Calculus of One Real Variable

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Lecture - 07
Continuous Function

Welcome to our course again. So, here is we are on the 6th lecture. So, here we are going to speak about continuous functions we are not going to fix up and try to always find functions which are continuous. If I can borrow the name from professor corner, I would like to call this course really that course for the ambitious calculus for the ambitious in the sense that here we are largely bother about concepts of course, there will be certain aspects of calculus where you know examples would be given. But this is not a course where again I want to repeat for those who would like to come in to the exams, for this course this is not a course in which you is going to make you pass in all India exam. This is something which is not the aim of this course ok.

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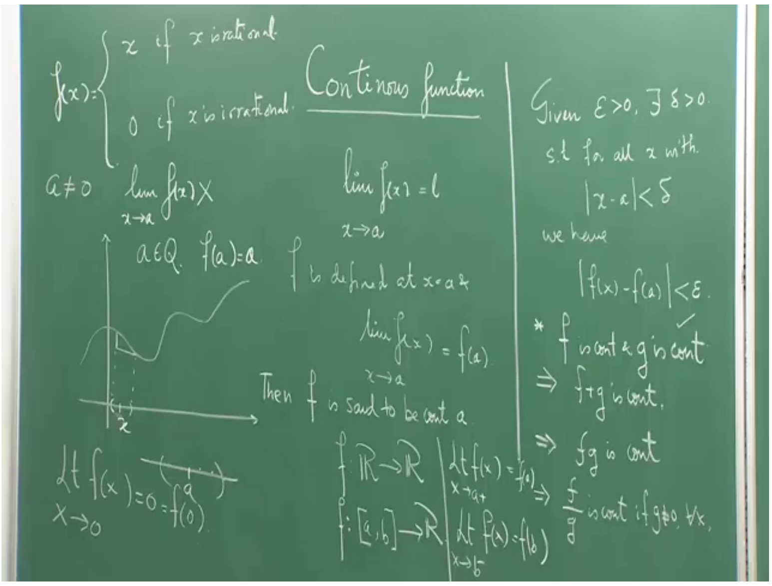

So, here we will use the book of spivak partly not fully to discuss what is the continuous function. When we were speaking about limits, when we wrote this statement we did not bother whether f was defined at a or not, but suppose luckily a is defined at a f is defined at a sorry. So, if f is defined at a, and we have limit f x as x tends to is f a. Then and only then we call function f to be continuous to be continuous at a. Of course, if you want to say it is continuous over the whole real line or the whole domain our definition of the function, then we are of course, the natural way to extend this idea is that we define the continuity and we expect the function to be continuous at each and every point. Now again you see that here we are talking about a limiting procedure. So, once I can talk about a limiting procedure I can of course, have an epsilon delta description of a continuous function.

Examples of continuous functions are abundant mathematics is actually the story of continuous function analysis is the story of continuous functions, because there are fast functions for which are not continuous. And they can have discontinuity at a certain points on jumps some oscillation right. So, limited exist existing does not at all mean continuity. We will discuss that issues discontinuous issues about discontinuities later on, but then how do I translate this in to a statement in terms of epsilon delta. Very roughly speaking in a very crude sense continuity simply means a small change in x must give a small change in y, in the sense that if I want to draw the graph of function. So, drawing graphs of function is something I urge you to spend some time with.

Those who love mathematics I tell you that so much better thing than being hooked up to video games and other devices through the internet other games to the internet. You can actually use the net to do the graphing. So, for example, if I can draw a function, without lifting my hand from the plane of the board, I like to call such functions such as continuous, but there little subtlety here.

In the sense that every function may not be graphed so nicely and continuity of functions for example, that we will soon give examples where functions could be just continuous at one point and not continuous at any other point. Now this again the idea is that a small change here in x if this is an x and if I move a little bit if this is my x and if I move a little bit to these 2 sides then you see the functional change in the function value. Change in the function value is not also very high is not that suddenly there is a jump. So, translated in to the epsilon delta language it says that given. So, I am writing the limit the definition I have for this, but in instead l it has not become f of a given epsilon greater than equal to 0, there exists delta greater than 0 such that for all x with We have.

Now you might ask me what about strictly bigger than 0 here which was in the definition of the limit. The idea is here idea here is that here x can take the value a and that is why it is defined at f a. So, it does not matter here, that here it we do not consider any more a punctured neighborhood, plus here x can take the value a because I where I can compute therefore, f values there. So, when I am talking a limit x tends to a, if I even if I do not have any mean the function may not be defined at a, but I am expecting that at least in our neighborhood around a all for every x the function value is well-defined. So, so this is the very basic definition.

So, that 2 things which you can immediately prove by using the epsilon delta definition about con about I about continuous functions 2 things which I would now leave separate for you. For example, f is continuous and g is continuous implies f plus g is continuous. Similarly if f is continuous and g is continuous. So, if this is my data then I will have f in to g which is f x in to g x when you evaluate it is continuous right. So, the function f plus g is continuous and f product f g is continuous, and f by g is continuous if g naught equal to 0 throughout, for a all x in the domain. So, that is that is something very standard. And so, we are not going to get in to any detail proof.

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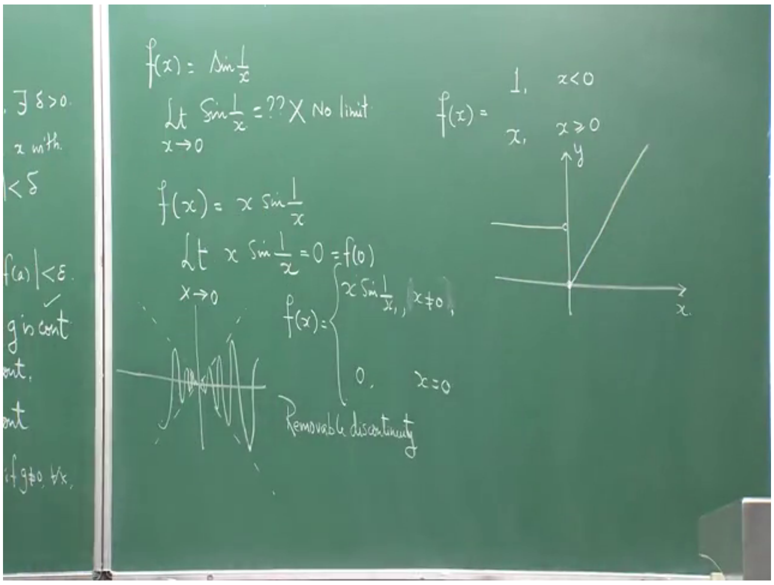

Now, look for examples the function f x is it continuous? Of course, it cannot be a continuous function at say at x equal to 0, because at x equal to 0 the function is not defined. There is a limit exists for it mean does limit sin of 1 by x tends to 0 this exists, answer is no because sin 1 by x keeps on oscillating as x approaches 0.

It does not us come to a stable value right. So, in this case answer is no the limit does not exist, there is no limit. It does not exist. Now if I take a function which you have already looked in to in the previous class as f x is equal to x sin x. So, you know that x equal to 0 is not in the domain, but what we had shown me using the epsilon delta technique in the last class, was that if you take the limit as x tends to 0 x sin 1 by x is actually 0. That is what we have shown.

If you remember the graph I just want you to recollect it. So, here we have drawn the lines y equal to x and y equal to minus x, and the graph essentially looks something like this. So, sudden, here a function is not defined at x equal to 0 this clear. And hence there is some sort of discontinuity there, continuity cannot be defined in the in this way. But here we can remodel the function or redefine the function in the following way, because I know that it is going to 0 this definition. Let me define f x now as a function is defined at x sin 1 by x when x is not equal to 0 and is defined as 0 when x is equal to 0.

So, once I define it like this then immediately I know that f 0 is 0. So, this is nothing but f 0, and hence the function is continuous at 0. So, I have redefined the function and the redefine function is now continuous. So, in such a cases where you can redefine a function a little bit basically the point where the function is discontinuous the and if a limit exists as the x values approaches that point, then if I take that particular that limit value as the value of the function at that particular point, then I can actually remove the discontinuity at that point that is called removable discontinuity. So, what we did is basically we remove the discontinuity. So, remove removable so but every discontinuity is not removable.

For example if you define a function like we look at this function. So, till 1 it comes to 1 like this and then from 0. So, f 0 here is not 1, f 0 is 0. So, there is a discontinuity here and here this discontinuity cannot be removed because the limit as x tends to 0 from the right is 0, but limit of the function value as x tends to 0 from the left is one. So, because the limit does not exist at the point x equal to 0 here the limit does not exist at the point x equal to equal, which means at that point that discontinuity is not removed removable.

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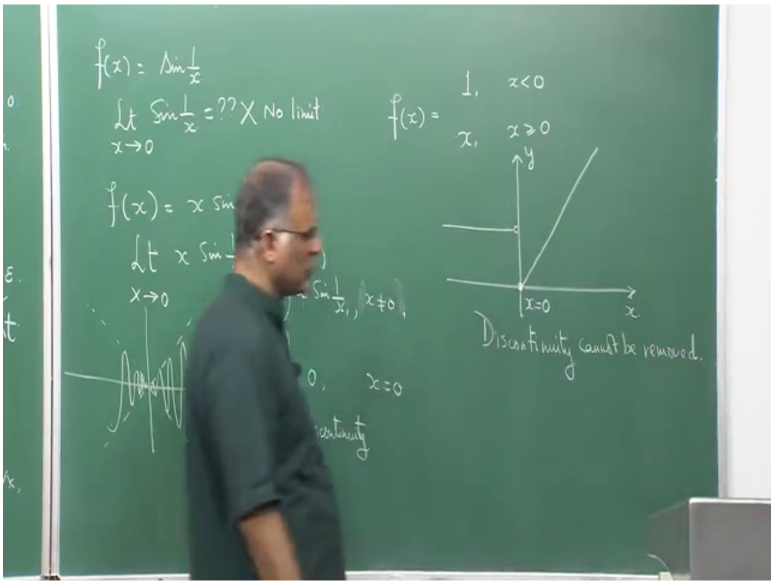

So, this is an example of a removal discontinuity and it is a case where discontinuity cannot be removed. Now I will give an example of a function an interesting one though, which will show that it is continuous only at one point and discontinuous at every other point they this continuity is not valid for any other point other than 0. So, we define that f x is equal to x if x is irrational and is equal to 0 if x is irrational.

So, if you take any point other than 0, any a not equal to 0, then limit f x as x tends to a does not exist. Why? Because the function values keep on oscillating. So, whatever wherever you are coming it is basically. So, you take the value say a and a is rational a is q, then the function value is a at that point. So, so f of a is a, but in that neighborhood there are infinite values where f of a is 0 and where f of x is 0 and f of x is a. So, you cannot say that I can always. So, here what would happen? You cannot say that I can given whatever; however, near the points I take in this particular case. Because of this oscillation given any epsilon mean given any epsilon I cannot always find a delta for which that relation of limit would hold true. If you want to do not if you do not find thinking an epsilon delta might be difficult just think of a point which is nonzero and the function at every point if it is rational it there will be only for any neighborhood around a point a, suppose even if it is rational. Any neighborhood only there are countable means rational points.

Uncountable many irrational points. So, what happens is that the function value keeps on simply oscillating between some value of x and 0 x and 0 x and 0 x and 0, mean it just it is it is not possible to stabilize that movement. And that is why limit does not exist, but if you take this particular issue when f x limit of f x, when x tends to 0. It does not matter whether we are looking at x is rational and x is not rational. If the x is not rational is anyway 0, but if the x is rational as I go near 0 from both side the function value goes toward 0.

So, when the limit of this function limit function at 0 is 0, and these is nothing but f 0 because 0 is a rational number, because f x is equal to x if x is rational say f 0 is 0. So, which means at no other point a non-0 the function has a limit which is the function value. So, there is not continuous at does not have a limit. So, no question of continuity it is not continuous at x equal to any a equal to non not equal to 0, but only continuous at the point x equal to 0. Here is an example of a function which is continuous at a point, but not continuous on the whole real line this is a very, very important result when a example which I am giving from spivak. And I think this is a very, very important thing. Now there are certain cases For example it becomes difficult to as spivak shows that it is very difficult to decide on the continuity of function such as this.

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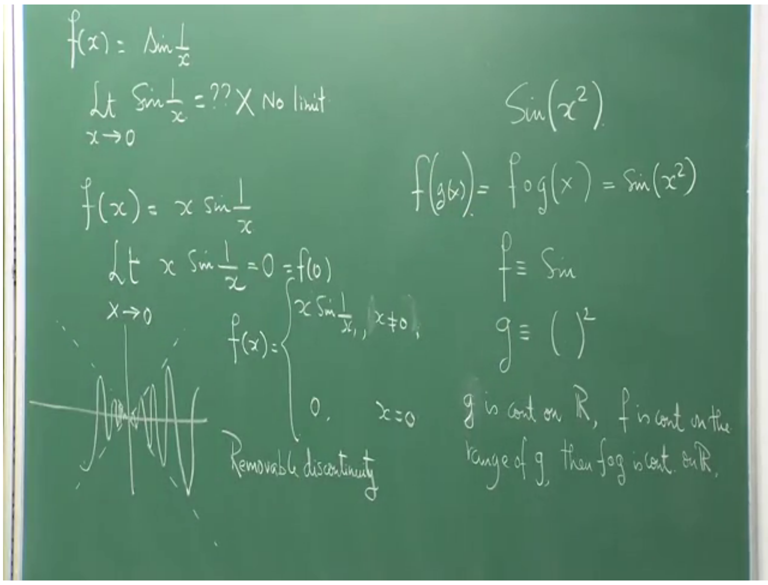

It might look quite obvious though it looks continuous looks nice, but behind the idea that this function is continuous has an idea of composition of functions because this function can be written in the following. So, first we apply the function g on x and then apply the sin function. So, if I take my f as a sin function and g as a squaring function. Then f compose g is nothing but sin of x square. So, I first take the x then square it.

So, that is that will give you g of x and then you apply f on it which will be sin of x square. So, if 2 functions, but individually sin is a continuous function, and individually g is a continuous function. See if I look in to the sine of if I look in to the sin function and the square function, individually they are continuous, but if I compose them will they remain continuous. That is if f is continuous on say R sorry, sorry I made mistake if g is continuous on R and f is continuous on the range of g, then f compose g is continuous on R.

So, this is a result which appears absolutely simple, but then you can actually go ahead and prove it you have to understand the continuity of f is not on R, but it is on the range of g. With f is acting on range of g, because f compose g means f of g x. So, f compose g if you know this thing it means f of g x. So, this is exactly what is going to happen. I do not want to get in to an epsilon delta proof of this, because nothing would happen if you may when is nothing great that I should be putting my time to prove, this is something I should leave you to handle around.

You can just play around a bit with this epsilon delta business, and it is very easy that this result will coming. What is possibly more important now is to understand here why I am talking about convexity about the function etcetera have never defined where is the function being defined from essentially implicitly in all these definitions I am I have been talking essentially a functions from R to R.

But no matter how much we want to restrict ourselves to R to R then very majority classes a function which has to be defined over a smaller domain. So, essentially a more general idea is used to study functions from of course, interval a b to R, but when you are talking about closed interval a b to R, you should be for one that here we are when you are talking continuity we are talking continuity only in the points interior to a and b, that is it that is in at all points other than a and b we are essentially talking about continuity in this sense. But when we are talking about the point a and point b we are essentially talking about left and right continuity.

So, limit in this particular case f x as x tends to a plus because it comes from the right side it should be f of a that is right continuity. And then there is left continuity because with b it goes from the left. So, x goes to b minus. So, this means I am coming to b from the left f x should be f b. So, whenever I am telling that a function is continuous over the closed interval a b you should be careful enough to understand that at the endpoints I am only talking about left and right continuity because. I do not know anything about the nature of the function beyond the interval a b.

This I knows about the nature of the function on an open interval beyond the interval a b and I am just x in to a b then it is obvious that I already know it is continuous on a so, but if you do not know anything at the end points this is the meaning of continuity left and right continuity. Of course, by the whole definition of the limit you might ask me at the notion of limit is valid a notion of continuity is valid, when this limit exists means right limit exist and left limit exists means right continuous continuity exist and left continuity also is there and the left continuity limit and right continuity limit for the right part on the left part should be equal to f a. But when you are at the end points this is what you mean by continuity, only the left part and the right part of the limit you do not either the left part of the right part because you do not have other things. Now we are going to prove and this has to be proved a very, very important result about continuity a very important result, it essentially shows the very nature of continuity.

There is a key result and it these ideas is been used even in very advanced research this idea actually helps. So, this ideas which are there in the calculus or analysis if you want to call it for a moment, is So, ingrained in the mathematical psyche or psyche of mathematicians, that these are the tools people want to use one after another these are the tools people want to use continuously.

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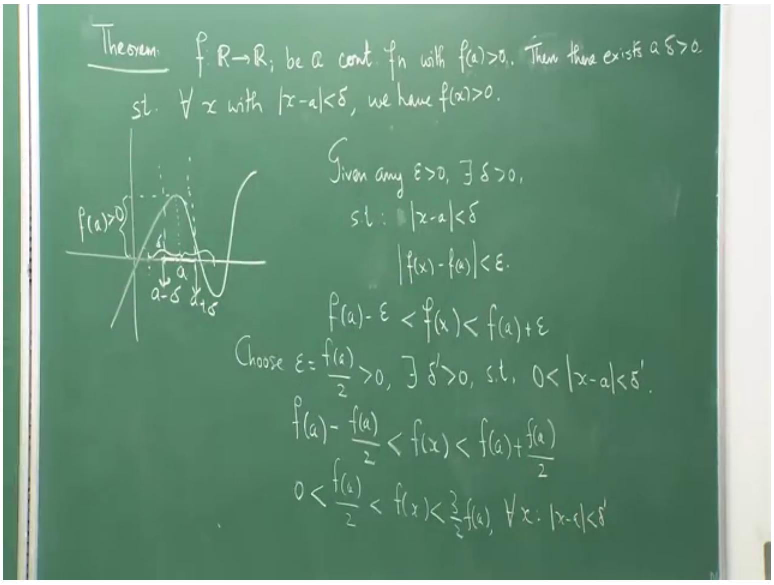

And so, we are going to now talk about a very important result which I want to write as theorem, theorem is the mathematicians way of telling you that he wants to state a very important result with important consequences. So, suppose f is continuous at a maybe I do not bother may be let me just for the time being make the thing look much more simpler let f from R to R be a continuous function. C o n t is my short for continuous and f n is my short for function, where continuous function with f of a strictly bigger than 0, where say a is an element in R for which f of a strictly bigger than 0.

What I want to show that there exists a delta then there exist a delta greater than 0, such that for all x with this property that x is lying in the delta neighborhood of a, we have f x strictly bigger than 0. So, this is the consistency of consistency property of a continuous function. That if I am at a point a which essentially should be an interior of the domain of the definition, or even at could be at the boundary then we you have to talk about the left and right intervals. So, if I have an a for which f of a is strictly bigger than 0, then for there is a delta neighborhood around a such that for every x we have f of x to be strictly bigger than 0 every x lying in the neighborhood, but if I change the delta if I make it bigger that is what you might just break. So, if you look at a continuous function like this.

So, here at this point. So, it does not matter the f a would be bigger than 0 f a could be less than 0 strictly less than 0. If f a is strictly less than 0 this everything in equality will change f a. So, this is a and this is my f a. And f of a is strictly bigger than 0. Now let me look in to look in to this interval look at this interval. So, this is a plus delta and a minus delta in this interval f of a is still continuous should even greater than 0. In fact, the interval can be expanded also a bit, but you maximum you can come here may be you can expand it a bit here and expand it a bit on this side not more. Because once you come a little bit more you will say I can extend there is f a is greater than 0 on this side, but because you want to make the thing symmetric that neighborhood you have to make it symmetric on this side also, but then a negative part comes in.

So, it is a So, sorry, sorry in this case I just I will make the drawing a bit more better to bring in a negative part to tell you that this might not always happen. So now, you see here f a is strictly bigger than 0 and now here I for example, if I take a neighborhood of this length I take a similar side this side. So, I will have a plus delta here. So, this point is my a plus delta this point is my a minus delta. And here also the function value you see remains strictly bigger than 0, but I might say you know this side I have more up to here I see I have f a to be strictly bigger than 0, but here if I increase it to this side to this amount and I have to increase the delta amount by also by this amount on this side it should come here, but then it encompasses of function value which is equal to 0 and In fact, less than 0. So, you, so for every delta it cannot work. So, it just works for a fixed delta. So, if I bring in delta 1 say this is my delta 1.

So, this is a minus delta 1 and then this will be a plus delta 1. So, in this case for this part for every x f of a strictly bigger than 0 part for this part of the neighborhood there is a part from which f of a becomes 0 and negative. So, here so, the result that it is only for R delta there is some delta that is the whole key idea it does not work for all the deltas it works for only one delta. Or may be more than or may be if you know for one delta you can keep on making it smaller then it will work. So, basically there is a threshold value of delta beyond which this will this result we break down. So, we increase the delta bit this result will break down.

You will of course, you will shrink it down; obviously, you will have the same thing, but if you make it bigger that thing will break down and that is that is the key idea here. And this idea would be used in the next class to talk about something very important called the inter intermediate value theorem, these ideas are used at that point right. So now, because f is continuous I would go and give a definition basically this that given any epsilon greater than 0, there exists a delta greater than 0 such that whenever I have x minus a strictly less than delta strictly less than epsilon.

So, I can just open this up a bit and write it in this form f x is lying between f a plus epsilon f a minus epsilon. Now so, this is for any arbitrary epsilon there will be a delta. So, choose epsilon equal to f a by 2 because f a is strictly bigger than 0 I can always choose epsilon to be f a by 2, half of f a. And this result mean epsilon strictly bigger than 0 corresponding to this epsilon there must be some delta. So, there exists a delta dash say such that whenever x minus a is in the delta dash, that is in the delta, delta dash neighborhood I would have the following to be true.

So, which means in this case what is happening? Here I have f a by 2 from here strictly less than f x strictly less than 3 y 2 f a. So, this is true for all x with x minus a strictly less than delta dash. Now these means, but f of a is strictly bigger than 0 f of f. So, this is epsilon this is nothing but the epsilon. So, they are strictly greater than 0. So, for every x here we satisfy this I have this. So, my required delta is this delta dash. So, f of x remains 0 within that neighborhood. Now sometimes what happens, if I know about the continuity of a function just at one point I can speak about the continuity of the function over the whole real line if we have some additional property a very standard sort of example that comes in I do not know many, many books. So, this comes as the exercise in spivak, but I think this is the very standard example, is given in many books I have seen in many books actually they are given examples also is the following.

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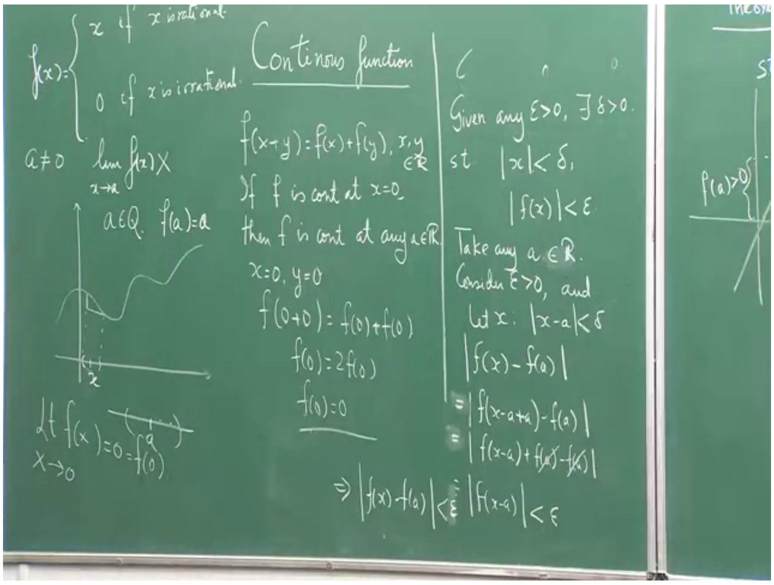

So, suppose a function is such that f of x plus y is f x plus f y. This function is from R to R and function satisfies this property is called the additvity property. So, this is true for whatever x y you choose does not matter whatever x y taking R this will be always hold if this happens for example, example take the line f x equal to x this happens f x equal to 3 x 4 x this will happen. Now if f is continuous at x equal to 0, then there is actually a linear function. Then f is continuous at any a element of R. So, how do I go about it? The first thing to observe is that first put x equal to 0 and y equal to 0, then by very definition f of 0 plus 0 is f 0 plus f 0. So, which means f 0 is equal to 2 f 0 which means f 0 is equal to 0 this something I this information is immediately observed.

Secondly so, what do you mean by continuity at 0? It means. So, given give given epsilon given an epsilon greater than 0, there exists a delta greater than 0, such that whenever this holds f of x minus f 0 which is 0 mod of f x is less than epsilon. This is what happens right. Now take any point in take any a strictly bigger than 0, sorry I am again is a element of R. Now take any epsilon. So, for that epsilon there is a existed delta for which this happens. So, take the same delta consider epsilon greater than 0. And let x be such that this is true. Now this if you choose this epsilon this epsilon will give me this delta because of the fact that it is continuous at 0.

So, I will take the same delta and look at all points in that neighborhood from a delta neighborhood away. Now I want to show that this will also be less than epsilon, how do I show that? Here again this I this fact we come to help. So, this can be written as f of x minus a plus a minus f x right. So, this is equal to equal to now what is this I will apply this one. So, I will this can be written as f of x minus a plus f a minus f x. So, f a f a cancels and this is equal to f of x minus a.

So, what this number x minus a the absolute value of this is less than delta. And whenever I have any real numbers absolute value is less than given that epsilon this absolute value is less than delta f of that value is strictly less than epsilon because of the continuity at 0 because I am now applying this instead of x I am writing x minus a here. So, this is strictly less than epsilon. So, what I found. So, f of x minus f a modulus of this is nothing but the absolute value of f of x minus a which is strictly less than epsilon. So, from here I conclude that given this epsilon either given this epsilon whatever delta works for the 0 case it will work for any other point a.

So here I have, so, once and it will delta is sorry once the delta is known for the point x given an epsilon once the delta is known for the point 0, the same delta will work for the point a. So, this is a very, so, you see how useful the epsilon delta definition is without this epsilon delta definition is not. So, easy to prove just writing the limit thing it is a very difficult it is not it is slightly it is a difficult game to prove this fact. So, epsilon delta definition immediately gives you the result. And that is why you should rather than getting worried about what is happened we should actually celebrate the power of mathematical thinking.

Thank you very much.