Management of Fixed Income Securities
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Module No # 05
Lecture No # 21
Determination of Theoretical Spot Rate and Forward Yield Curves

Welcome back! So, in the previous class we discussed about the features of the pure expectations theory then as well as the liquidity preference theory or the liquidity premium theory. So, in this particular session, we will be discussing about the how to determine this theoretical spot rate and the forward yield curve?

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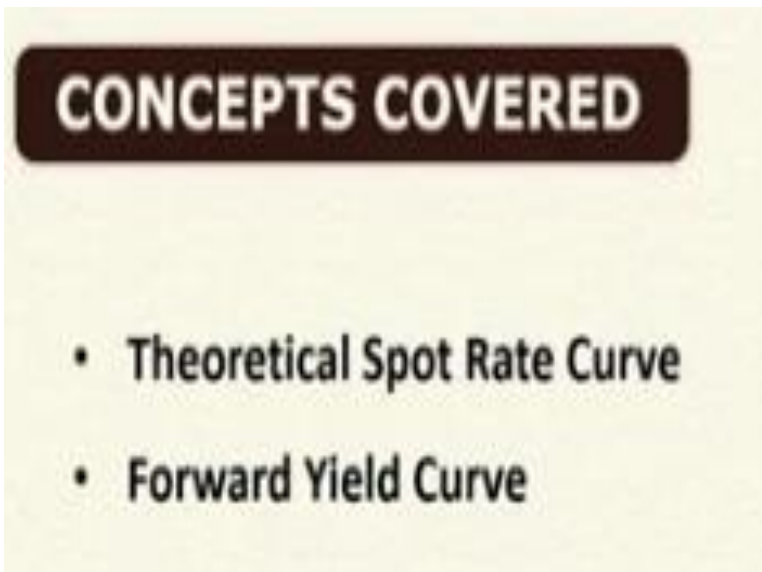

So, here mostly the 2 concepts will be covered up. one is your derivation of the theoretical spot rate curve and the forward yield curve.

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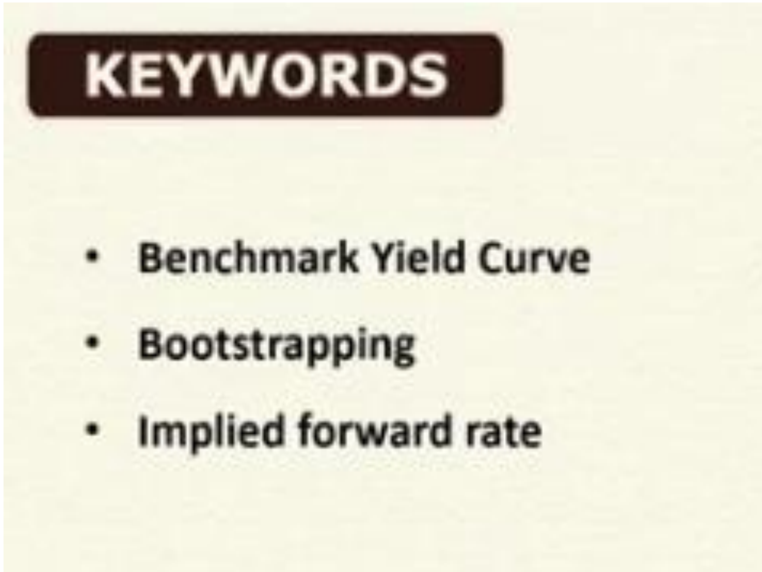

And you will come across certain keywords like your benchmark yield curve, bootstrapping, implied forward rate. again, so all these things basically you will come across in this particular discussion.

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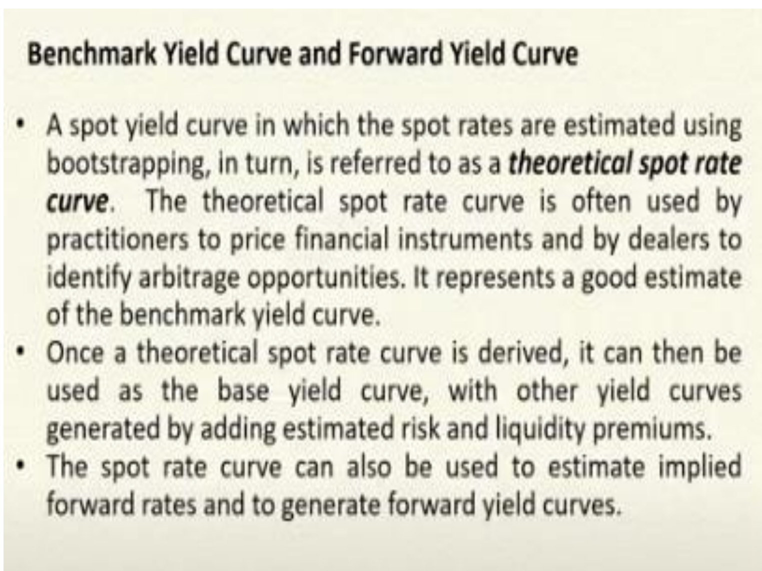

Let me tell you what do you mean by the theoretical spot rate and why it is called a theoretical spot rate? The concept of the theoretical spot rate, basically comes to understand or to derive a base rate. Because, whenever the market participants invest in the market, they have some idea about the basic characteristics or basic behaviour of the yield curve from where they will start. So, to understand the basic behaviour of the yield curve, always we need a kind of base yield curve and that base yield curve is derived from these, portraits which are prevailed in the market.

So, if there is a current spot rate, may be 2-year spot rate or one-year spot rate which is prevailed in the market. And using that particular spot rate we try to determine the shape of the curve in the beginning. And, after that the role of expectations and other things will come into the picture; the liquidity premium and other concepts will come into the picture. But, in the beginning, we have to understand that, how the shape of the yield curve will look like using the spot rate which is prevailed in the market.

So, if you recall to some extent, we discussed about this part. whenever I was discussing with you about the methods of bootstrapping, which basically used to calculate this kind of rates which is going to prevail in the future. The same thing I will just repeat. but why basically we are discussing this? because that concept is basically used to also derive the forward yield curve. So, using a bootstrapping method, a spot rate yield curve can be derived. So, that is basically what in our language, we call it the theoretical spot rate curve.

And already I told you. this is used by the practitioners to price these financial instruments and also used by the dealers to identify the arbitrage opportunity. That is why it is called a bench mark yield curve, using the bootstrapping method, whatever spot rate yield curve we are deriving, that is used as a benchmark yield curve. So, once the theoretical spot rate curve is derived or determined then, this can be used as a base rate yield curve or base income and other yield curves are generated by adding the estimated risk and the liquidity premiums.

Very minutely you observe that, unless that base yield curve is not available, then where are these premiums and to which rate this premiums will be added? So, because of that, the base rate yield curve is very much important. And also, the particular spot rate curve is used to estimate the implied forward rates and also it will generate the forward yield curve. So, it has lot of use. So, the benchmark yield curve or the spot rate or the theoretical spot rate yield curve determination is very much important.

Whenever we are trying to determine the long-term rates in the market or the investors are trying to price the securities which are available in the market, which are basically depending upon the interest rates which are prevailing in the market.

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Benchmark Yield Curve and Forward Yield Curve

Security	Type	Maturity	Semi-Annual Coupon	Annualized YTM	Face Value	Current Price	Spot Rate
1	Zero Coupon	0.5		5%	100	97.561	5.00%
2	Zero Coupon	1.0		5.25%	100	94.9497	5.25%
3	Coupon Bond	1.5	2.75	5.5%	100	100	5.551%
4	Coupon Bond	2.0	2.875	5.75%	100	100	5.577%
5	Coupon Bond	2.5	3	6.00%	100	100	6.03%
6	Coupon Bond	3.0	3.125	6.25%	100	100	6.30%

There are two zero coupon with maturities of six months (0.5 years) and one year, trading at yields of 5% and 5.25%. These rates can be used as spot rates (S_t) for maturities of 0.5 years ($S_{0.5}$) and one year (S_1). The other bonds shown in the exhibit are coupon bonds assumed to be trading at par and therefore with yields equal to their coupon rates.

So, in that context let us take one example and we will see that, how this particular thing is working? here we have taken 6 assets. The first 2 assets are the zero-coupon bonds and next four assets are the coupon bearing bonds. And the maturity period is given. let the first zero coupon bond the maturity period is 6 months, then next one is 1 year, then 1.5 years, then 2 years, 2.5 years and 3 years only 6, 6 months maturity is increasing.

First 2 bonds are the zero-coupon bonds so they do not pay any coupons in between, that is why there is no coupon involved in that and another four bonds they are giving certain coupons. So, because these are all 6 months basis given, that is why here it is the semiannual coupon payments have been given to you 2.75, 2.875, 3, 3.125. so, these are the semiannual coupons.

Now, the YTM; the yield to maturity of all those bonds are given. For 6-month zero coupon bond yield to measure it is 5%, for one year it is 5.25%, for 1.5 coupon bearing bonds, it is 5.5, 5.75, 6% 6.25%. only 0.25% are increasing on the basis of the maturity. The face value of the zero coupon bonds are 100 rupees and we are assuming that coupon bearing bonds are issued at par. That is why, the phase value of the bond is basically is given 100 and the current price is also 100.

So, these are the current prices. So, these are also 100 for the coupon bearing bonds but because we are assuming these bonds are issued at par. And already, you know that these are the price of the zero coupon bonds because the face value is given, this annualized YTM is given. Then, you can find out your price on this face value divided by 1 + r to the power t your r is given here. Right?

So, in that case you can find out the price of the zero-coupon bond and here, we have calculated that the price of zero these 2 zero coupon bonds with the different maturity is 97.561, one is 94.947. Now our job is or our basic objective is to calculate this, these rates how these rates are calculated? Already, the rates we have mentioned here but our basic discussion is how to calculate this spot rates of the different zero-coupon bonds or the coupon-bearing bonds which are traded in the market?

So already, below it is already highlighted there are 2 zero coupon with maturity 6 months. zero coupon bonds basically with a maturity of 6 months and one year which are trading at yields of 5%, 5.25%. you can mention, it is zero coupon bonds. And these rates can be used as a spot rate because generally the zero-coupon bonds rates are used as a spot rate that already we have discussed in the previous sessions.

So that is why the, for maturities of 0.5 here we are representing at s 0.5 and for 1 year maturity we are representing at s1. Right? And the other bonds which are reflected here these are the coupon bonds that trading at par. So, that is why they therefore their yield is equal to their coupon rates. And also, the price is also equal to the face value of the bond. is it clear? So, then our objective is to how to calculate or how to find out these spot rates which are already given to you?

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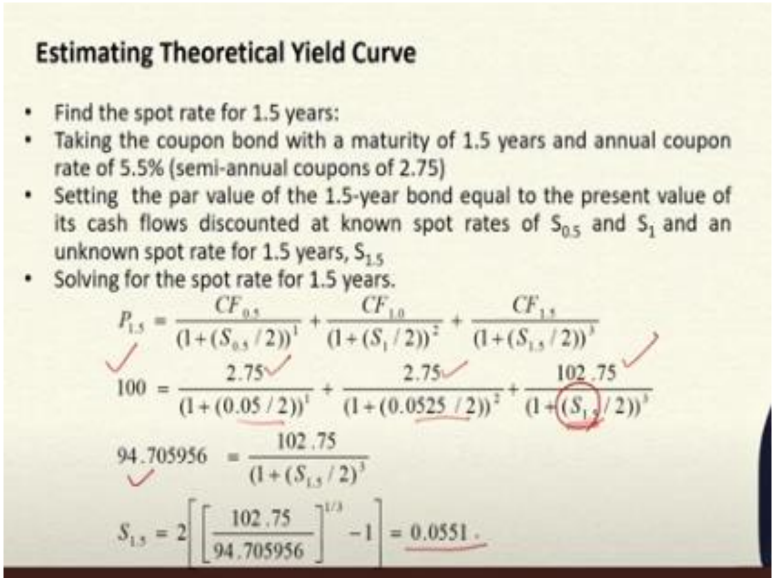

So how we can calculate? let find the spot rate for 1.5 years. let the question is find the spot rate for the 1.5 years? So, if you are going to calculate the spot rate for 1.5 years, how we calculate? Taking the zero-coupon taking the coupon bond with a maturity of 1.5 years and your annual coupon rate of 5.5 % that means semi-annual coupon is 2.75 .right? 5.5 divided by 2 that is 2.75. Then, you set the par value of this 1.5 year bond equal to the present value of its cash flows which are discounted at known spot rates.

There are 2 spots are available one is with a maturity period 0.5 and another one is 1 year S 0.5 and s 1. And one annual spot rate for 1.5 years. is it clear? that is s 1.5. So, now what basically you can you can say? that p 1.5 years the price of the bond of the maturity 1.5 years should be equal to your cash flow at 6 months. That is 0.5 divided by your 1 + s 0.5 divided by 2 to the power 1 + C F 1 divided by 1 + S 1 divided by 2 to the power 2 + c f 1.5 divided by 1 + S 1.5 divided by 2 to the power 3. Because it is semiannual coupons what we are considering.

Now we say we know the price of the bond which is issued at par that is 100, price of the bond is known. And the cash flow at 6 months for a first zero coupon bond which is S. what basically we are getting? that is 2.75 second also we are getting 2.75 are getting my point. The zero-coupon bonds rates will be considered below but above the cash flows what you are getting from the coupon bearing bonds that is your 5.5% annually that means that is 2.75 semiannually.

The first 6 months you will get 2.75, second 6 months you got 2.75 and obviously at the maturity you will be getting your face value that is 100 + 2.75; that is 102.75. Which are the rates at which it will be discounted the spot rate for S 0.5 that is given here. That is 0.05 divided by 2, that is the spot rate for the 0.5 years bond, that is the spot rate for the one year bond zero coupon bond. And this rate we have to determine that s 1.5 that basically we have to determine.

So, now what basically we have seen whenever these 2 things we have calculated we got 94.705956, that basically we have taken from the other side. That is 100 - this will we are getting this much 94.705956 that will be equal to 102.75 divided by 1 + S 1.5 divided by 2 to the power 3. Then from there we can calculate that your s 1.5 that will be calculated on the 6 months basis you multiplied by 2. That will give you 0.0551, which is basically spot rate for the coupon bearing bonds, which maturity is 1.5 years. is it clear?

Estimation of spot rate for 1.5 years bond:

$$P_{1.5} = \frac{CF_{0.5}}{(1+(s_{0.5}/2))^1} + \frac{CF_{1.0}}{(1+(s_{1.0}/2))^2} + \frac{CF_{1.5}}{(1+(s_{1.5}/2))^3}$$

$$100 = \frac{2.75}{(1+(0.05/2))^1} + \frac{2.75}{(1+(0.0525/2))^2} + \frac{102.75}{(1+(s_{1.5}/2))^3}$$

$$94.705956 = \frac{102.75}{(1+(s_{1.5}/2))^3}$$

$$S_{1.5} = 2\left[\left[\frac{102.75}{94.705956}\right]^{\frac{1}{3}} - 1\right] = 0.0551$$

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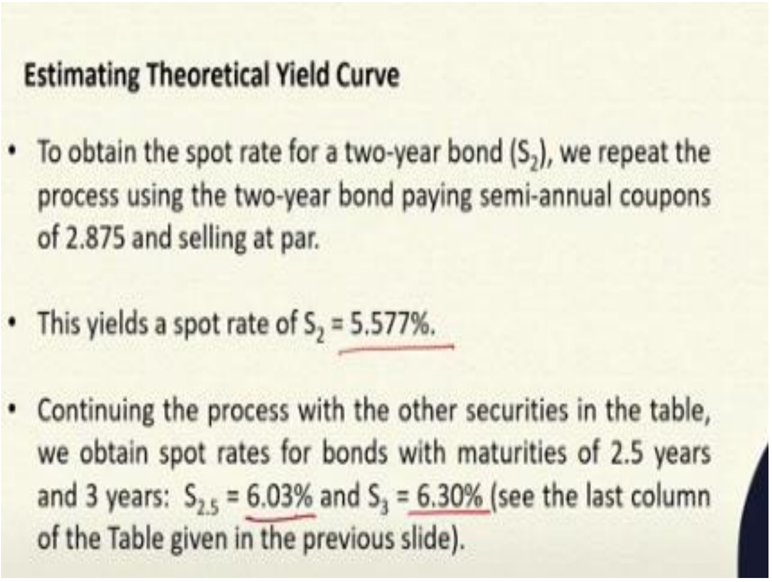

Like that, you can calculate also for the other maturity period bonds the spot rates can be calculated. The spot rate for the 2-year bond, we repeat the process using the 2 year bond paying semi-annual coupons, that is 2.875 selling at par. Then, now you got your spot rate for 1.5 years. that spot rate now can be used for calculations of the spot rate for the 2-year maturity period. And whatever to your maturity spot rate you will find that is your 5.577%, then that can be used to calculate your 2.5 years maturity bonds.

Then again, whatever rate you will get it for the 2.5 years that we got it S 2.5, 6.03% that 6.03% can be used again to calculate for the 3 years spot rate that is 6.3% like that. You can see all these things are already mentioned in the last column of this particular table which has given in the previous slide.

So, this is the process whatever we have followed to calculate the spot rates of the different maturity and only we have 2 zero coupon bonds which are available, which we consider at S 0.5 and S 1 and from there the other rates are calculated.

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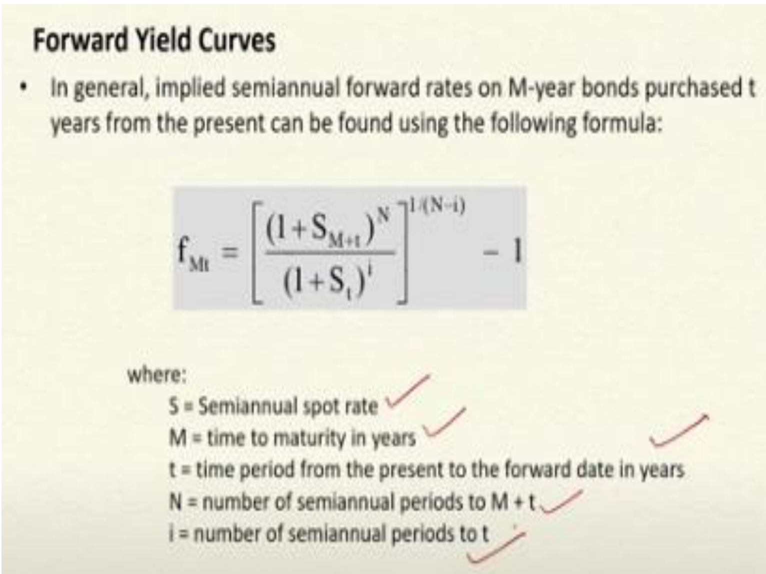

Once this is calculated what basically we can do? we can calculate the forward yield curves? So, I am here basically, we are giving you the general formula for deriving this forward yield. So, in general, if you are using implied semiannual forward rates on m-year bond purchased t years from the present. Let then this can be always found by using this formula your f m t = 1 + S M +

t divided by 1 + S t 1 + S M t to the power n divided by 1 + S t to the power i whole to the power 1 by N - i – 1.

Forward yield = $[\frac{(1+s_{M+t})^N}{(1+s_t)^t}]^{1/(N-i)}$- 1

And here, your S represents your semiannual spot rate, m represents time to maturity in years, t represents the time period from the present to the forward rate in years. N is equal to your number of semi-annual periods to m + t then i is equal to your number of semiannual periods to the t. So, this is the general formula. So, let us apply this and find out the forward rates and from this the forward yield curves can be determined.

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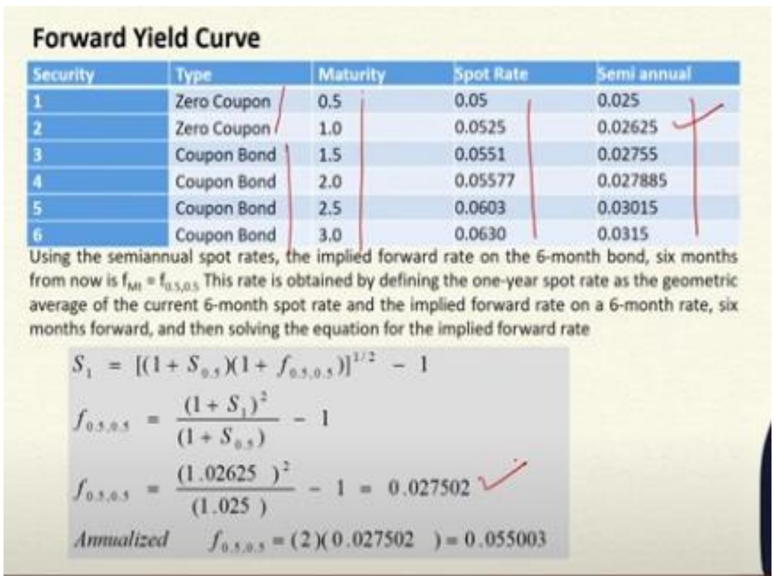

We continue with our same example. Already, you have seen that we have 6 bonds, 2 zero coupon bonds and another four coupon bonds. These are the maturity which has given to you these are the spot rates whatever we have calculated. So, these are the spot rates and these are the semiannual rates from here this divided by 2 we have basically calculate the semiannual rates.

6 months from now, the maturity is 6 months and 6 months from now that is basically t so let we are representing it f 0.5 0.5. Forward implied forward rate on a 6 months bond, 6 months from now then, how basically we calculate it? This can be obtained by defining the one-year spot rate as the geometric average of the current 6 month spot rate. Let the first question is using the semi-annual spot rate what will be the implied forward rate on the 6 months bond?

And the implied forward rate that already we know this 6 month rate implied forward rate on a 6 month rate that is called the 6 months forward. Then your S 1 is nothing but your 1 + S 0.5 into 1 + f 0.5, 0.5 to the power 1 by 2 - 1 that already you know. We have already discussed that then from there if you want to calculate your f 0.5 0.5 that means your implied forward rate on the 6 months bond the 6 months from now. That will be 1 + S 1 whole square divided by 1 + S 0.5 - 1 and your S 1 = 0.2625 0.02625 the semiannual basis this one.

Then it is 1.02625 to the power 2 divided by 1.025 which is basically your spot rate for a 6 months bond that is 0.025 in the semiannual basis. Then if you solve this then you will find this is 0.27502 right. So if you make it annualized then you will get your 2 into 0.027502 that is 0.55003 and approximately this values are calculated because this is a very the fractions are quite long. So, like that you can calculate the other rates the other implied forward rates can be calculated.

$S_1 = [(1+S_{0.5})(1+f_{0.5\ 0.5})]^{1/2} - 1$

$f_{0.5\ 0.5} = \frac{(1+s_1)^2}{(1+s_{0.5})} - 1$

$f_{0.5\ 0.5} = \frac{(1.02625)^2}{(1.025)} - 1 = 0.027502$

Annualized $f_{0.5\ 0.5} = (2)(0.027502) = 0.055003$

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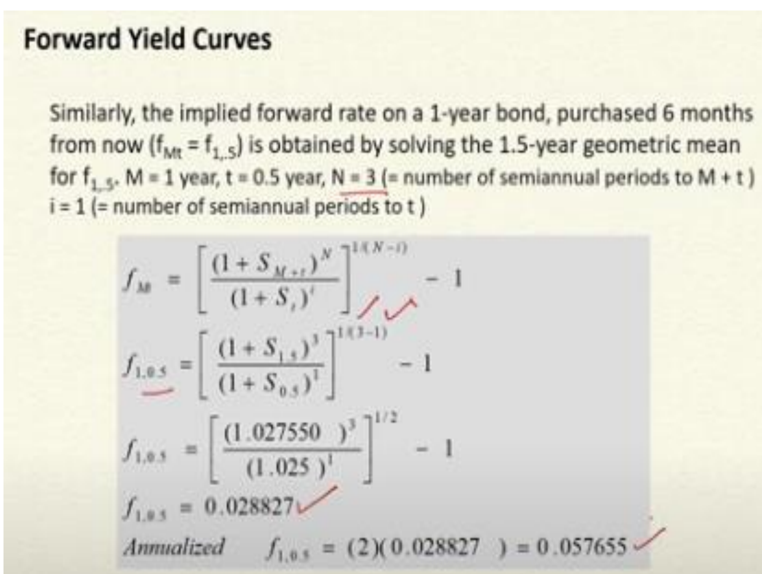

For example, if you see what is the implied forward rate on a one-year bond purchase 6 months from now f M t = f 1, 0.5. Then, how you can solve it? You can solve it by 1.5 were geometric mean for f 1.5 you now. we can use this formula here m = 1 here t = 0.5 n = 3. That is basically the number of semiannual periods to M + t and i = 1. So then your f M t become your m = 1 year t = 0.5 years so that is why f 1, 0.5 right. So that means it is M + t your t = 0.5 years then it is 1 + S 1.5 to the power n.

N = 3 to the power 3 divided by 1 + S 0.5 to the power 1 right. Your t is equal to t = 1 sorry t = 0.5 i = 1 that is why it is 1 + S 0.5 to the power 1 divided by to the power 1 by n – i. N = 3 i = 1 - 1 then if you solve this then S 1.5 already we have that is 0.027550 then 1.027550 divided by 1.025 and here you take the cube of this that 1.027550 to the power t 3 divided by 1.025 to the power 1 to the power 1 by 2 - 1 then your f 1, 0.5 = 0.028827.

So if you make it annualized then it will be multiplied by 2 then you will get 0.057655. So this 5.6755% is the implied forward rate on a one year bond purchased 6 months from now is it.

$f_M = [\frac{(1+s_{M+t})^N}{(1+s_t)^t}]^{1/(N-i)} - 1$

$f_{1.0.5} = [\frac{(1+s_{1.5})^3}{(1+s_{0.5})^1}]^{1/(3-1)} - 1$

$f_{1.0.5} = [\frac{(1.027550)^3}{(1.025)^1}]^{1/2} - 1$

$f_{1.0.5} = 0.028827$

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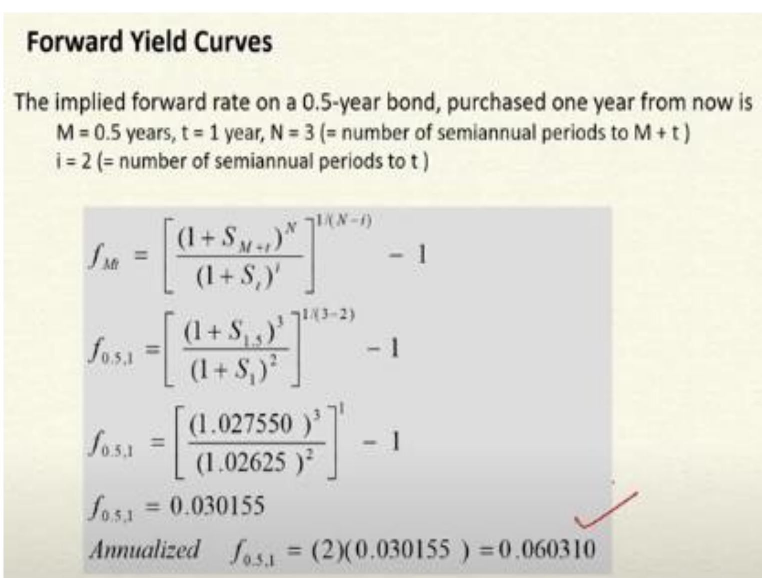

Now for example I will tell you to calculate implied forward rate on a 6 months maturity bond 0.5-year bond purchased one year from now. Then, here your m = 0.5, then your t = 1, n = 3 because the number of semiannual periods to m + t and i = 2. Because the number of semi-annual periods to t because it is a purchased one year from now. Now your S 1.5 that already given that is 0.027550 so 1.027550 to the power 3 divided by your 1 + S 1 to the power 2 because here your t = 1 then it is 1.02625 to the power 2 to the power 1 by n – i. N = 3 and i = 2 that is why it is to the power 1 only - 1 that will give you 0. 030155.So, if you make it annualized then it will become 2 into 0.030155 into 2 that will give you 0.06031. So, like that the other rates can be calculated from this, particular spot rates whatever we have derived.

$f_M = [\frac{(1+s_{M+t})^N}{(1+s_t)^t}]^{1/(N-i)} - 1$

$f_{0.5.1} = [\frac{(1+s_{1.5})^3}{(1+s_1)^2}]^{1/(3-2)} - 1$

$f_{0.5.1} = [\frac{(1.027550)^3}{(1.02625)^2}]^1 - 1$

$f_{0.5.1} = 0.030155$

Annualized $f_{0.5.1} = (2)(0.030155) = 0.060310$

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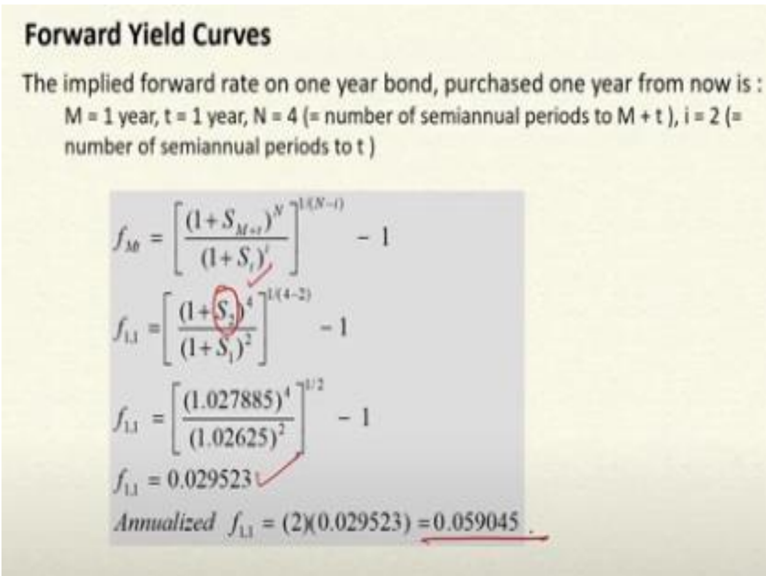

Let I will give a little bit more, longer period or the other kind of forward rates let you are calculating the implied forward rate on 1 year bond percentage 1 year from now. Your M = 1 and also t = 1 so then here what is happening your M = 1 t = 1 n = 4 because the number of semiannual periods to M + t. M = 1 and the t = 1 that means total how much 4 periods 6 months basis we are paying right then i = 2 because the number of semiannual periods to t

So now what will happen your f 1 1 whenever you are calculating you have to consider S m + t, m is equal to 1 t = 1 then we have to calculate s 2 to the power n. N is equal to your periods that is the 4 periods that is why we are taking 4 divide by 1 + S 1 to the power 2 because this is s t. T = 1 then 1 + S 1 to the power 2 whole to the power 1 by 4 - 2 – 1.

Then your f 1 1 will become we have the S 2 rate already available the spot rate we have derived it then 1 + 0.027885 to the power 4 divide by 1.2625 to the power 2 whole to the power 1 by 2 - 1 and that will give you your 029523. Then your if you make it analyze then you can multiply 2 here then you will get this particular rate like that the other rates basically you can calculate from here.

$f_M = [\frac{(1+s_{M+t})^N}{(1+s_t)^t}]^{1/(N-i)} - 1$

$f_{1.1} = [\frac{(1+s_2)^4}{(1+s_1)^2}]^{1/(4-2)} - 1$

$f_{1.1} = [\frac{(1.027885)^4}{(1.02625)^2}]^{1/2} - 1$

$f_{1.1} = 0.029523$

Annualized $f_{1.1} = (2)(0.029523) = 0.059045$

So, we have given the 3 cases the 1-year bond purchased from 1 year bond purchased from 6 months from now 6 months bond purchases 6 months from now. So, like that there are different alternatives what basically, we have taken here and accordingly the implied forward rates are calculated using the spot rates the benchmark rates whatever we have derived in the beginning.

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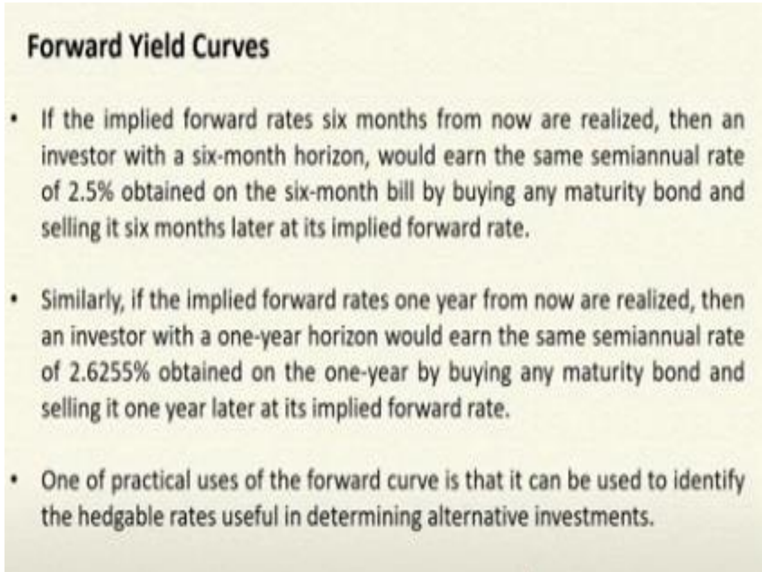

So, how it can be used so if the implied forward rate 6 months from now are realized then an investor with 6 months horizon would earn the same semiannual rate. That is 2.5% obtained on the 6 months bill or the coupon whatever we are taking coupon bond by buying any maturity bond and selling it 6 months later at its implied forward rate. Similarly, if the implied forward rates one year from now are realized then the investor with a one-year horizon would earn the same semi-annual rate of 2.655% obtained on 1 year by buying any maturity bond and selling it one year later I did simply forward it.

So, one of the practical uses of this forward curve is that it can be used to identify the hedgable rates which are useful in determining the alternative investments. So that is the practical use of this particular forward yield curve.

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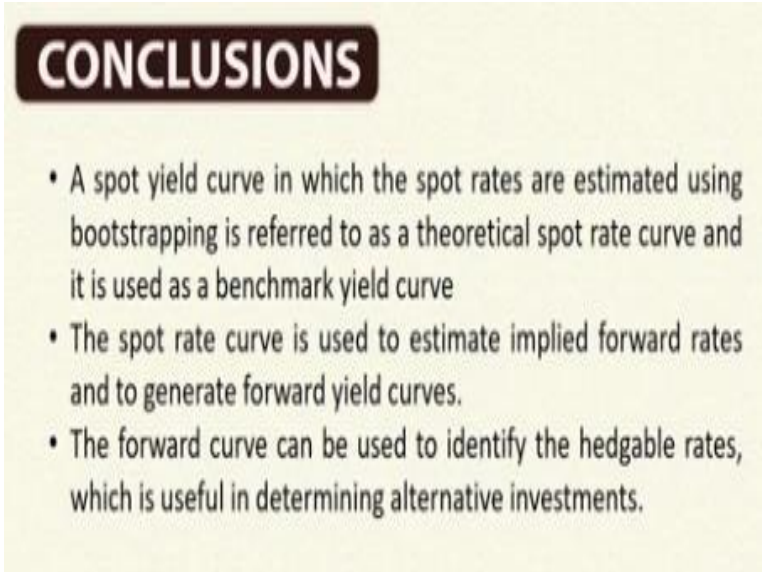

So, what basically we discussed, we derived a spot rate yield curve where the in which this spot rates are estimated using the bootstrapping, which generally called the theoretical spot red curve and also it is used as a benchmark yield curve. And the spot rate curve is generally used to estimate the implied forward rates and also generate the forward yield curves. And already, just now we see that the forward curve can be used to identify the hedgable rates, which is very much important in terms of the determination of the alternative investments.

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So, these are the references you can go through for today's session. thank you.