Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

Lecture – 09
Maximum/Minimum Theorem

Welcome to the ninth lecture of this course on calculus. Today we are going today we are going to learn a very important result. Result of great consequence which can be applied in many places in many ways in fact, can be used to prove the intermediate value theorem which shows that how things are actually connected.

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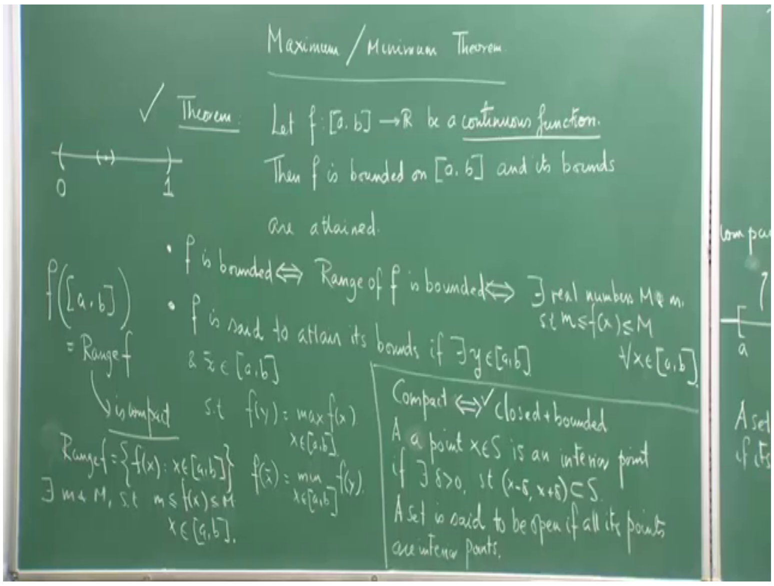

So, let me write down the theorem which is an important one the result. If I write it as a theorem which is the way mathematicians express important results, and then we will try to explain each and every part of it. We are not going to completely prove this result as I said, but show some applications of this, and let me I write it down first and explain it to you.

So, let f from a b to R be a continuous function a continuous function, then f is bounded on a b, and it is bounds are attained, I will explain each of this term separately. So, first we have to talk about what is the meaning of this term bounded. So, we will first discuss what is the bounded set.

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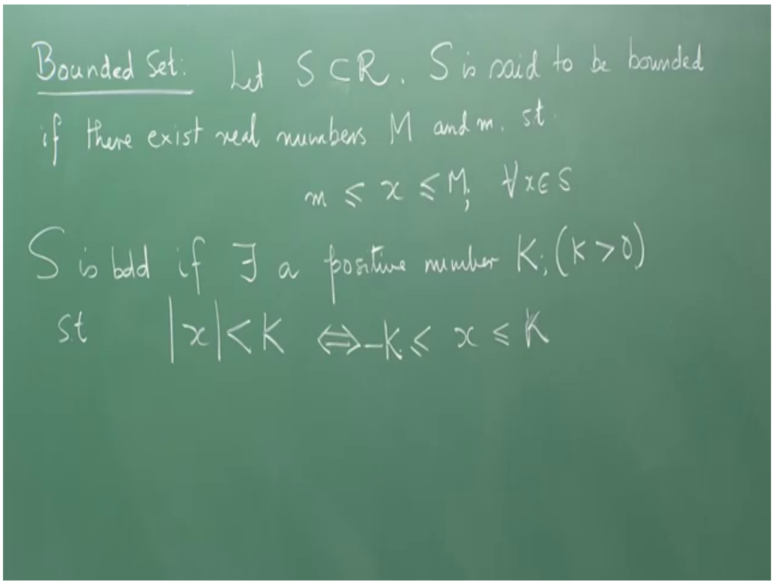

This is the first step to understanding this result. So, we will take a subset s of R. So, s be a subset of R, proper subset by the way. And then we will call s to be bounded if given any x in s. So, or rather I just put it in this way that I will call s to be bounded if I can find 2 numbers, capital M and small m 2 real numbers, such that whenever I take an x from s x must be bigger than m or bigger than small m and less than capital m. So, s is said to be bounded, bounded if there exists real numbers m capital M and small m such that x, whatever x you choose from s the value of x is always lying between these 2 2 real numbers. So, in that sense x is called bind bounded.

Now, there is another way of expression boundedness many people would like to prefer that way, that s is bounded I am bringing a short of bounded bdd if there exists a positive number say k that is greater than 0, such that the absolute value of x is strictly less than k. There is no real difference between these 2 definition of boundedness of a set, because this implies this is same as writing x is lying between k and minus k. So, that is the thing they wouldn’t want to say that there is really no major difference. So, whichever way you can use it. I introduce 2 terms call the maximum and minimum of a function. So, a point y in a b is said to be the maximum is said to be the maximum of f over a b, if f of x is less than equal to f of y for all x in a b.

So, sometimes we write this as f of y is equal to the max of f x x element of a b. This is another notational way of writing what we have just written down. So, that goes for the maximum. So, I would expect that you would definitely define beneath the notion of minimum. So, that if you are you do not you are not sure how to do it. So, I just again do it a point x bar in a b is a minimum of f over a b if f of x bar is less than f of x for all x in a b.

So, one is the smallest functional value one is the largest functional value. So, these things sometimes also written as f of x bar is the minimum value of f of x, where x ranges over a to b. It might look just a symbolical thing, but it has this (Refer Time: 08:28) class of problem. So, are usually important in optimization and in many many practical applications. So, you have this basic idea now written down. So, first what is mean by f is bounded? So, f is bounded this thing this term implies that the range of f, f is bounded.

Obviously you know by now what is the meaning of range of f. And so, I am not going to write down what is the meaning of range of f, but f is bounded implies this and if range of f is bounded then we say f is bounded. So, this is the link. So, boundedness of a function is linked to the boundedness of a set. So, what is the meaning of been attained? What is the meaning of is bounds are attained? F is said to attain it is bounds if there exists y element of a b and x bar element of a b, such that f y is the maximum value in the way we have just defined. And f of x bar is a minimum value that is a meaning of f at any x bounds.

So, basically if you have to prove that f is bounded on a b, then you really have to show that the range is bound the bounded set in our. And then you have to show that there are points y and x bar for which these are true. And that is the maximizing point of an x bar is the minimizing point. So, at any is bound means at there exists an y n x for which this holds for these are the maximum and minimum values. So, we will have a better idea of these when we study the notion of supremum and infimum tomorrow. In tomorrow class in lecture ten. So, what let me let me start giving a hint of how such a thing is proved. The key idea behind this whole story is that we have f a to b to be a continuous function. This is a very, very important fact, without the continuity your result might fail. So, we will first demonstrate an example and then try to use then try to show how the proof of this thing is done. So, I am going to assert that this idea of continuity of the function, that is there in hypothesis is very important you cannot just say I do not care it need not be continuous hump every function should have this, no every function do need not have this.

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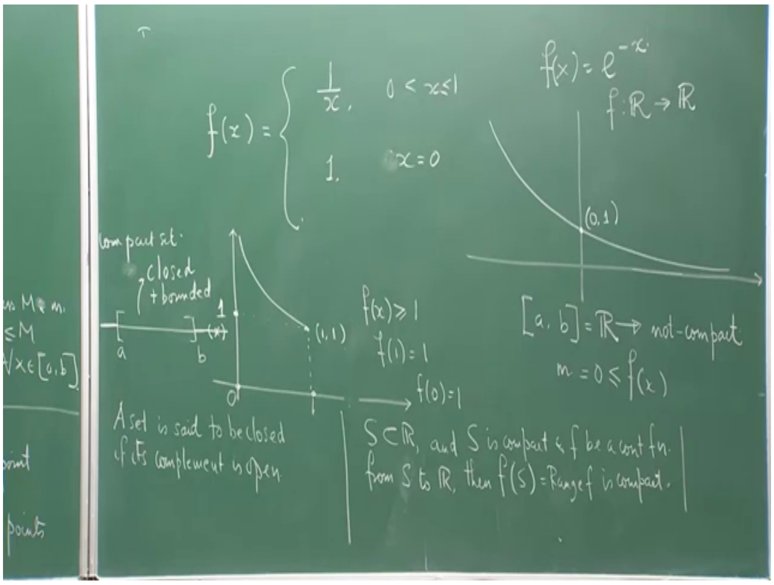

Let me define this function, let us look at this function. So, when I say f is bounded, it essentially means f is both bounded above and below. So, this actually means there exists real numbers capital M and small m such that f of x lies between these 2 numbers is f of x less than capital M f of x, bigger than small m this has to be true for every x that is choosing a b. That is essentially the meaning of boundedness. So, if you have this part, but do not have this part then we call the function is bounded below, if you have this thing to be true, but not this thing you cannot find any number m for which this is true then it is said to be bounded above.

So, apart from just been bounded they are 2 more additional terms which is called bounded above and bounded below. So, these things have to be kept in mind for example, let us look at this function this is as you can immediately in your minds I draw the graph. So, it is this is 1. So, this point is 1 1, may be I should draw it the scale is very bad may be I should have one here, because this is 0, this is 1, this point is 1 1, and at 0 the value is 1.

So, you can understand this a discontinuous function, but you see the function is bounded below, because here for any x here f of x is bigger than equal to 1. So, you found your small m, but is there are large m here. No because all the values of the function. So, here you see my minimum value is attend at this is this one value, this is the bound I will show that you see f f of 1 is 1 and f of 0 is 1. So, f of 1 is one and f of 0 is also 1. So, these is your x bar. So, it is attained the minimum value is attained this is bounded below, but it is not bounded above you cannot find the capital M for this function.

The key fact is that you have lost continuity. Loosing continuity is a very, very important issue. Loosing compactness could be important issue to even if continuity is there, I will give you a very straight forward example. One of the my most favorite ones you should look at a graph of f x equal to e to the power minus x. I am sure that you have learned a something about this number e a mathematical constant and it is a presentation through an infinite series here we will discuss it in more detail. If you e e to the power 0 is 1. So, if you wanna right this thing then this is like this.

So, when put 0 the functional value is 1. Now I am looking at the function here my a b is now whole over, I am trying to find whether a function defined here from R to R this is the function from R to R, whether this function is bounded or not. Look at the fact the range of the function is actually R plus the non negative real line non negative values of y. And non negative real line is never ever bounded. So, you can see that this function the value goes down towards 0, may it almost touches the x axis you feel as if it is going to touch the x x axis, but it never does. So, right it almost it is going to touch 0. So, the minimum value of this the lower bound you can say 0 is a lower bound here, f of x as you see everything is nonnegative is greater than equal to 0. So, your small m is 0, but is there an up upper bound the answer is no, because as you keep coming towards negative side the functional value will keep on increasing it will be e to the power some positive number, and then it will just keep on increasing.

So, here I have continuity of the function, over the whole real line, but I do not that have the compactness this whole line R is not compact. Now you might ask me that you have given examples, but etcetera etcetera told a lot of things, but what is this compact business what does it mean? So, let me explain to this these things to you. So, there are 2 terms compact this term compact is the short hand for a set we just 2 important properties closed plus bounded.

So, when is a set in the real line closed and what you mean by that, right? So, here what we have shown that even if the function is defined over this interval, a continuity is done. Here we have defined over whole R. So, basically this the property of this 0 1 interval is no longer there on the whole R, and thus continuity does not guarantee the existence of the bounds. Here you see there is a lower bound, but there is no x from which this lower bound is attained there is no x as the f of x is equal to 0 or e to the power minus x equal to 0.

So, we know what already is a bounded set which we have told. Compact is a name given to a set which is both bounded and closed. Now what is a close set? So, we are going to look at this point. So, before that we look in to the nature of an a set called open set. So, we are going to look a this bound. So, before that we look into the nature of an a set called open set. So, a set is open a set or rather a point x element of s is an interior point is an interior point if there exists delta greater than 0 such that x minus delta x plus delta this gain is a subset of s. So, such points are called interior points they are inside for example, if you take a set like this. So, a 0 1 open interval take any point here do not take the this points 1 0 take any point here you can always make it a small neighborhood around it. So, and this small neighbor is inside this set 0 1 inside the open interval 0 1.

So, a set is said to be open if every all it is points are interior points. A set is said to be closed if it is complement is open. So, here a set is said to be closed if it is complement is open. For example, if I take this intervals say a b of course, it is bounded every x is lying between a and b, but is it closed just look at a look at the exterior part of the other part of the set the complement. Take a point in the complement, you see you can have a small neighborhood around that point which lies in the complement. So, the complement of these set is open. So, this set is closed. And plus it is also bounded and hence the bounded this close bounded set. So, the an closed interval is a compact set. The real number line is actually both closed and open. Because the empty set is assumed to be both closed and open by convention.

So, if you take the complement of the real number line it was real number line every point is interior point, the complement of the real number line which is 5 is also a closed a is close is a is a you take the real line then it is complement is open the empty set. So, which means that real line is closed recklessly in the sense that it is; obviously, it satisfies the all the definition.

Now, definition of closeness. So, a set which is said to be closed has at the has it is complement, then open set if a set is said to be closed if it is complement is open when a set is said to be open if it is complement is closed. Though you can have this sort of definition for and they are actually same you can have the this sort of definition that if all it is points a interior point the set is open. So, if you look at the complement of a given set and if you find that all the points a interior points then the set itself must be closed, but the real line here for example, in this example is actually closed. The real line is closed, but never the less keep in mind that is not bounded. You cannot say that I can give 2 number between which all real numbers lie this is something you know very well this is the story of the infinite.

So, you really need to need not, but again you need to take care of these little things. So, continuity breaking, but set been compact can lead to problems, continuity is there of the function set been not compact can leave to problems. Now what we will do will try to explain to you why that this result is true. Why such a result is actually true. The key idea here is the following. If you take a compact set on a real line that is the closed, and bounded set and then if you operate a continuous function on it. Then the image of that set right is also compact. So, take s subset of R n R sorry, R and s is compact. And let f be a continuous function from s to R and f be a continuous function from s to R, then f of s the image of s under f the f of s which is also called the range of s range of f that is range of f range of f is compact.

So, continuous mapping of a compact set is compact, and that is a whole idea. So, here you can immediately see since f is continuous, the image f of a b actually is a range of f. So, range of f is compact because the closed interval as we have seen is compact. So, the range of f is compact, range of f is compact. And when the range of f is compact it is both closed and bounded and since the range of f is bounded. So, there would exist. So, range is a set which consist of all. So, what is range of f the set of all f of x where x is belonging to a b.

Now, what would happen? What is the meaning of this been bounded? That there exists small m and capital M such that f of x value lies between this, 2 numbers when x is in the interval a b. That is that is a key key idea. So, essentially then we have proved this fact by using a important result, or do you are not going to prove this result. You know sometimes these theorems which are very important you can use it in many many places and they they should be stated at the very beginning the because that this is one of the key ideas of calculus itself.

So, these results these more ideas more tools to prove which we are not doing of course, because this is of very basic calculus course. So, let us slightly high level, but. So, here we have proved the first part once we know that. Now about the second part, for the second part we really have to go back to something more fundamental which is called the supremum and infimum. And that is the topic of the next lecture, that would complete 2 weeks of lecture.

So, with this very basic facts I this today's lecture was very conceptual, it really told you what this theorem means. And this idea can be used in many many places. For example, if you know anything of if you know the definition of a polynomial I do not know, how many of you know for those of you who are ambitious here and who knows the definition of polynomial. If you take a polynomial of even degree on the real line there would always exist a minimizer, that is you will always find you will always be able to prove the existence of a minimizer.

So, that is a very, very interesting fact, but that for that proving that you will actually use this idea. That would be given in the notes as I prepared the notes I am gradually put it online. It may not be exactly at the same time, but little later, but this is exactly this results a some more application would be show in the notes because we have limited time when we are in the class. So, I would rather request you that from tomorrow we are getting into a slightly conceptually higher level. So, before you really look into the lectures from the next one supremum and infimum, please have a look or recall into the lectures that we have given now. If there is any mistake if you find something is not fine you can always right down in your portal and you can always ask the questions to us and we would answer back. My tears would definitely answer back and in case even I will also answer back.

Thank you very much, and all the best.