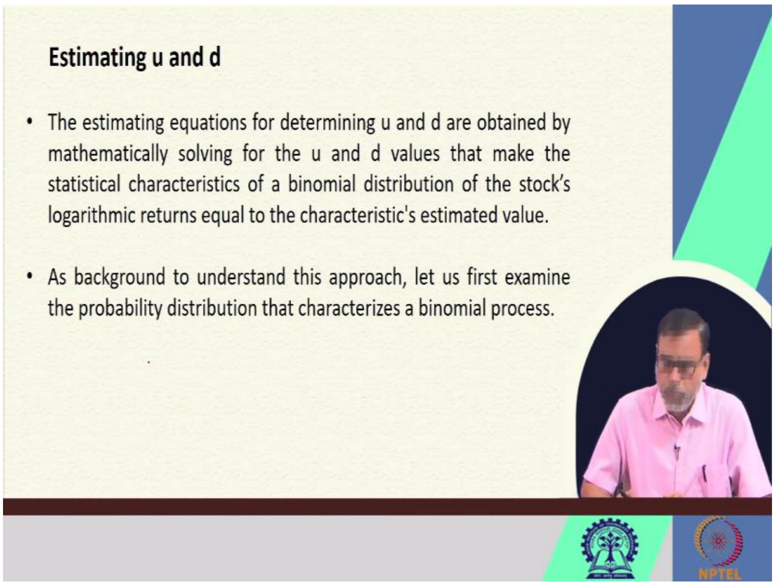

How, you can determine this u and d. Mathematically, if you want to solve this u and d, which make this statistical characteristic of a binomial distribution of the stock's logarithmic returns equal to the characteristics estimated value. So, that is what the basic notion of the equilibrium model. So, before that what basically we have to do to? understand this particular approach, to understand this model.

Let us first examine the probability distribution that characterizes a binomial process. What is the probability distribution which characterizes a binomial process? So, whenever you talk about the probability distribution of a particular data, then how this binomial process basically looks here or how basically it is going to explain, how it can be derived?

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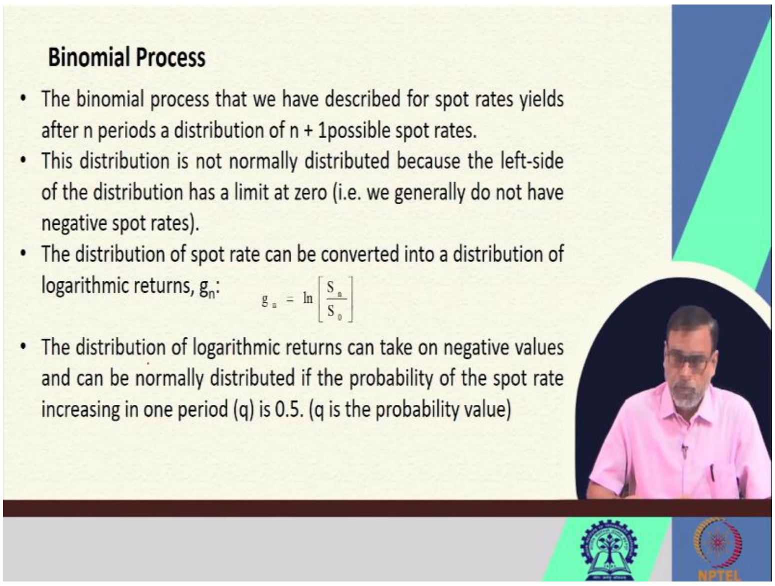

So, here we are trying to derive a binomial process of the spot rates and this distribution generally is not normally distributed because the left side of the distribution has a limit at 0 because interest rate cannot be less than 0. If you are depositing the money, nobody will deduct some kind of interest rate from your money to keep the money. it can be 0 you may not get anything, but it cannot be 0 because we do not have negative spot rates.

So, because of this but whenever we are going for a statistical analysis if you are assuming a binomial distribution then some values can be 0 also. So, that is why the distribution of the spot rate can be converted into a distribution of logarithmic returns. Let we are assuming a g n, then what is the g n? g n = the ln of S n by S 0 and it can be 1, 2 any numbers the initial spot rate whatever we are considering and one of the period spot rates let S n = S 1.

So, g 1 = ln S 1 by S 0. So, the distribution of the logarithmic returns can take on the negative values and can be normally distributed if the probability of the spot rate increasing in one period which is generally, we consider q is 0.5; q is basically the probability value.

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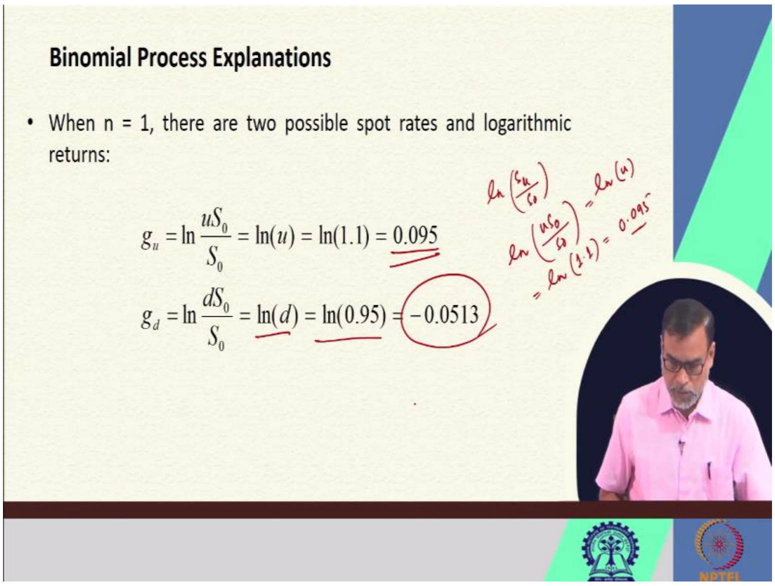

So, now let us see what exactly how it explains how the binomial process can be explained? Let when n = 1. There are two possible spot rates and the logarithmic returns that means your g u. if you want to calculate that means, whenever the interest rate or the spot rate is going to be up your g = what? Your ln S u divided by S 0 and what is your ln S u that is nothing, but your ln u S 0 divided by S 0 S u is nothing, but u into S 0.

So, S 0 S 0 can be cancelled out you can find out ln u. So, ln u means let we have to consider u = 1.1 then ln 1.1 = 0.095 that is what basically we have mentioned here. The same thing if you are going for the lower side or the downward side, then it is ln d S 0 by S 0, then it will be ln d then ln 0.95 then it will be – 0.0513 like that for other periods also you can look at this.

When n=1, there are two possible spot rates and logarithmic returns:

$g_u = \ln \frac{uS_0}{S_0} = \ln(u) = \ln(1.1) = 0.095$

$g_d = \ln \frac{dS_0}{S_0} = \ln(d) = \ln(0.95) = -0.0513$

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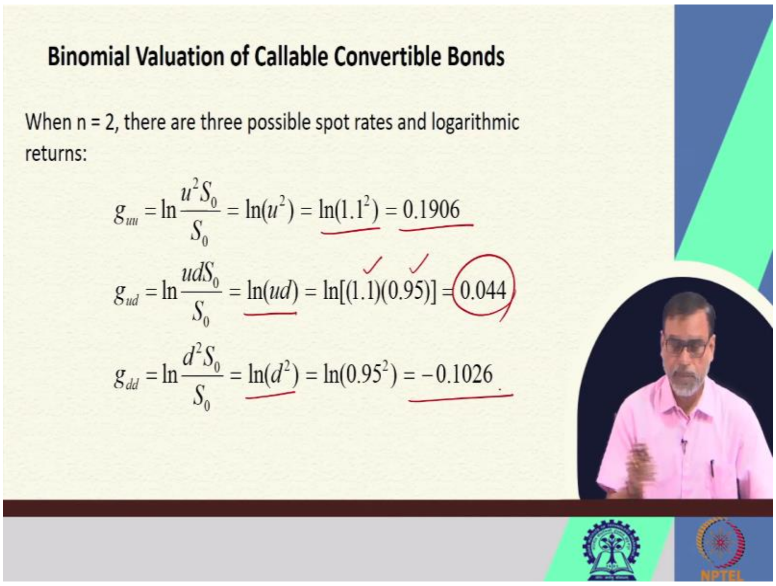

If n = 2 then there are three possible spot rates and the logarithmic returns then your g double u = ln u square S 0 by S 0 which is nothing, but ln u square then ln u square means it is ln 1.1 square that is 0.1906 then ln g u d = ln u d S 0 divided by S 0 then it will be ln u d then ln u d means it is ln 1.1 into 0.95 that is 0.044 like g d d if you are calculating then ln d square S 0 by S 0 that is ln d square then it will be – 0.1026.

When n=2, there are three possible spot rates and logarithmic returns:

$g_{uu} = \ln \frac{u^2 S_0}{S_0} = \ln(u^2) = \ln(1.1^2) = 0.1906$

$g_{ud} = \ln \frac{u d S_0}{S_0} = \ln(ud) = \ln[(1.1)(0.95)] = 0.044$

$g_{dd} = \ln \frac{d^2 S_0}{S_0} = \ln(d^2) = \ln(0.95^2) = -0.1026$

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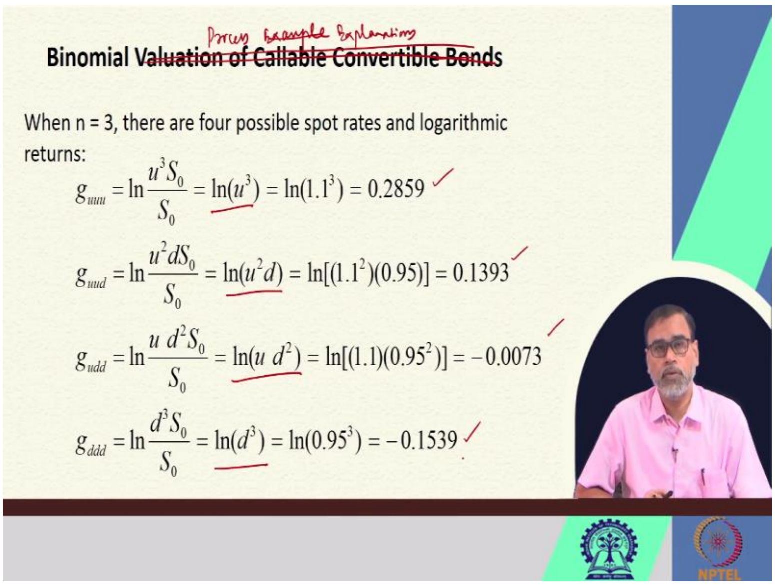

So like that if your n = 3 there are four possible spot rates in the logarithmic returns. One is your ln u cube ln u square d ln u d square and ln d cube. So, like that you can get these values like this. So, already we said that the binomial process; so here there is the title is it is basically not the convertible bond it is the binomial process the example whatever we are basically considering there is a typo error here.

So, here your g, ddd you can find out – 0.1539. What basically we have seen here the binomial process if you are explaining in 1 over n = 1 you find out these values. Here also it is binomial process explanations here also it is binomial process explanations. So, there are some type errors please keep that thing in mind. So, here we have seen that these are basically the values what we are getting whenever we are taking the log of this particular values.

When n=3, there are four possible spot rates and logarithmic returns:

$g_{uuu} = \ln \frac{u^3 S_0}{S_0} = \ln (u^3) = \ln(1.1^3) = 0.2859$

$g_{uud} = \ln \frac{u^2 d S_0}{S_0} = \ln (u^2 d) = \ln[(1.1^2)(0.95)] = 0.1393$

$g_{udd} = \ln \frac{u d^2 S_0}{S_0} = \ln (ud^2) = \ln [(1.1)(0.95^2)] = -0.0073$

$g_{ddd} = \ln \frac{d^3 S_0}{S_0} = \ln (d^3) = \ln(0.95^3) = -0.1539$

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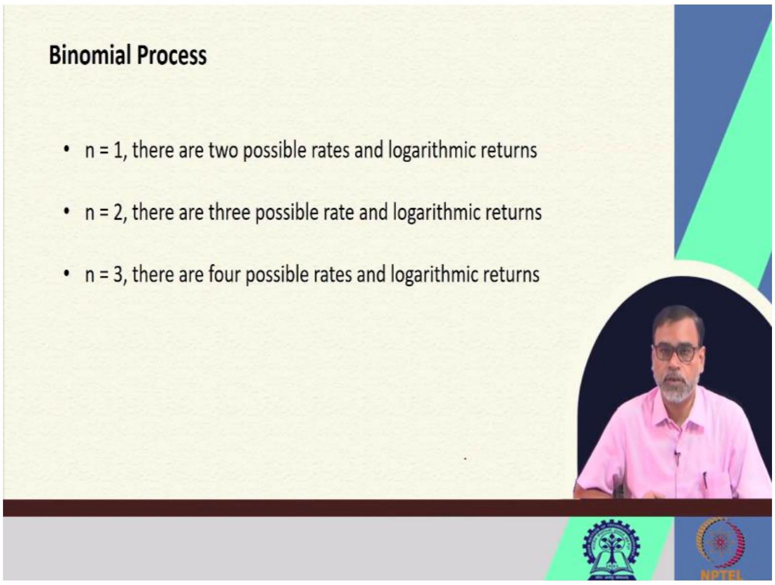

Then what we have seen here, whenever n = 1 there are two possible rates and logarithmic returns 1 over n = 2 there are three possible rates in the logarithmic returns. When n = 3 there are four possible rates in the logarithmic returns.

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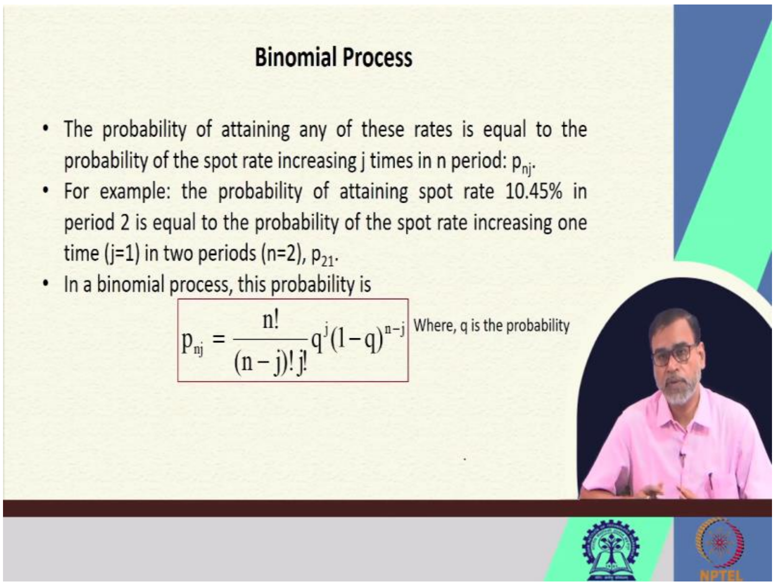

So, now what basically we can do? You can generalize this. So, the probability of attaining any of these rates is equal to the probability of the spot rate increasing j times in n periods which can be represented as P n j. For example, the probability of attaining spot rate of 10.45% whatever binomial tree we have just explained before in period 2 is equal to the probability of the spot rate increasing one time that is j = 1 in two periods that is n = 2 that means it can be represented as p 21.

In a binomial process, the probability is

$P_{nj} = \frac{n!}{(n-j)!J!}\, q^j (1-q)^{n-j}$

So, p 21 is basically represented in this particular case. So, in a binomial process this probability is P n j = your n factorial divided by your n – j factorial into j factorial into q to the power j into 1 – q to the power n – j where q is the probability where q is basically the probability that we are considering. So, now if you have a binomial process and you are taking the log of that it follows a normal distribution then what basically we can do?

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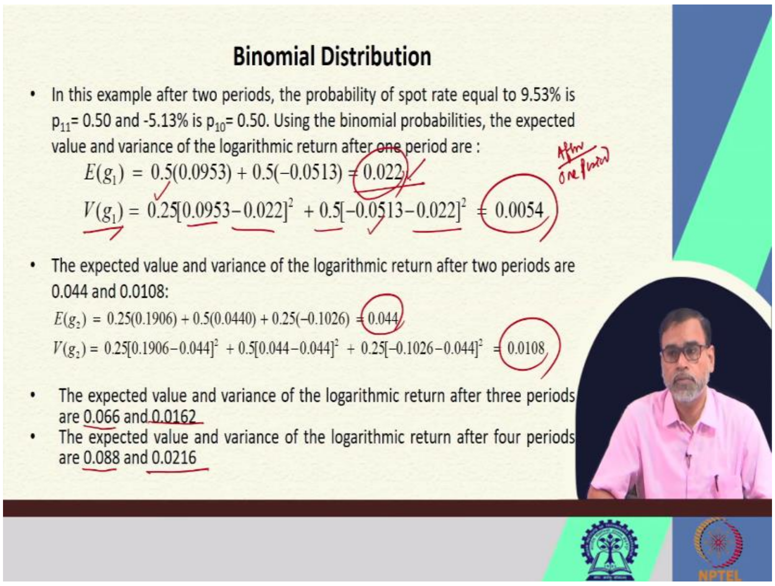

We can see that how this binomial distribution looks like and how this mean and variance of that particular distribution can be calculated? In this example, if you look at after two periods the probability of spot rate is equal to 9.53% and your p 11 is 0.5. Your probability of the spot rate = 9.53% is basically 0.5 that means p 11 = 0.5. If you are going to get -5.13% the probability is also 0.5.

So, using this binomial probability the expected value and the variance of the logarithmic return after one period can be calculate that is E g 1 = 0.5 into 0953 + 0.5 into – 0.0513 that is 0.022 and your V g 1 = 0.25 into 0.0953 – 0.022 square + 0.5 into – 0.513 – 0.22 square that is 0.0054 and what do you mean by this 022? that basically mean this basically your variance. So, this is the actual data deviation from the mean the square of that multiplied by the probability + the probability – the actual spot rate – this mean that will give you the variance.

the expected value and variance after one period are:

$E(g_1) = 0.5(0.0953) + 0.5(-0.0513) = 0.022$

$V(g_1) = 0.25(0.0953-0.022)^2 + 0.5(-0.0513-0.022)^2 = 0.0054$

It is after one period. If you are going for the expected value and variance after two periods then again you can calculate this on the basis of the rates what you are getting that will be 0.044 and the variance will be 0.0108.

the expected value and variance after two periods are:

$E(g_2) = 0.25(0.1906) + 0.25(-0.1026) = 0.044$

$V(g_2) = 0.25(0.1906-0.044)^2 + 0.5(0.044-0.044)^2 + 0.25(-0.1026-0.044)^2 = 0.0108$

So, the expected value after three periods if you see that will become 0.066 and it will be 0.0162 after four periods it will be 0.088 and 0.0216. So, that means if you see that it is basically n times this value whatever you are getting in the first period.

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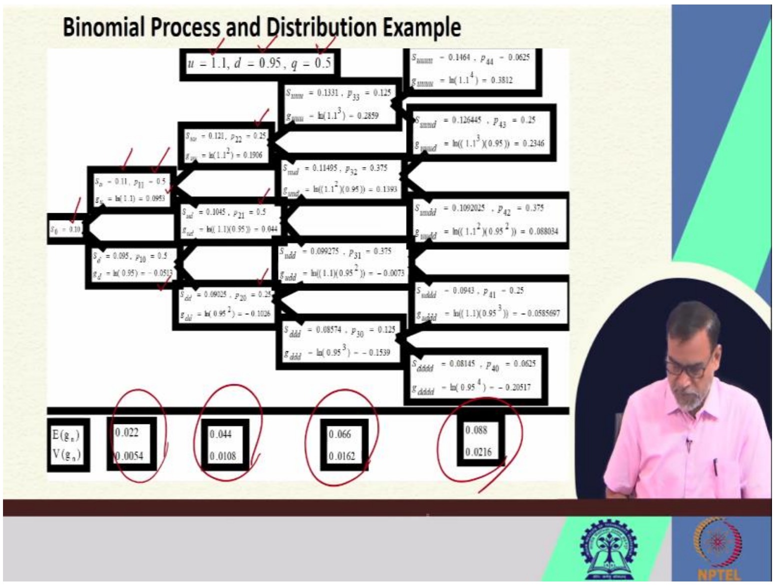

So, that means just now if you look at just now whatever calculations I have shown you this thing has been reflected in one picture. You start with 0.1 it has become 0.11 so here because you have assumed u = 1. 1, d = 0.95, q = 0.5 so p 11 = 0.5 so g u = this here your g d = this and so on. Everywhere we have calculated this. So, here your probability has become 0.25 it is 0.5 it is 0.25 and so on.

So, these are the expected mean value and the variance figures what basically here reflected. So, now on that basis what is the property we got?

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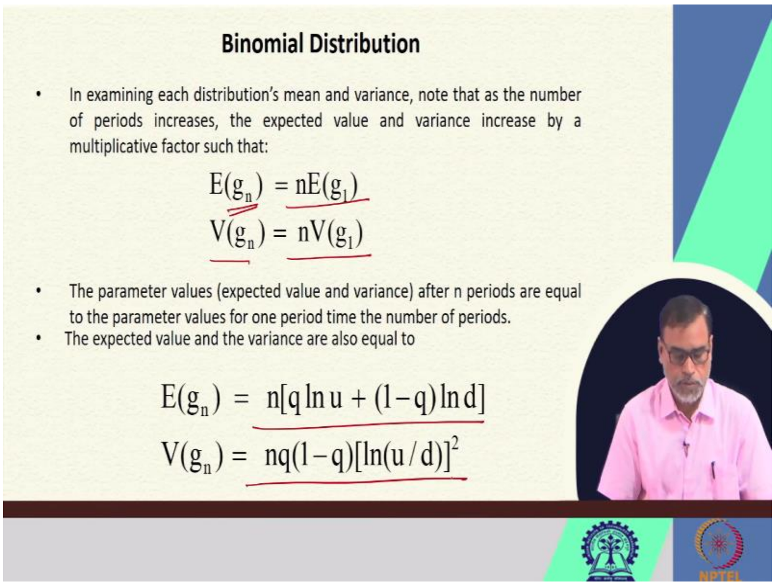

In examining each distribution's mean and variance note that as the number of periods increases the expected value and variance also increase by a multiplicative factor such that your expected g n n = the period will be n E g 1 and V g n = n V g 1. So, the parameter values after n periods are equal to the parameter values for the one period time the number of periods and the expected value and the variance are equal to n into q ln u + 1 – q ln d.And if you are going for the variance your V gn = nq 1 – q into ln u – d whole square. So, this is your mean this is basically your variance.

The expected value and the variance are equal to

$E(g_n) = n[\ q \ln u + (1-q) \ln d]$

$V(g_n) = nq(1-q)\ [\ln (u/d)]^2$

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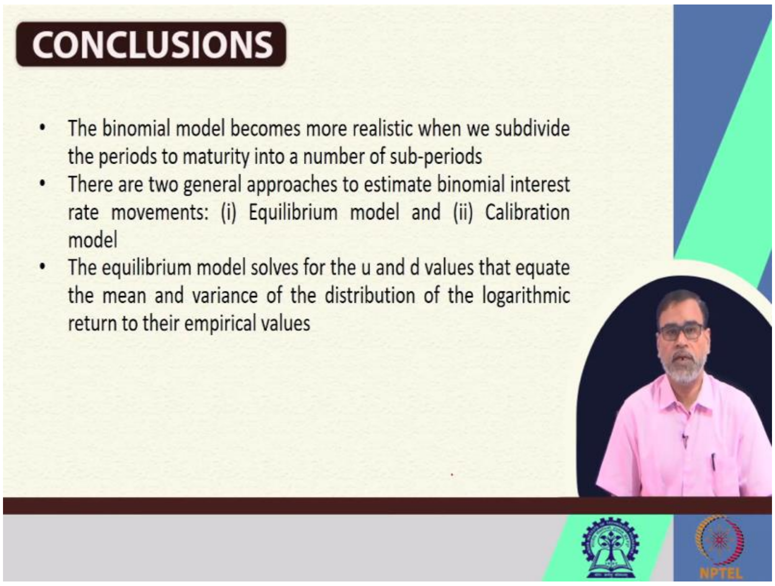

So, what basically we have discussed here? the binomial model becomes more realistic when we subdivide the periods to maturity, into a number of sub periods. There are two general approaches to estimate the binomial interest rate movements. One is your equilibrium model and the calibration model and the equilibrium model basically solves for the u and d values which equate the mean and variance of the distribution of the logarithmic return to their empirical values.

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This is the reference you can go through. Thank you.