Calculus of One Real Variable
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Lecture – 08
Intermediate Value Theorem

Welcome once again; as we continue moving towards more advanced things, do not get so worried about the word advanced. It simply means that you have some more additional information; that is what advancement means. Today we are going to speak about a very important result, now many of you might think though mathematicians are only bothered about some results and it has no consequence possibly in life, it might not have always a consequence in practical life, but sometimes these results can be used to prove some other results which might have consequences in your life; what does this intermediate value theorem says? It essentially tries to tell you the following.

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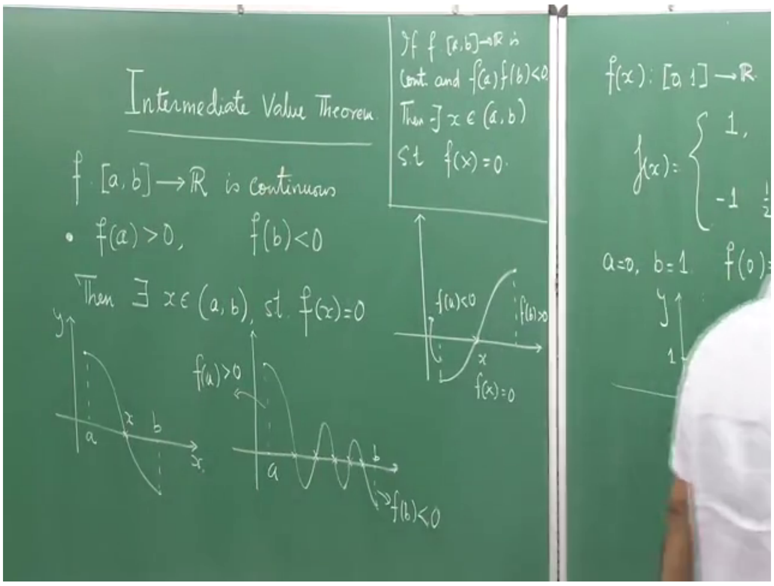

So, suppose if we have a function f from a, b to R, right; assume that it is continuous, once you assume that you have a you know huge; you have huge information of property of the function, the nature and then it is says that suppose you have the following information, this is a given information that f of a is strictly bigger than 0, f of b strictly less than 0.

Then there exists; this is the symbol for this opposite; capital E is the symbol for there exists x in a, b open interval a, b such that f of x is equal to 0; obviously, this x cannot be any of the n point because n points are not 0. So, this is what is called the intermediate value theorem does there exists 1 x only or does there exists more than 1 x. Let us draw some pictures to confirm it, for example, if you takeout term like this, if you have a continuous function like this between a and b. So, this is your a and this is your b; f of a is positive and f of b is negative and here is this point x; where it cuts the x axis means you start from the top of mountain and go below the river, then you really have to cross the surface of the river; once that is a basic idea, then that is basically it means you want to travel on the number line from minus 2 to plus 2; you have to cross 0 once that that simple idea is in built here.

Now, let us look at a situation where we can have more than one point where f; f of x would be 0. So, the function could be like this never mind you see f of a is still bigger than 0 and f of b; this is b and this is a and f of b is still less than 0, but you have 1, 2, 3, 4, 5 points at which the function attends the value 0. So, here the statement says that there exists an x, it does not say that there is exists only 1 x. So, here I want to make this fact clear that there can be more than 1 x possibly could be countably infinite or un countably infinite xs. So, functions would look very complex we are not getting into that.

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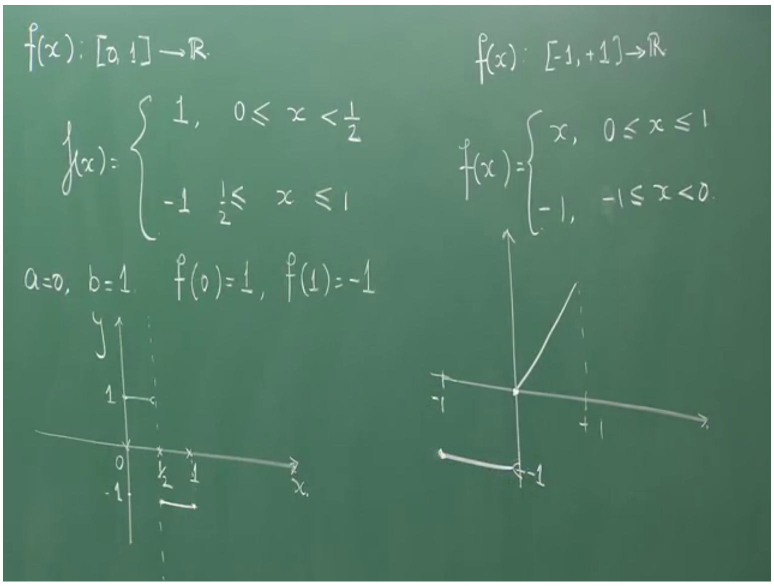

But this idea of continuity is very important; why I will tell you, for example, I will take a function f x which I will define from 0 to 1 to R and I will define it as follows at f x is equal to 1 when it is between 0 x and half and is equal to minus 1 when it is between half x and 1. So, you would you me immediately notice that here a is equal to 0 and b is equal to 1, a is equal to 0 and b is equal to 1 and what is f of a at 0? F of 0 is 1 which is positive and f of 1 is minus 1 which is negative, but let us look at the function values. Let us draw the diagram, but you can immediately see from the function; the values at the function is taking there is no x for which the function is taken he x value 0 value why it has not; it is simply because of the fact that in this line 0 to 1. So, at this is the point half. So, up to half the value; so, this is the demarcating point. So, here the value was one, this is 1 and this is minus 1. So, it comes here and here. So, at half exactly at half it takes the minus 1 value and not the 1 value.

So, this is the function value. So, you see immediately that there is a gap; this is a discontinuous function and for a discontinuous function, the intermediate value theorem need not hold true, it may hold true also I do not say it does not; for example, I will say f x is from minus 1 plus 1 to R, sorry to R and let me define it in this way; f x is equal to x when see here; you must understand that I am always not restricting f a to be strictly bigger than 0 and f b to be strictly less than 0, it could be the opposite, it could be for example, that f b is strictly bigger than 0 and f a is strictly less than 0 then also. So, here your either f b to be strictly bigger than 0 and f a to be strictly less than 0, then also you would have the same conclusion; there would be an x for which f of x would be equal to 0.

What now I am trying to show you that this conditions f a when that this f a and f b are opposite of opposite sign. So, if I take the product of f a into f b that would be negative that is just a sufficient condition and it is not a necessary condition for such a thing to happen that whenever I find that there is a x between a and b such f x is equal to 0, it does not mean that this is going to happen or the product of f a into f b would be strictly bigger than 0. So, this is one form of the theorem to be more precise. Now you can write it in a better form that if from a, b to R is continuous I am writing c o n t for short and f a into f b the product is negative, then there exists an x lie in the open interval a, b such that f of x must be equal to 0. So, this is the form which is much more acceptable, right.

Now, here let me put f x is equal to x, when x is lying between 1 and 0 and f x equal to minus 1, when it is lying between minus 1 and the interval minus 1 0 excluding 0. So, this is the functional value. So, if I draw this; what does this mean? So, here is my say minus 1. So, here is my plus 1, here is 0. So, you come from here. So, in this case f of b is plus 1, if I take b to be plus 1 and a to be minus 1 and then from 0 to minus 1, I put the value minus 1. So, here is minus 1.

So, this is my function, this is of course, a discontinuous function, it is not continuous, but you can immediately see that the conclusion that f x is equal to 0. So, here is an x between minus 1 and plus 1 for which f x is equal to 0. So, this and also you can see the product f of a which is 1, f of b is 1 and f of minus f of a which is minus 1 is minus 1. So, product is negative. So, this what I am trying to say is that if this condition, for example, continuity has been failed, but this is true then also this can remain true. So, these conditions that you see imposed here that its continuous and this product is negative is a sufficient condition means if this happens, then I am guaranteed that such a thing will definitely happens. So, this is something you have to keep in mind.

These small examples actually give you much more better inside into what is really going on and let us see, I will not immediately try to give a idea of proof. So, I am going to do the proof of this because everybody might not be adopted to or getting used to mathematical proofs though it is a necessary tool, but I would rather insist on concept building here in this course which some cases coming to proofs, but I will try to give you some first show some ways by which we can apply this result, let us see one of the first conclusion consequence which I can draw from these intermediate value theorem.

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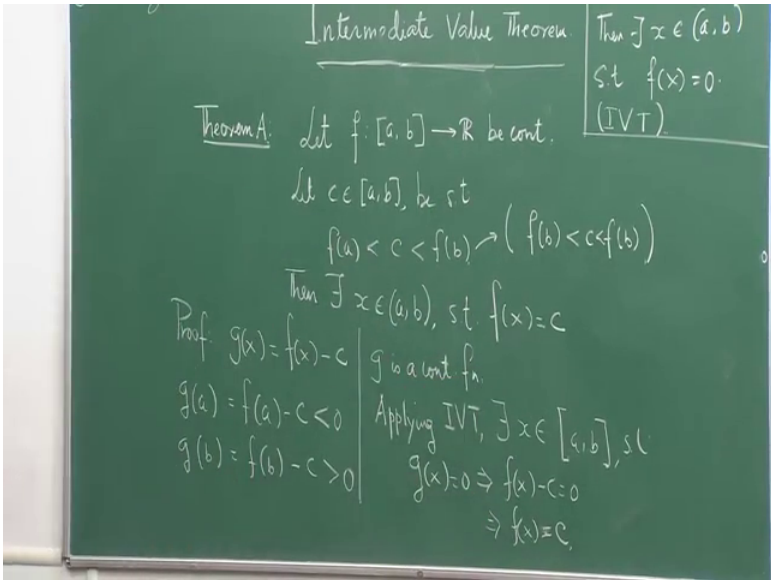

So, this is what I would call in short the IVT. Now here is another form of intermediate value theorem. So, this intermediate value theorem; if I go a little bit of the history was first shown or rather proved by Bernard Bolzano and Bolzano by the way was a priest in the beautiful city of the Prague which is now in the capital of Czeh Republic and believe it or not; lot of very great works in science for example, has been done by priest or monks because for very poor people at those times, monastery was one of the places where they get an education they could get a food, etcetera. So, in Europe lot of very talented people, but not from a rich background went to this monasteries and they really had they had time to practice science and. So, this was this was the contribution of Bernard Bolzano. So, let us see; what is an immediate effect of that IVT.

So, may be first conclusion I should write corollary actually, but I am writing as a theorem because this is what is written as IVT by many many books, but what I am going to write down is IVT. So, let f from a, b to R be continuous, let c element of a, b actually open interval a, b be such that c strictly lies between f b and f a you see it could be opposite, it could be lie on f b and f a could be strictly bigger than f b here I am just telling that let us take it like this then there would exists x in a, b such that f x is equal to c, never mind, I want to write tell you that this condition can also be written as it could be of this form also, but just for the sake of showing how the results applies, we will just consider this form; now how do we go around and really prove something of this nature.

So, in this case; the natural tricks. So, here is a trick which mathematicians are used what I know is that there would be a point f x is equal to 0, if I have 0; if I have the product of f a into f b to be negative that is 0 should strictly lie between f a and f b; 1 is negative, 1 is positive then, but Neumann have see what I do is. So, if I want to prove it. So, consider a new function g x g whose all values are evaluated like this f x minus c, then let us look in to what is f g a g a is f a minus c or what is f a minus c f a minus c is negative because f a is strictly less than c. Now what is g b f b minus c which is strictly positive because from this condition that f b is strictly bigger than c. So, the, but because f is a continuous function; if you add a constant to a continuous function, it remains a continuous function.

So, the first observation that we get from here is that g is a continuous function now what is the product between g a and g b their product is strictly less than 0 because 1 is g a is negative g b is positive now this means. So, hence now applying IVT there exists x in this interval a, b such that g of x is equal to 0 which would imply f of x minus c is equal to 0 which would imply f x is equal to x is equal to c. So, what are we proved that if you do this; this is what going to happen you will go going if this is the condition by applying IVT by doing this little trick a little translation, we can show just by a simple application of that that there would always exists an x for which this happens.

Let us look into another interesting consequence of this, right. So, can I prove this fact, every positive number has its square root if I tell you prove to me that every positive number has it is square root you would immediately try to take positive numbers and do the square root and so common; it is obvious you can calculate the square root of positive numbers what is so big about it, but if I tell you to demonstrate to me that every positive number has a square root right, then which means that you have to mathematically demonstrate that whatever positive numbers; I give to you there is a number whose square is that number.

So, if I give you any number alpha you have to mathematically show the existence of n x for which x of square is alpha it is not that just you will sit down and start writing those you know start trailing on one has the square root 2 has the square root; it is not like that in mathematics you do not do such exist or exhaustive things you can cannot do an exhaustive checks because in calculus everything is infinite here this is the real of infinite hence a mathematical demonstration is very important, but we will see how this intermediate value theorem or even this one; if you want to say actually shows you that you can really prove that every positive numbers has a square root. So, that is exactly what I am going to show you.

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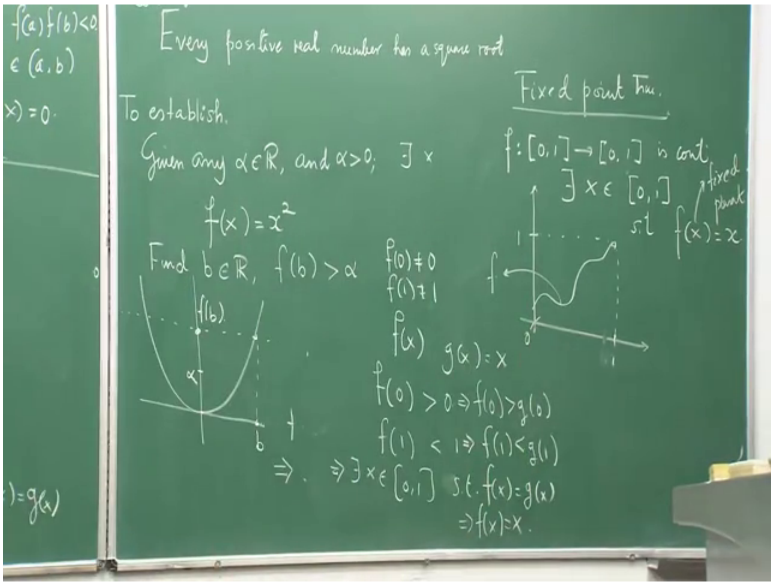

So, there is a beautiful proof from the book of Spivak and so may be if I want to call the theorem or mathematician would call corollary whatever you want to call it; it is a nice result every positive numbers every alpine that natural numbers set has a square root every positive real number not in n, sorry, in R plus means in the positive part of R every positive real number has a square root; say in order to prove this; what do I establish? I have to establish the following fact that to establish. So, what is the thing that I have to establish; I have to establish the following fact that given any alpha element of R and alpha greater than 0, there exists another real number x whose square is this number. So, x would be the square root of this number. So, there exists x element of R such that x square of x is alpha which means x is the square root of alpha, right.

Now, let us write down the functions, here let me consider the function f x is equal to x square and alpha is a given number to me. So, now, find b in R such that f of b is strictly bigger than alpha if alpha is strictly bigger than 1, you could take b equal to alpha because alpha square root be strictly bigger than 1 and if you want if alpha is less than 1 and b to 0, then you could take alpha equal to 1 because alpha square root be definitely less than 1 where alpha square would be less than alpha. So, whatever it is you can always find such a number. So, here let me draw the diagram f x is equal to x square. So, here is my alpha and here is my f b is my b and here is my f b.

So, f b is strictly bigger than alpha, but alpha being a positive number is, but since alpha is greater than equal to 0, it implies that alpha is strictly bigger than f 0 because f of 0 is 0 here because it is x square f of 0 is 0. So, what do I find f of 0 strictly less than alpha strictly less than f b, but in the interval 0 b, but in the f, if I just restrict it to the interval 0 b to R, then this is also a continuous function. Now once I do know that I know this has happened; I can just apply; what I have learnt here with a is equal to 0 and b is equal to b there the same b.

So, this would imply that there exists an x in the open interval 0 b such that f of x is equal to alpha where this and this; obviously, implies where the f x is nothing, but x square x square is equal to alpha. So, you see how beautiful mathematics says that a. So, called advanced idea can is actually needed to prove something which looks very obvious to you now we shall go towards the understanding of the proof of this results how do you prove it actually, but before doing.

So, I will tell you a story associated with this result there is a interesting story in many books write this story, but here I will speak from a book called the concept of mathematics by young steward about which I have already mentioned to you, but I still; I would like to remind you and write down the name of this book and I expect that every maths student would get an access to this every school library should have it maybe high school level; not school, rather college library should have it in India.

Unfortunately not every; any college which can has money to buy books should have these book at least.

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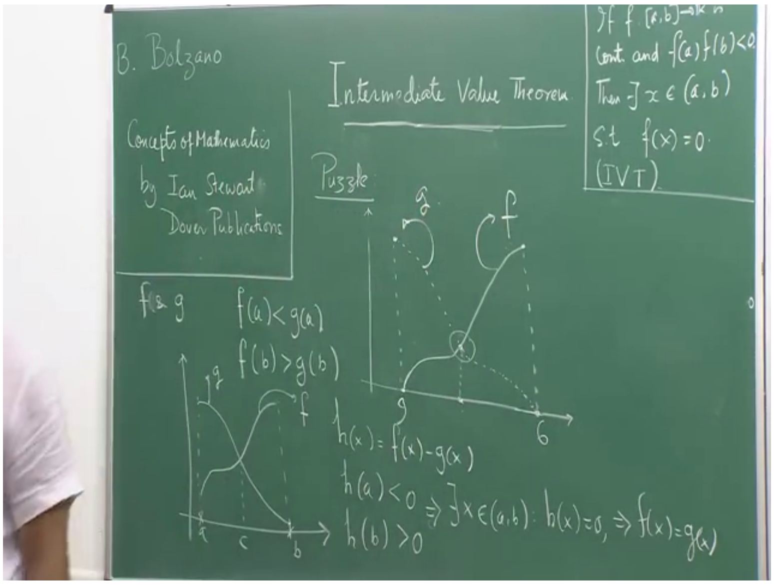

It is a very cheap book at this at present he is one of the most passionating writer in mathematics; he is one of the leading mathematician over times, sorry, Ian Stewart from Warwick university in the United Kingdom and this book is published by Dover publications.

So, I begin by really showing you this book on the screen I hope you see that. So, please consider if thus some even teachers watching this program please consider having this book in your library, it is must for students and even infact, teachers to have a read of have a reading of this book, math is not always about doing calculations or doing manipulations math is about developing ideas brilliant mind developing ideas in biology or mathematics is a brilliant mind, it is not that a mathematician who manipulates too many things is much; he is much more intellectual faculty and than a biologist developing a break through concept.

So, here is the puzzle the puzzle is the following. So, I read out from this book which says that at exactly 9 o clock on Monday morning, a man sets of up the mountain arriving at 6 ‘o’clock in the evening. So, a man starts. So, here I can view on a horizontal plane the topography of the mountain. So, here at 9 pm, he starts climbing and 6 pm, he reaches the top of the mountain and in the next day at 9 ‘o’clock next morning, he descends from the same route arriving at a point which he set out at 6 ‘o’clock in the evening.

So, 9 ‘o’clock next morning the graph would be like that at 9 ‘o’clock in the next morning, he sets out like this and climbs back to this point. Now the puzzle is the following prove that at some time he is at the same place both the days. So, there must be some point here, say some point on the mountain route where he is present; he arrives at the same time in both the days.

So, if he was at 12 ‘o’clock in some place on the mountain route while going up, he is at 12 ‘o’ clock at that same place while going down might looks; it might looks like dis-strange to many of us; who has travelled up and down mountains the idea is the following that. So, this guy is now climbing down the mountain in the morning. So, assume that there is ghost of this man who is climbing up this mountain and he does the opposite journey some man start 6 in the morning and does the opposite journey climbs up the mountain. So, this is the path of this guy. So, he starts at 6 and this guy has started at say 9 in the morning and he started climbing down.

So, another guy is actually now another guy has started at 6 and he is climbing up. So, there is a time you see. So, here is up path here is a down path. So, in the down path what the guy is doing he is climbing down you can think there is a ghost down path the; so, in the down path this guy is coming down at starting at 9 and he is coming down, but think that there is a ghost; who is also walking up the same path the next day towards the top f the mountain, then you see that there is a time when both of their paths cross which means the ghost and the man should be face to face at that point.

So, that this is nothing, but the intermediate value theorem and what is this form of intermediate value theorem. So, we will just write down; how can I write this thing. So, this is the particular way of talking about the intermediate value theorem. So, this is something like this; see this curves are not continuous if the mountain profile was not continuous such things would not happen as I already shown, but here the mountain profile is also continuous. So, this is an application of the intermediate value theorem that there exists a point between this where you get my whatever be the point there would be a there would be a point where time point where this particular point would be reached in both ways.

So, this can be actually expressed in the following way this idea this is the trick behind proving the intermediate value theorem then I can write the intermediate value theorem as the following. So, I left 2 functions f and g and they have very opposite behaviors. So, behavior is this f of a is strictly less than g of a, but f of b is strictly bigger than g of b. So, this let me just draw the graph of this here say if 2 functions say between a and b. So, here is my a and here is my b right f a and f b. So, the function value at a; suppose this is my f, the graph of my function f is my f function f x and incase of g you know that f g of a must be strictly bigger than f of a, but g must be strictly lesser than f of b. So, this is my g.

So, there exists a point c between a and b where f x must be equal to g x and that is exactly this puzzle right here because if you look at the mountain in this way it has one functional graph if you look at the mountain in this way it is a another function of graph and they have this same properties. So, this idea; so, what happens; now I consider a function h whose h x is given by f a minus g a; right is given by f sorry f x minus g x. So, this function h is the continuous function between a to b because f and g are continuous function between a to b and difference between 2 continuous functions is again a continuous function.

So, this is a simple facts now what is h of a here h of a here is strictly less than 0 because f of a is strict strictly less than g of a, but h of b is strictly bigger than 0 because f of b is strictly greater than g of b which would imply again by using IVT that there would exists an x in a, b such that h of x is equal to 0 which implies f x is equal to g x at that point and that is exactly the solution to this puzzle because while going up the mountain profile would look as f while going down the mountain profile would look as g.

So, there is many-many ways to view this simple thing. So, how do you actually would prove it I not really get into the I mean; idea the proof essentially is the following you divide say the interval a, b into say 10 or 20 equal partisans equal small intervals look into now for the first starting intervals you will observe because of the property of continuity that if the difference of if h a is strictly less than 0. So, in never hood h a would continue to remain strictly less than 0. So, for sometimes you will have f x value strictly less than g x values, but then look at the first interval where you find a value of x for which f x value is bigger than g x value keep on dividing. Now divide that interval again and among that interval first find. So, first find the interval where again f value is less than g value.

So, you make these intervals smaller and smaller and smaller, right. So, so you in this interval first find those values for which f means your you are not divided equally again find the case where you first find an x for which f x is bigger than g x. So, take that small make take another take that small interval equal on both sides and then again divide that. So, you will keep on having this smaller intervals and smaller intervals basically; you will taking intersectional intervals say; we will have smaller intervals and smaller intervals and smaller intervals and which will silently close in on the point c at which f x is equal to g x.

So, this is the idea at which we have not getting in to, but one application the last application a very important application is the idea which we will read; we now talk to you now and that will be the last thing that I speak here and then the application of intermediate value theorem is called the fixed point theorem fixed point theorem tells me suppose I have a function of this nature that f is from 0; 1 to 0 one and is a continuous function. So, here is my 0 1 on the x interval here is my 0 one interval on the y interval, it will be slightly come this side. So, basically I have this square. Now I have some continuous function some f is continuous insist domain is 0 1 co domain is also 0 1; what I am trying to say is that the fixed point theorem says is that there would exists an x element of 0 one such that f of x would be equal to x.

Now, of course, this function could have x 0 equal to 0. So, it is done or it could have f 1 equal to 1. So, it is done the question arises if not if f 0 is not and f 1 is not one the function should always have a positive values. So, this whenever a point remains fixed under the application of the function such a point x is called a fixed point. So, what does it got to do with the intermediate value theorem, but look at what we have just learnt consider my function f x is equal to f and consider another function g x is equal to x f x is the one which I have given and considering the function g x is equal to x. Now f 0; if the f 0 is not equal to 0, then f 0 must be strictly bigger than 0 because the function has to lie between 0 and 1, it cannot be negative.

So, f 0; so, the function value f 0 must be strictly bigger than 0, right that is f 0 is strictly bigger than g 0. This implies this and what is f of 1? F of 1 cannot be more than 1; f of 1 has to be strictly less than 1, either it is equal to 1, then it is done, if it is not equal to 1, then it has to be strictly less than 1. The function value must be within 0 and 1 and in that particular case, I just rub this part. So, just do it for you again. So, what is happening here? So, I have taken f x as the function f is the function functional values of f and for me g x is equal to x.

Now, let me say that f of 0, let me be in a situation where f 0 is not equal to 0 and f 1 is not equal to 1 because if any of these conditions happens then this result is true, then in this case f of 0 must be strictly bigger than 0 which implies that f of 0 is greater than g 0, then f 0 is not equal to 0 and the function is between the 0 and 1 which is non negative. So, it has to be strictly bigger than 0, but f of 1 has to be now strictly less than 1 because the function value cannot exceed 1 and f 1 is not equal to 1 which implies that f of 1 is strictly less than g of 1 which is 1 because g x is equal to x.

So, this is exactly in this situation it is exactly in the situation of this mountain climbing situation. So, you will immediately prove that there would be an x between 0 to 1 such that such that f of x would be equal to g x, but g x is equal to x. So, this would imply that f of x is equal to x and voila the theorem is proved voila actually the French word which means that is it done. So, if you see how a built up the whole story and surrounded with this simple man climbing of the mountain which can come and prove this fact; it is; it looks very obvious because you can draw this line y equal to x here and a continuous function you would feel cut across this y equal to x line if it is; if the function range is also within 0 and 1.

Now, fixed point theorem you might say what is what story you are talk talking about, but let me tell you a huge amount of modern economic theory and economics is of course, you know practical stuff it is based on fixed point theorems. So, with this, I would like to end this class and I hope you had enjoyed it.