Management of Fixed Income Securities
Prof. Jitendra Mahakud
Department of Humanities and Social Science
Indian Institute of Technology, Kharagpur

Module No # 05
Lecture No # 23
Bond Duration

Welcome back! and good morning. So, in the previous class, we discussed about the concept of the market risk which is nothing but, the due to the interest rate change how the price of the bond basically changes or the total return of the bond changes? So, there we have seen there are 2 effects work, one is your price effect other one is the reinvestment effect or interest on interest effect.

So, depending upon the interest rate moments and as well as the 2 effects which are working are the depending upon the dominance of those particular effects. Whether price effect dominance, or the interest on interest rate effective dominance, the total return of the bond basically will change. So, today we will be discussing about the other concept which is related to the bond risk, that is basically we call it the bond duration.

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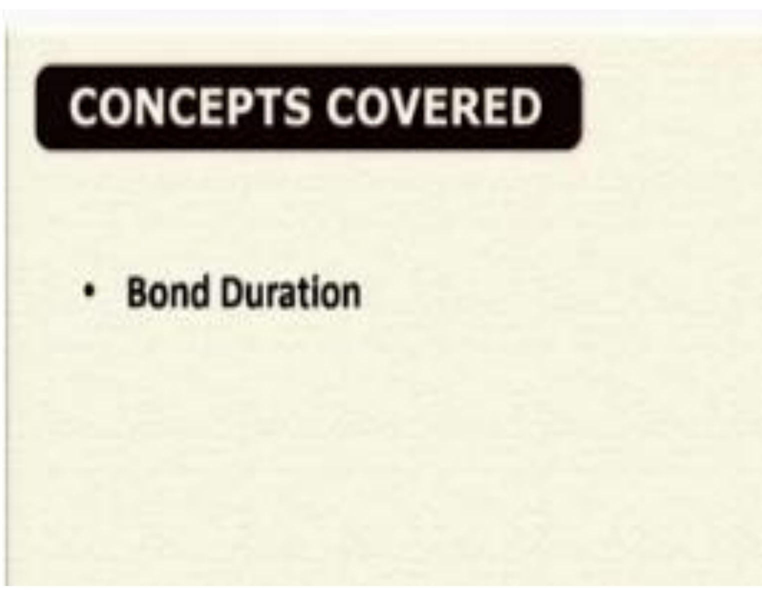

So, today we will be covering of the concept of the bond duration. So, in this particular context, we will discuss about what exactly the bond duration is. And how the bond duration is measured and what is the use of the bond duration? And also, how we can calculate the duration of a portfolio of the bonds or we can say that portfolio duration or duration of the bond portfolio. So, these are the different concepts, what basically will be discussing in today's session.

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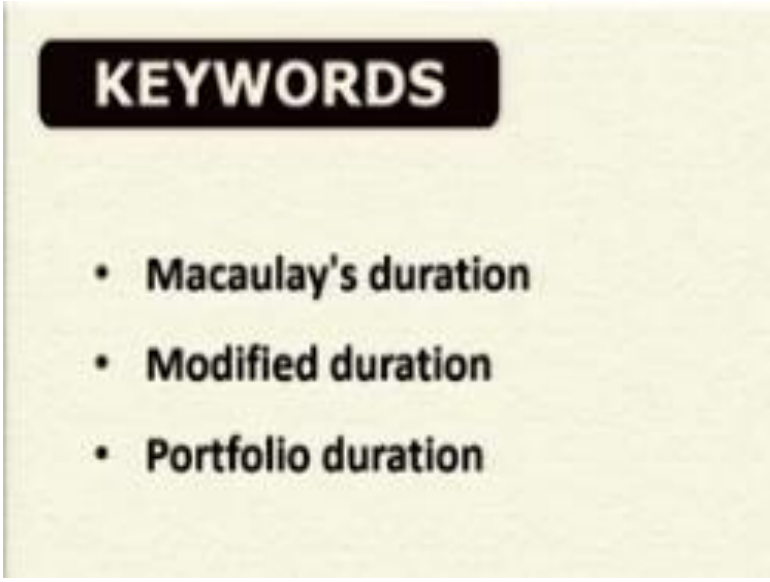

So, you will come across certain concepts or certain keywords with respect to this particular session that is basically your Macaulay's duration, modified duration, and the portfolio duration. So, these are the different keywords you will come across throughout this particular session.

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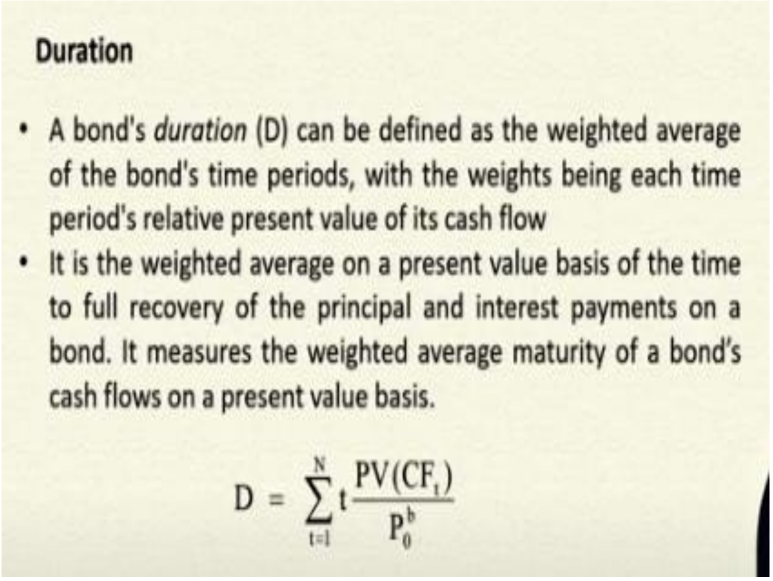

So, let us see that what exactly the bond duration is? So, all of you know that we have a concept called the term to maturity. term to maturity is nothing but the period at which the bond is going to be mature. So, like that the duration is also a time measure, bonds duration is also a time measure but it is basically a weighted time measure. So, in this case what basically we can say that the bonds duration is basically a weighted average of the bonds time period.

Bonds duration is defined as the weighted average of the bonds time period. So how we can give the weights? The weights are basically given on the basis of the time periods relative present value of its cash flow. So, in general, if you see it is nothing but a weighted average on a present value basis of the time to full recovery of the principal and the interest payments on a bond. So, therefore it generally measures the weighted average maturity of a bonds cash flow on a present value basis.

If you look at this particular formula, this formula is used to calculate the concept of the duration. So, here if you see it is written summation t = 1 to n t into the present value of the cash flow at the time period t divided by the price of the bond. So, what basically we do? we calculate the present value of the cash flow in each period. Then, we try to find out this weight by dividing that particular each period time period with respect to the price of the bond then we multiply that with the time.

$D= \sum_{t=1}^{N} t \frac{PV(CF_t)}{{P_0}^b}$

So, then we take a summation of all then we can calculate the duration of this particular bond. So that is why it is called a weighted average of the bonds time period and the weights are basically given on the basis of the present value basis.

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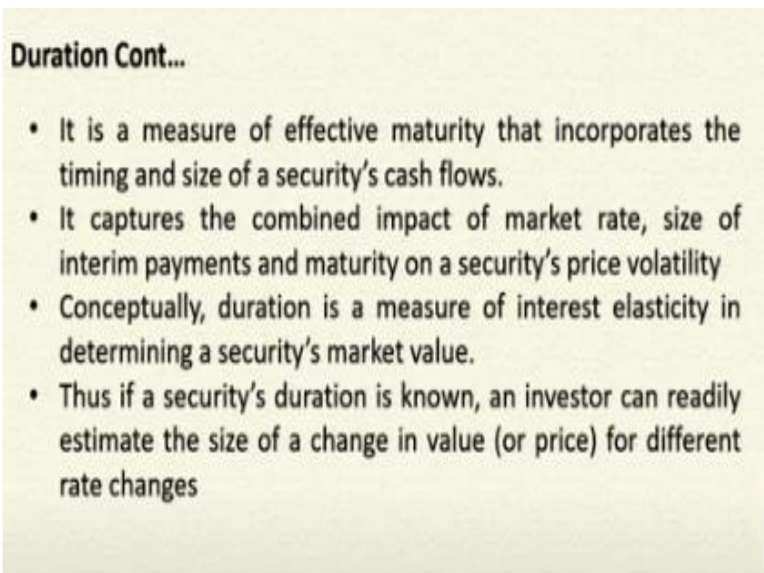

So, let us see that what exactly it measures? If you see that the bonds duration is basically a measure of the effective maturity that incorporates the timing and the size of the bonds cash flow. The size of the bonds cash flow and also the time both the factors are taken into consideration whenever we calculate the duration of the particular bond. So, it captures the combined impact of the interest rate size of the interim payments which is coming out of the coupons and maturity on the bonds price volatility.

Conceptually if you see duration is nothing but it is a measure of the interest elasticity in determining the bonds market value. it is basically measure of the interest elasticity in determining a bonds market value. So, if the securities or the bonds duration is known, then the investor can estimate the size of change in the value or the price of the particular bond for different interest rate changes.

So, if there is a interest rate change, how the bond price is going to be changed, that you can judge if you have the duration data available with you. So that is the basic use of the particular duration. So, that is why the concept of duration is quite important from the risk management perspective. In the previous class if you remember, we have discussed that a particular bond where the price effect is equal to the interest on interest effect or the reinvestment effect.

So, how basically you can choose that bond and what type bond basically will give this kind of scenario, in that case your duration generally helps. So that is why that concept or that particular strategy is called the immunization strategy, that will discuss further. But you just keep in the mind duration is very much relevant. duration is very much important from that perspective.

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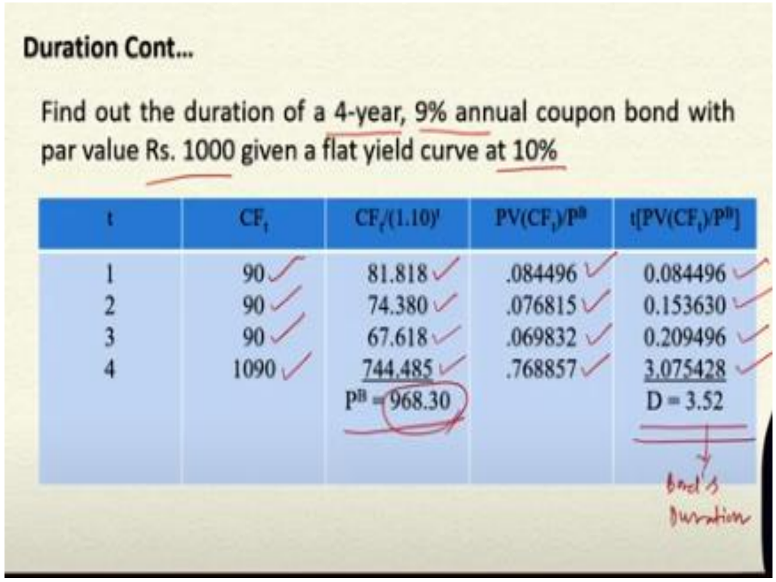

So let us see that how exactly the duration is measured? because I have given you this particular formula. But here if you see that whenever we calculate this duration let we have taken a bond where the term to maturity is 4 years, coupon is 9%, par value of the bond is 1000 rupees and the yield to maturity is 10%. So, in that case if you if you observe let the coupons are paid annually, it is an annual coupon payment bond.

So, in the first year you will get a coupon of 90 rupees. in the end of the first year, second year you get 90, in the end of the third year you get 90 and in the end of the fourth year you get 1090, because 1000 is the par value of the bond. So now what you can do your yield is 10%? so you find out the present value of that particular cash flow, what you got in the first year. So, then it is 90 divided by 1.1 yes then it will be 81.818. then second year end of the second year it will be 90 divided by 1.1 square, then you will get 74.380.

Third year 90 divided by 1.1 to the power 3 you get 67.618. And in the fourth year you get your coupon and as well as the principal. then if you are using your yield for discounting it then you can get your 744.485 right. So, then if you add up all these then you get this price of the bond. So that is basically your price of the bond. so price of the bond has come 968.30. So now what you can do as per the formula what basically we discussed in the previous slide, you find out the proportion of each cash flow with respect to that particular price.

Then this one will be 81.818 divided by 968.30, that will give you this one this will give you 74.380 divided by 968.30, you will get this. Then 67.618 divided by 968.30 will give you this then finally 744.485 divided by 968.30, that will give you this. Now, what you can do you multiply this value with respect to the time so 0.84496 by 1 you get this 0.76815 multiplied by 2 you get this then 0.69832 multiplied by 3 you get this then 0.768857 into 4 you get this.

Then approximately your duration has come 3.52 if you add up all these, then you get a summation of all. You get a summation of all then that basically will be 3.52. then this 3.52 is basically the bonds duration, it is a bonds duration .so that is the way the duration of the bond is calculated.

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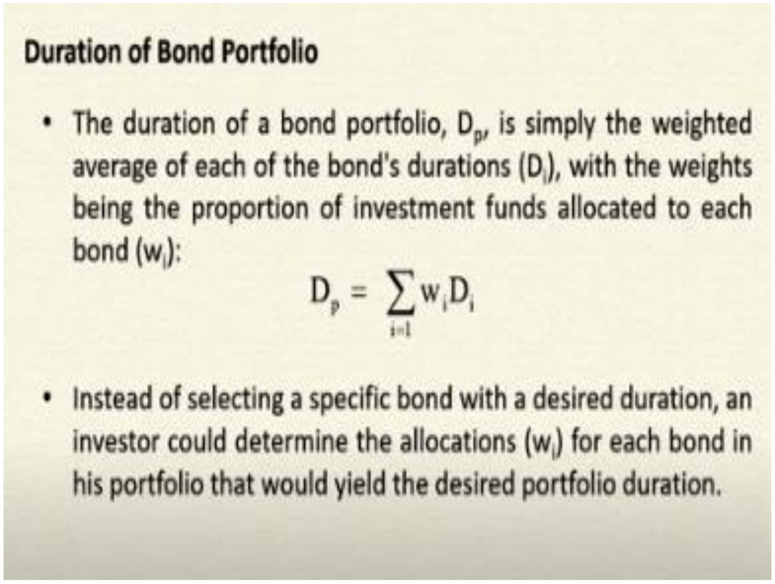

So, now you see how we can calculate the duration of a bond portfolio? here what calculation we have shown? that is basically the duration of a single bond. So, whenever you are calculating the duration of a bond portfolio that is also very simple. it is nothing but the weighted average of each of the bonds division. And how the weights are given? the weights are given on the basis of the proportion of bonds, you have invested in each bond.

$D_p = \sum_{i=1} w_i D_i$

Let you have 3 bonds available. one bond duration is 2, another bond is 3, another bond is four. then let total 1000 rupees you are investing. Then you are allocating let 300 rupees for bond 1, 400 rupees for bond 2, and 300 rupees for bond 3, then in that case what basically you can do? you find out the each proportion 30% 40% and 30%. Then each bond has the duration available, then

0.3 multiplied by 2 plus 0.4 multiplied by 3 plus 0.3 multiplied by 4, that are the duration which has been given to you for each type of bonds. then your portfolio duration can be calculated.

So, instead of selecting a specific bond with a desired duration, one investor also could determine the allocations for each bond in his or her portfolio, that would yield the desired portfolio duration. That also somebody can do instead of investing in a single one, you can choose the multiple bonds and where you can get your desired portfolio duration from that particular context.

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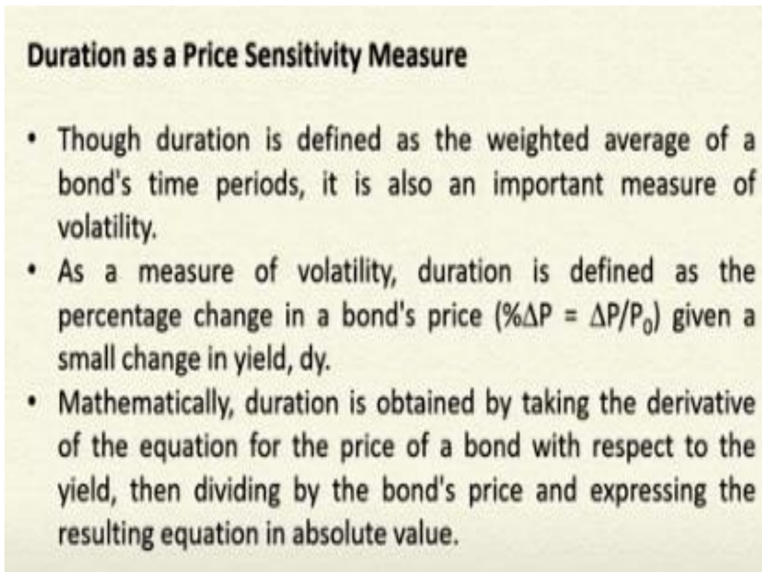

So otherwise already i told you, because the duration has a different use, also it talks about how the movement of the price of the bond is going to be changed. That you can judge if you have the duration data available with you. So, that is what duration is also considered as a price sensitivity measure. So, already we have defined that duration is nothing but a weighted average of a bonds time period. but it is also important measure of the volatility; price volatility of the bond.

So, whenever we are thinking from that perspective, the duration as a volatility measure, in that case, what basically we do or how we can define the duration? In that case, duration is basically defined as the percentage change in the bond price given a small change in yield. That percentage change in the bond price is percentage of delta p. delta p is the change in the bond price. p 0 is the original price of the bond. then your d y which is represented as the small change in the yield or small change in the interest rate.

So, in that case if you want to calculate the duration of the bond mathematically, then what you have to do? you have to take the derivative of the price equation? You take the first derivative of the price equation of a bond with respect to the yield. So, that means with respect to the yield how the price, if there is a change in the yield how the price of the bond is going to be changed?

You take the first derivative of that particular bond price formula and divide by the bond price and express that particular value in absolute form, so that will also give you the value of the duration. So, it is nothing but the first order derivative of the bond price equation with respect to the change in the yield divided by the price of the bond. That basically will give you also the concept of the duration or the measure of the duration.

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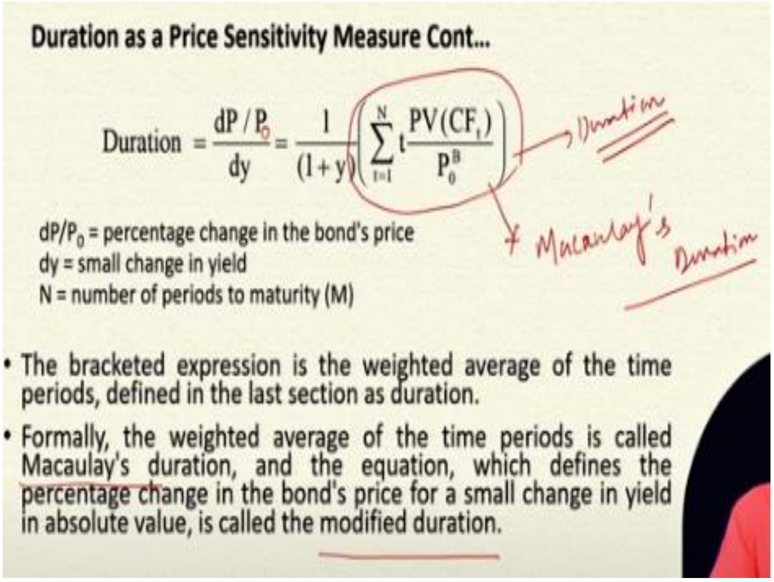

So, in this case just now what basically we discussed then how we can duration can be defined. Duration is equal to your d p by p divided by d y. So that is basically nothing but your 1 by 1 + y the summation of t = 1 to n t the present value of the cash flow at the period t divided by the price of the bond. So, this particular term already we have seen that is basically your duration in the formula whatever we have seen.

$$\text{Duration} = \frac{dp/P_0}{dy} = \frac{1}{(1+y)}\left[\sum_{t=1}^{N} t\,\frac{PV(CF_t)}{{P_0}^B}\right]$$

So, here you have, you are getting extra term, that is 1 by 1 + y you are getting a 1 extra term that is 1 by 1 + y. So, that is why that is little bit different. what formula we have just discussed in the previous slide and now what formula we are deriving here. So, here already I told you d p by p is a percentage change in the price of the bond. So you can write your 0 I think I have so d p by p 0 is the percentage change in the price of the bond.

And d y is equal to your small change in the yield and your n is equal to the number of periods to the maturity that is basically we can also say m. So, with bracketed expression that just now whatever I have written, that is basically is the weighted average of the time period which is defined in the last section as the duration just now what I told you. So, here why there is a change in the formula because the 2 duration concepts are different.

In the first case, whenever we have calculated the duration on the basis of this formula, that is called basically your Macaulay's duration**.** And the equation which basically defines the percentage change in the bond price due to a small change in the yield in the absolute value that is called basically the modified duration. So, this part is called the Macaulay's duration and this divided by 1 + y or multiplied by 1 by 1 + y that are called the modified duration.

So, whenever we are calculating the duration as a price sensitivity measure using the concept of the first order derivative, generally we are calculating the modified duration. And using this formula whenever we are calculating the duration that is basically called the, what we call it that is basically called this Macaulay's duration. These are the 2 different concepts what basically you have to keep in the mind.

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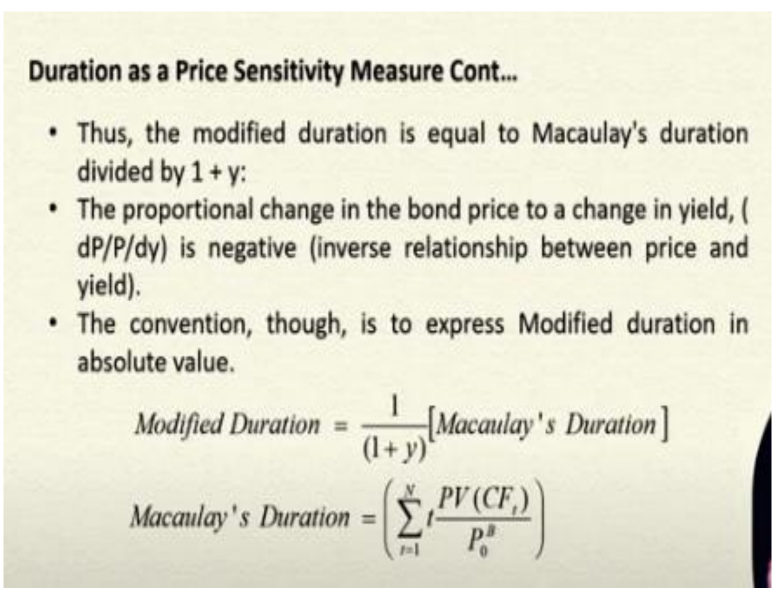

So, what we have seen here? the modified duration is equal to the Macaulay's duration divided by 1 + y. where y basically is nothing but the yield to maturity. So, the proportional change in the bond price to the change in yield that is your d p by p by d y is negative, that means there is a inverse relationship between the price and yield. So that is why there is a convention to express the modified duration in the absolute form right.

Modified Duration = $\frac{1}{(1+y)}[\,Macaulay's\ Duration]$

Macaulay's Duration = $\sum_{t=1}^{N} t\,\frac{PV(CF_t)}{{P_0}^B}$

So already, I told you that modified duration is nothing but the Macaulay's duration divided by 1 + y and the Macaulay's duration is t = 1 to n t into p v or the present value of the cash flow or the period t divided by the price of the bond that part already we have discussed.

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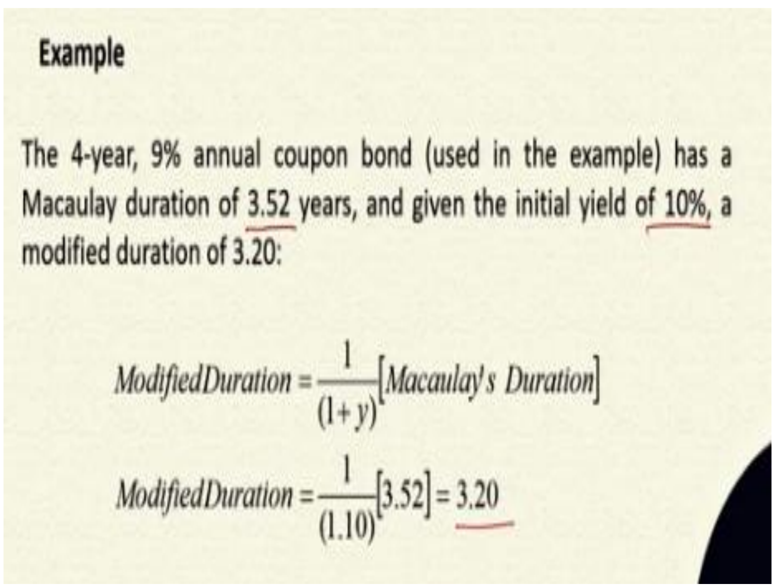

So, let in the same example, whatever we have taken if your Macaulay's duration, we have calculated that is your 3.52 and your yield is 10% then your modifier duration will be 3.52 divided by 1.1 that will give you 3.20. 3.20 is the modified duration and 3.52 was the Macaulay's duration. So, whenever we are using duration as a price sensitivity measure mostly, we are calculating this modified duration not the Macaulay's duration.

Modified Duration = $\frac{1}{(1+y)}[\,Macaulay's\ Duration]$

Modified Duration = $\frac{1}{(1.10)}[\,3.52] = 3.20$

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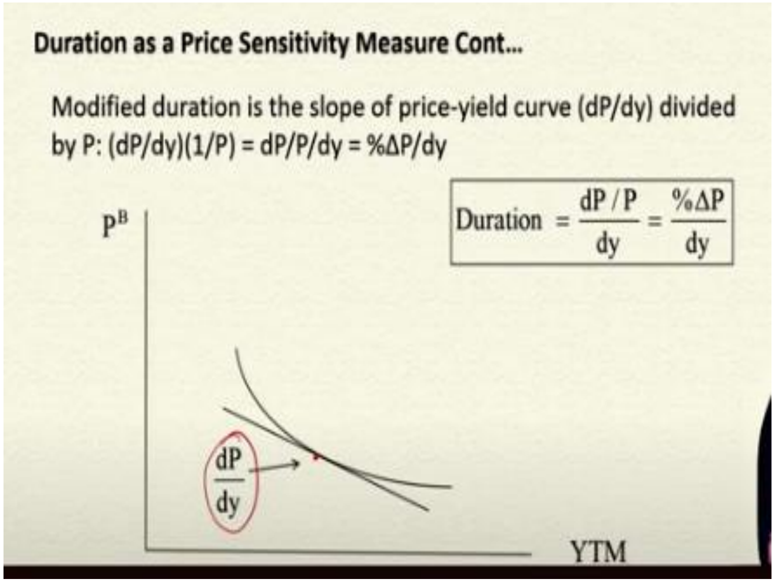

So, that is why if you show it in a diagrammatic way, the modified duration is nothing but is the slope of the price yield curve divided by the p. Because, the slope of the price yield curve is nothing but d P by d y divided by p, that is basically called the modified duration. so the percentage change in the price that is percentage change in percentage of delta p divided by d y. So, this is basically wherever this particular this is the slope of this particular curve price yield curve that is basically your d P by d y.

Duration = $\frac{dp/P}{dy} = \frac{\%\Delta P}{dy}$

Or you can use d P 0 by d y, whatever depends on the notations, whatever you are using then divide by your p that will give you the modified duration of this particular form. So, that is why, if you are measuring the slope of the yield curve and divide it with respect to the price then you can calculate the duration. So already, you know that what is your yield price yield curve; one is your price one axis other axis is the yield to maturity. So, that is basically you call it the price rate curve. that is the way basically the modified duration can be formulated or can be calculated.

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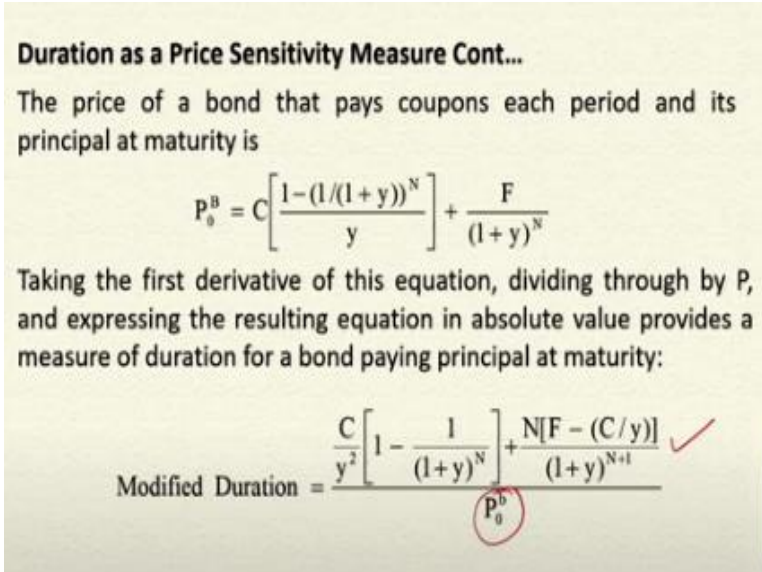

So, if you see that the price of a bond which basically pays the coupon in each period and its principle at the maturity. Already, we know the formula how the price is calculated? that is your p 0 b is equal to c into 1 - 1 by 1 + y to the power n divided by y + f by 1 + y to the power n. N is the term to maturity, that already we have discussed.

$$P_0^B = C\left[\frac{1-\left(\frac{1}{1+y}\right)^N}{y}\right] + \frac{F}{(1+y)^N}$$

So, if you take the first derivative of this equation and divide that particular thing through p then express that particular resulting equation in the absolute value that will basically provide you the measure of the duration for a bond paying the principal at the maturity.

So, that basically will get your modified duration is equal to your c by y square into 1 – 1 by 1 + y to the power n + your n into f - c by y divided by 1 + y to the power n + 1. So, that is basically the derivatives first order derivative what you are calculating from this equation. then you divide it with respect to the price that will give you the modified duration of that particular coupon paying bond and the principal is basically paid at the maturity. So, directly you can calculate your modified duration using this particular formula. this particular equation that basically you can do whenever you are using duration as a price sensitivity measure.

$$\text{Modified Duration} = \frac{\frac{c}{y^2}\left[1-\frac{1}{(1+y)^N}\right]+\frac{N\left[F-\left(\frac{C}{y}\right)\right]}{(1+y)^{N+1}}}{{P_0}^B}$$

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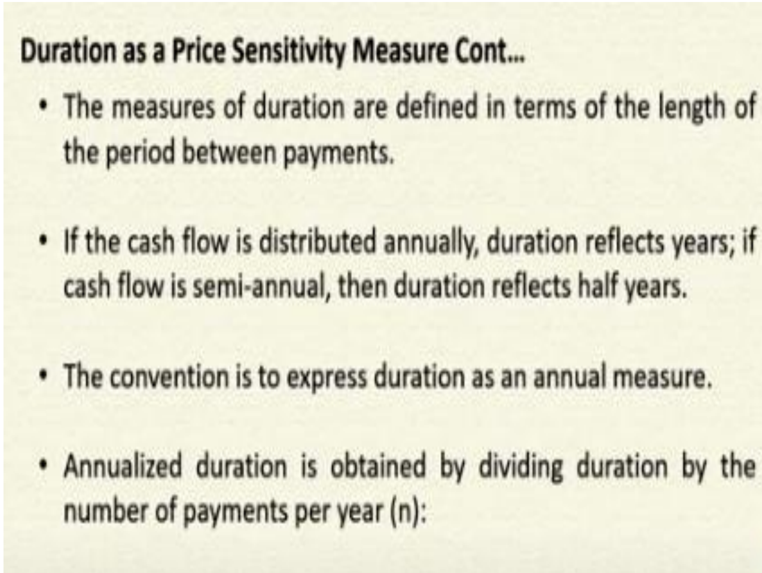

Now, if you see what kind of properties we can get it in terms of the duration? So, the measure of the duration basically is defined in terms of the length of the period between the payments, it is a time measure. If the cash flow is distributed annually, duration is reflected in years. if cash flow is semiannually, then duration basically reflects the half years. But general convention is always duration is expressed as an annual measure.

So, how you can calculate the annualized duration? The annualized duration is nothing but whatever duration you are calculating on the basis of the frequency of the coupon payments that you first calculate and divided by the number of payments per year, that will give you the annual duration of that particular bond.

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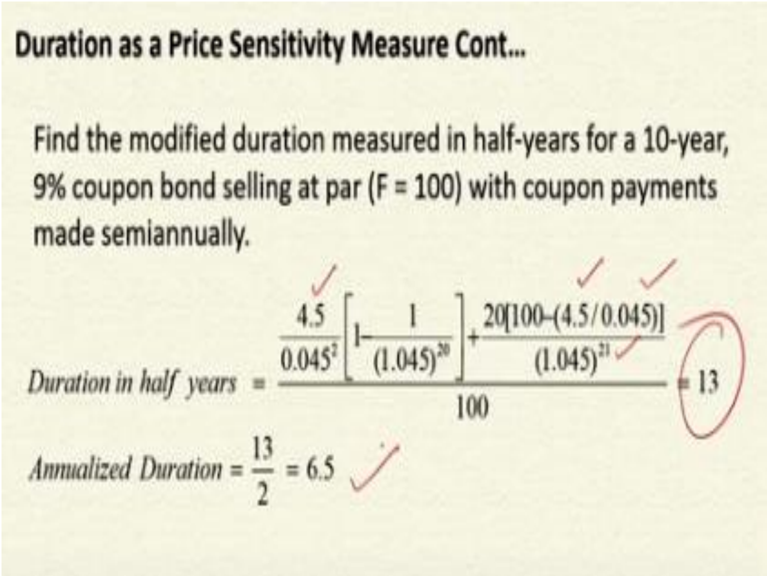

So, let us come to an example. so if you question is let find the modified duration measured in half years for a 10 year, 9% coupon bond selling at par. let f is equal to 100 with coupon payments made semiannually. Then 9% of the coupon so that is why half yearly coupon is c that is 5.5 then your divided by 0.045 square 1 - 1 by 1.045 to the power 20 because 10 years is the maturity.

Now, it is halfway payment that will become n become 20 + your n into a 20 into 100 - f – 4.5 that is the coupon our half yearly coupon divided by your 0.045 it is 9% divided by 2 then divided by 1.045 to the power n + 1 that is your 21 divided by 100 that is the basically the price of the bond because bond is issued at par. Then, you will get 13 approximately it will get 13 so this particular is the duration in the half years duration in half years. Then your annualized duration will become 13 by 2 that are 6.5.

Duration in half years = $= \dfrac{\frac{4.5}{0.045^2}\left[1-\frac{1}{(1.045)^{20}}\right]+\frac{20[100-\left(\frac{4.5}{0.045}\right)]}{(1.045)^{21}}}{100} = 13$

Annualized return = 13/2 = 6.5

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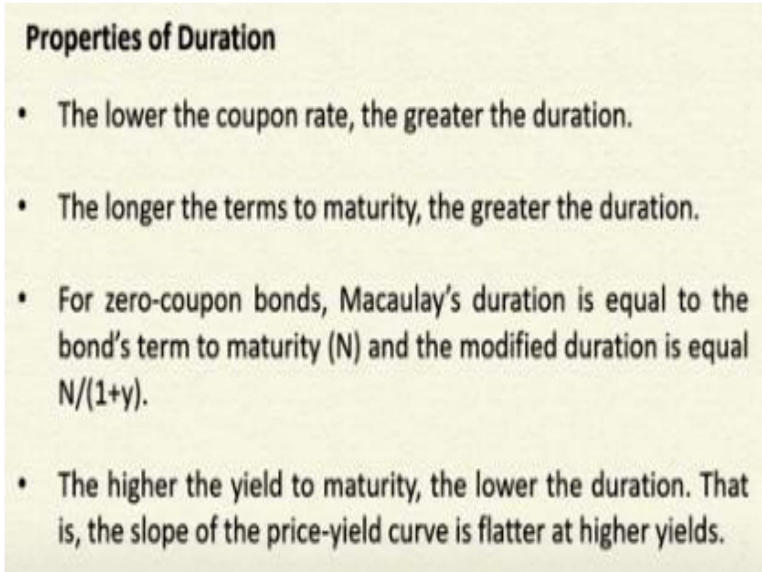

So, then what are the properties of the duration? we have certain properties; lower the coupon, greater the duration. longer the term to maturity, greater the duration. Other things remain same if, you are comparing between the 2 bonds, lower the coupon rate greater will be the duration, longer the term to maturity. Obviously, if the term to maturity will go, over more than duration will be more.

For a zero coupon bond, this Macaulay's duration is equal to the bonds term to maturity. anyway there is no periodic coupons available. And the modified duration is n by 1 + y that yield what you are calculating? that yield if you divide it with respect to that term to maturity or the duration, then you can get it the modified duration.

Higher the yield to maturity, lower the duration that means the slope of the price will curve is flatter at the height the slope of the price. Yield curve is generally flat at the higher yield to maturity. So, these are the different observations or different properties, what you can get with respect to the duration principle.

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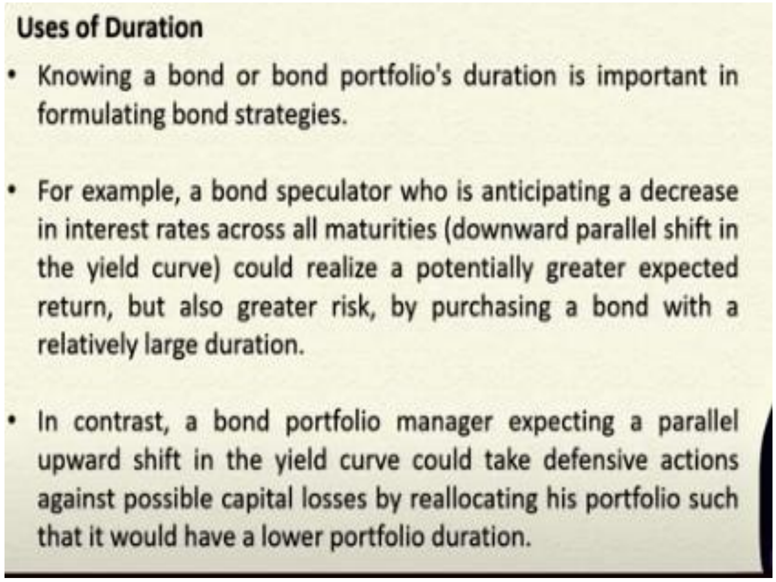

So, if you know the bonds duration, you can formulate the bond portfolio strategy how? For example, let a bond speculator who is anticipating a decrease in the interest rate across all the maturity that means there is a downward shift of the yield curve. In that case, they could realize a potentially greater expected return but also greater risk by purchasing a bond with a relatively large duration.

Just now, we have seen but in contrast a bond portfolio manager who is expecting a parallel upward shift in the yield curve they can go to for a defensive strategy. Defensive action against the possible capital losses by reallocating their portfolio such that they will have a lower portfolio duration. So, this is the way duration can be used as a bond portfolio strategy measure, that actually you have to keep in the mind.

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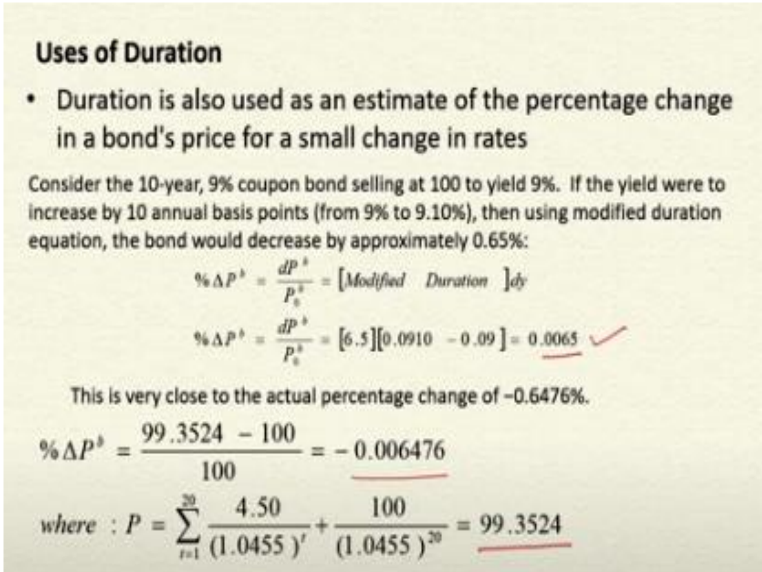

So, the duration is also used as an estimate of the percentage change in the bonds price for a small change in the interest rate. For example, there is a 10 year bond ,9% coupon which is selling at 100 rupees to yield 9% if the interest rate will increase by later 10 annual basis point, that means from 9% to 9.1%. Then if you go by the using the modified duration equation if you find this bond price will decrease approximately by 0.65%.

$\%\Delta P^b = \frac{dp^b}{{P_0}^b}$ = [Modified Duration] dy

$\%\Delta P^b = \frac{dp^b}{{P_0}^b}$ = [6.5] [0.0910-0.09] = 0.0065

So, that is 0.65% what you can get so that is very much close to the percentage change in the actual percentage change at the bond price. Because, the bond price is you can calculate let, we are using the same example then their coupon is 4.5 semiannually if you calculate the price of the bond that will you will get 99.3524 and percentage change in the bond price will be 99.3524 - 100 divided by 100 that is approximately 0.6476 %.

$\%\Delta P^b = \frac{99.3524-100}{100}$ = -0.006476

Where: P= $\sum_{t=1}^{20} \frac{4.50}{(1.0455)^t} + \frac{100}{(1.0455)^{20}}$ = 99.3524

But, if you are using the modified duration directly you got your duration modified duration is 6.5. in the previous example,e directly you can put this percentage change in the price of the bond then you get 65.65%, which is a very much close to this.

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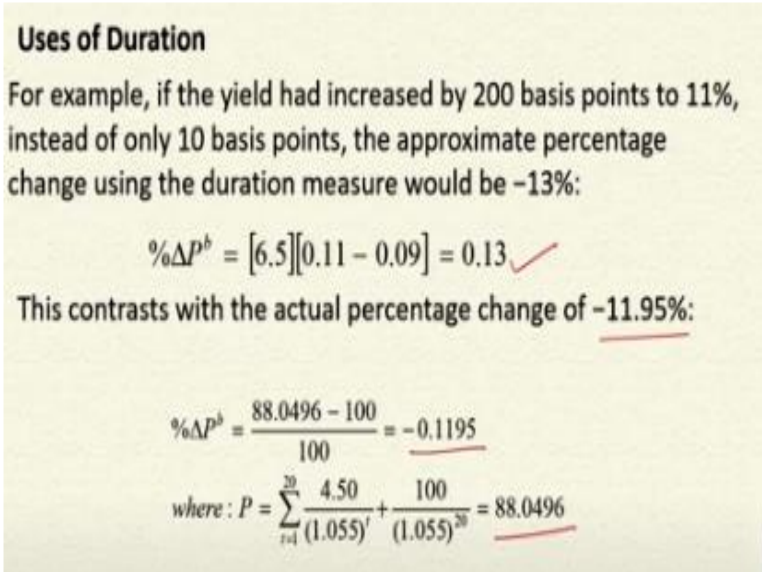

But little bit, if you increase your interest rate change let 200 basis point let it has increased to 11% in that case what you find? You find that the price will change by point you will get this .13 but in true sense it is basically 11.95%. If you calculate the price of the bond you will get 88.0496 then effectively it is 11.95%.

$\%\Delta P^b = \frac{dp^b}{{P_0}^b}$ = [6.5] [0.11-0.09] = 0.13

$\%\Delta P^b = \frac{88.0496-100}{100}$ = -0.1195

Where: P= $\sum_{t=1}^{20} \frac{4.50}{(1.055)^t} + \frac{100}{(1.055)^{20}} =$ 88.0496

So, that means what we can conclude, if the interest rate will change by in a larger amount then this modified duration may not help you or may not give the exact value of change in the bond prices but in the small change that thing can be used.

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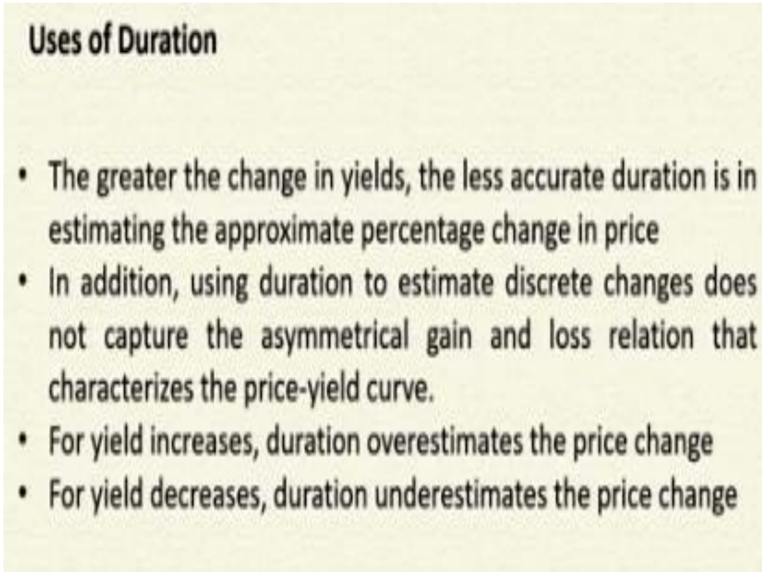

So, greater the change in yield the less accurate duration is estimating the approximate percentage change in the price. So, using the duration to estimate the discrete changes does not capture the asymmetrical gain and loss relation that characterize the price yield curve. For yield increases duration over estimates the price change for you decreases duration underestimate this price change.

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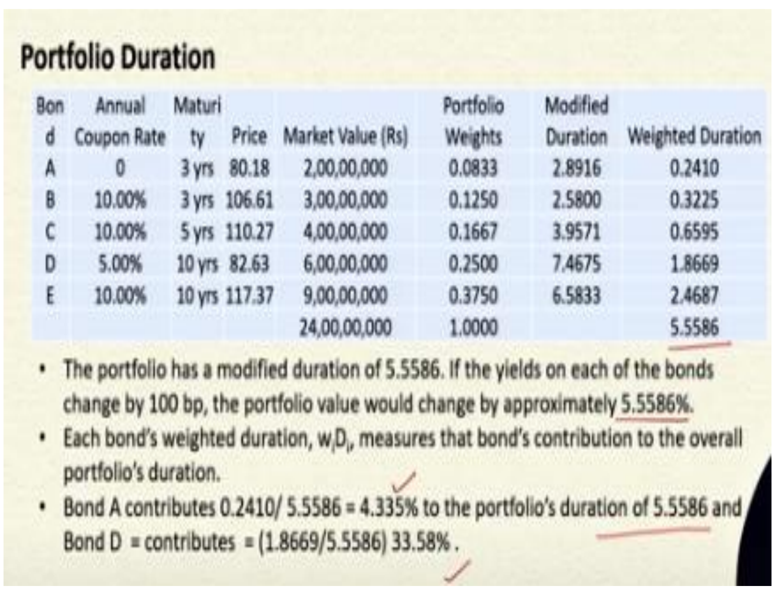

Bond	Annual Coupon Rate	Maturity	Price	Market Value (Rs)	Portfolio Weights	Modified Duration	Weighted Duration
A	0	3 yrs	80.18	2,00,00,000	0.0833	2.8916	0.2410
B	10.00%	3 yrs	106.61	3,00,00,000	0.1250	2.5800	0.3225
C	10.00%	5 yrs	110.27	4,00,00,000	0.1667	3.9571	0.6595
D	5.00%	10 yrs	82.63	6,00,00,000	0.2500	7.4675	1.8669
E	10.00%	10 yrs	117.37	9,00,00,000	0.3750	6.5833	2.4687
				24,00,00,000	1.0000		5.5586

So already, we have explained this portfolio duration here one example has been given. So here, it is the coupon rate it is the maturity, it is the price these are the market value of this particular bonds. So, this on the basis of the market value total market value the weights are calculated. So, modified duration of each bond has been given then this multiplied by this weight, the weighted duration of each bond is calculated. Then finally basically we are calculating this portfolio duration of that particular bond.

So, the portfolio has a modified variation of 5.5586 that means if yield on each of the bonds change by 100 basis point; then the portfolio value will change approximately by 5.55886%. So, each bonds weighted duration measures the bonds contribution to the overall portfolios duration. So, bond a, contributes this percentage that is 2410 divided by this portfolio duration of 5.86 and bond d contributes this that is the way basically we can calculate this or use this concept of the portfolio derivation.

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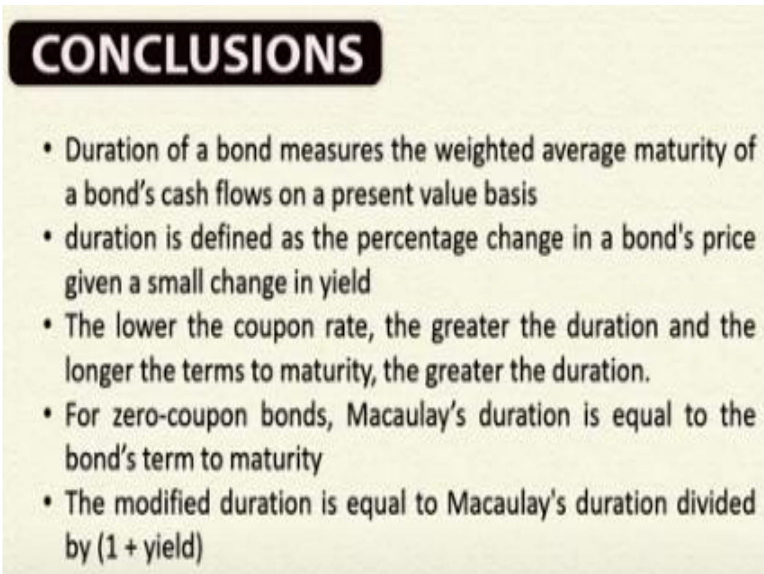

So, in the conclusion what we find? that the duration of a bond measures the weighted average maturity of a bonds cash flow on the present value basis. Or it is also defined as a percentage change in the bond’s price given a small change in the yield. Lower the coupon, greater the duration. longer the term to maturity, greater the duration. For a zero coupon bond Macaulay’s duration is equal to the bond’s term to maturity. And the modified duration is equal to the Macaulay’s duration divided by 1 + yield.

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So, these are the references what you can go through. thank you.