Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

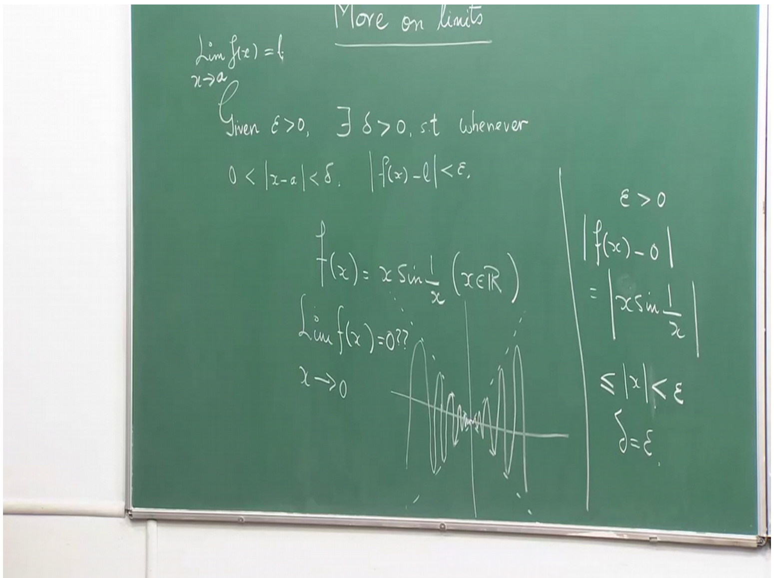
Lecture - 06
More on Limits

So, we are now going to speak about limits, the second part of the limits we had been speaking about it is definition in terms of epsilon delta which is now the standard one. And we had tried to explain to you may be it was not very clear that what would happen if I change the roles of mean change the roles of epsilon delta. That is I fix delta and then try to figure out the epsilon. It is not always possible and we will show we will discuss that more in the notes or rather getting in to the details, as why such a thing is not possible that if you have, if you say that I will have all those y's such that y is equal to f x such that this happens then I might not always have access lying in this. So, this change of epsilon and delta is always not a good idea.

So, this definition of a limit is what this means. Or rather we want to say that this means this. Once this happens I will make this notation. Sometimes definitions cannot be questioned, but sometimes for what happens we want to go in to the literal meaning the English meaning of that, and try to see whether it fits in with the mathematical statements. Now here if you look at the last example where we have given we have taken a function f x equal to 3 x and try to show that if I know the epsilon I will find the delta. So, there of course, at where it was we were going at x was going towards 5, and at 5 of course, you know the function was well defined.

Now, let us take an example of a function where very typical function this called the topology sin curve which is given as follows.

(Refer Slide Time: 02:01)



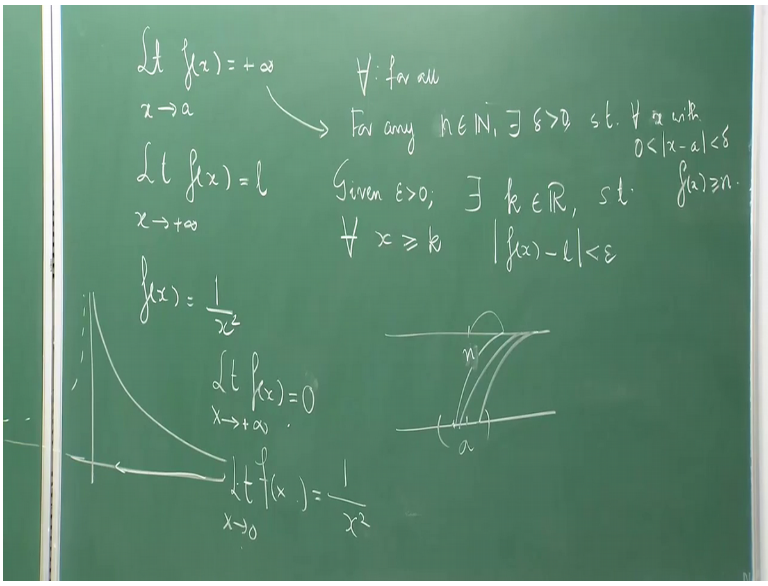
So, you understand that this function of course, x is in r I am not going to write this thing repeatedly. Of course, you know that this function is not defined at x equal to 0 because 1 by x would x would blow up. And so, what is what would happen now? So, if it is not defined at 0. So, what would happen if I take the limit of f x as x tends to 0?

This function if you draw the graph would look something like this may be I should draw it here. Now the function oscillates like this and as it sorry, it from this it oscillates means it touches the this lines y equal to x and y equal to minus x. And oscillate heavily near 0 of course, there is a wonderful symmetry of this stuff and so on. So, this is the graph of x equal to f x equal to x sin x sometimes we want to define a let x equal to 0 v 0 that is because the limit actually becomes 0. So, at x equal to 0, you want to define this function is as 0. So, this limit is actually 0 it looks obvious because it does not matter because sin x is always checking value between one and minus 1. And so, if I take x toward 0 that should just take me to 0 which is some sort of a logic which is alright, but if you want to validate it through it is definition, then you would see there is tremendous amount of oscillation of this function.

So, how can we really find a delta given an epsilon? But here it turns out that is very simple that take any epsilon greater than 0. And I know that this would be strictly less than epsilon, I want to make this strictly less than epsilon. Because now I am just looking at this if it is a limit l is 0 and I am making looking at this is same as which is less than, now mod sin 1 by x is less than 1. So, I can write this as mod x in to mod sin 1 by x or mod sin 1 by x is less than 1. So, this is less than mod x. Now because this is this in to this is nothing but mod x in to mod sin 1 by x mod sin 1 by x is less than equal to 1. So, we will straight away get in to this inequality. Now if I choose mod x to be always less than epsilon in this case my delta is epsilon you see the function looks complicated, but your epsilon delta is very simple. So, if I choose this to be less than epsilon, my delta to be less than epsilon then I get f to be less than epsilon. So, in this case my choice of delta is epsilon.

Of course here when we are writing like this we are expecting l to be a number, finite number n a to be a finite number 2, but what would happen because we are already using the term infinite infinity, what would happen if I write a statement like this?

(Refer Slide Time: 06:12)



Does this have any meaning? Secondly, also if I can write a statement like this does this have any meaning? Now the second one says that if I make x very large. So, in f x here I cannot consider left limit and right limit, I have to consider only the left limit, because there is nothing like a right limit from for infinity, but there is nothing beyond infinity right. It simply means that as the value of x becomes larger and larger and larger the functional value is smaller and smaller and smaller or smaller. Functional value is not smaller function value is going towards l it is stabilizing gradually towards l.

For example if you take a function, this is a very famous curve hyperbola. So, if you take the limit of this function. So, my limit symbols could be sometimes lim would be sometimes l t it depends on possibly the mood. I could have at that moment there is not a very good thing to do, but that happens it just mathematician are sometimes you know they just keep on changing symbols, but they mean the same thing.

Now So, if I x if this tends to infinity if you look at the graph of this you can understand it this is going toward 0. So, this one this f x goes to 0. So, what it means it means that I it becomes very large the function value is going towards 0 function value is becoming very small. So, how do you write this statement for example, this statement what is the meaning of this? It says if I give like given epsilon greater than 0 no matter how small, there exist say a real number k element of r such that for all x this is a symbol of for all this is the standard mathematical symbol means, for all for all x which is bigger than or equal to k.

So, k could be as large for every epsilon there would be a case such for every x which is bigger than equal to k. This distance would be less than you have chosen epsilon, that is the meaning of this. While the other one means if I look in to this it means here I cannot talk about epsilon, because my functional value is going to expand. So, for any now. So, what the idea is that for whatever natural number you take any n here. So, I can always show you a neighborhood around a such that for every x in this a the functional values of whatever x you take f x values would always go to this, yeah that is that is the sorry for any epsilon. Sorry, any n element of n there exists a delta greater than 0 such that for all x with f of x must be bigger than your n.

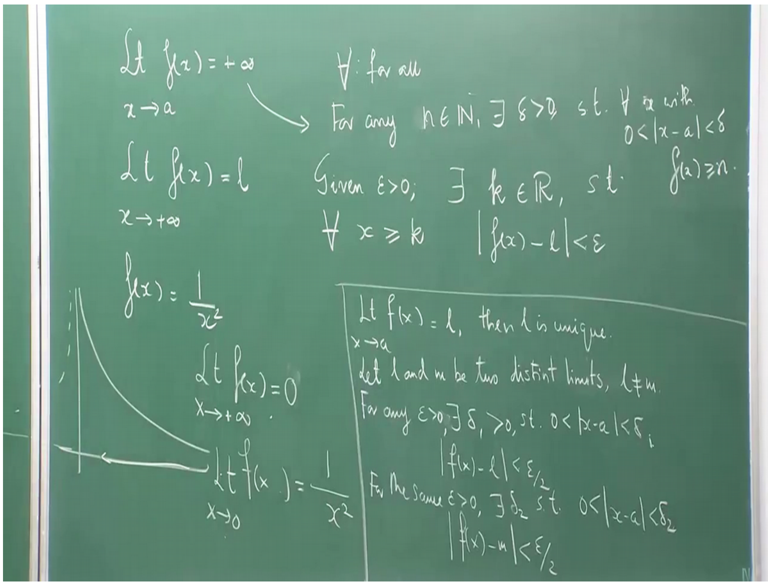
So, that is the meaning of So, it is a blowing of situation so; however, big n you choose one; however, n; however, large this n you choose it does not matter it could be 1001 1002000 10,000 I can always find x is near a such that the functional values are just going up for example, if you take the same function 1 by x and if you look at this thing limit of f x as x tends to 0. I will just change the function of it I will take it 1 by x square. So, the value would be positive whether it is on the plus side or minus side.

So, let me take f x. So, suppose I now take f x equal to 1 by x square then this is also true and then limit x tends to 0 whether it is coming from the right side or left side it does not matter this 1 by x square would start going up the value of becoming. So, here you see as you come towards 0 the functional and write 1 by x square the other part is here other part of the graph is here. So, if you come to 0 from this side also the functional value as you come nearer and nearer the functional value is going up. So, this is So, these are the various ways a functional value function can approach a limit, as x takes various x goes towards either infinity or towards the finite number.

Now, is this epsilon delta formulation helpful in telling me some more information about the limit. For example, you wouldn't bother much in the high school level that whether what would happen mind will a function of 2 different limit. It looks intuitively So, it cannot have 2 different limits, but how do you prove because there is so many functions Now, un countably infinite of them.

So, you cannot check for each and every one that whether it has a single limit or more than one limit. So, in that case you really have to base your statement on a proof and that is why proofs are So important in mathematics.

(Refer Slide Time: 13:10)



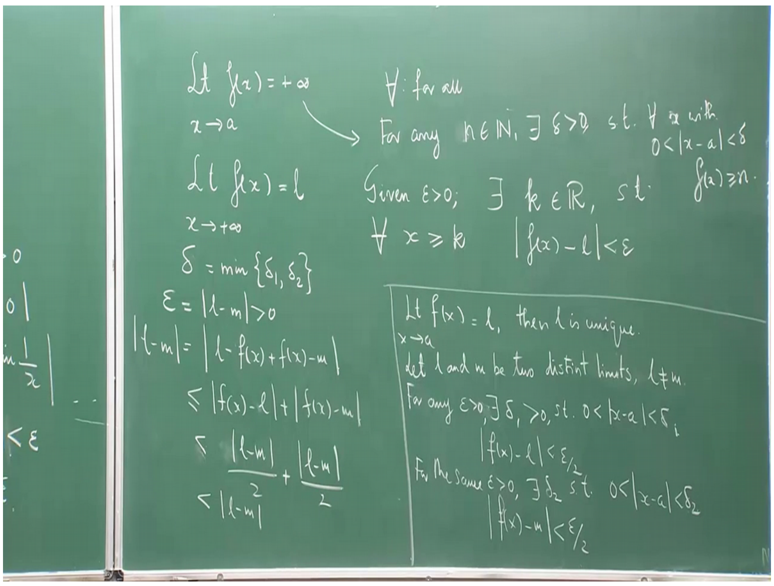
So, here we want to say that the limit. So, if I say that I have observed this limit f x as x tends to say a is l then l is unique. Now how do I prove this? The only way it can be proved very easily is using this approach. So, this epsilon delta method instead of getting worried about it is, it is it is writing that oh it looks very fearful I think you should simply go ahead and try to use it. A lot of things in mathematics is done just by using it we are getting used to things in mathematics is very important.

So, if you just get used in the first case to try to use that and get results other than bothering exactly about the philosophical issues or what was it whether it is really a good definition. So, we can leave those things with a philosophers and we can really concentrate on the main issue at hand. So, what does it mean now let us take that there are 2 limits of this. So, let l and m be 2 limits 2 different limits 2 distinct limits. So, the limit goes to l as well as m distinct limit means l is not equal to m. So, what do I have now? So, I will take an epsilon once I know epsilon I know epsilon by 2 [FL].

So now, what I do? For any epsilon there exists delta 1 great. So, I am greater than 0 there exists delta 1 greater than 0 such that whenever x is lying here I have f x minus l less than epsilon by 2 right. So, this delta 1 is a delta 1 corresponding to epsilon by 2. Similarly for the same epsilon there exists delta 2, such that f x minus m the other limit such that whenever x is in this range this holds.

So, you simply blindly applied this definition here. Now you might say what am I going to do with this definition. Oh fine let us see what we can do. So, we are going to prove that l is equal to m. So, can there be a delta for which both of these definition both of these things hold together. It is obvious because if this works for certain delta and if this works for certain delta 2, and if you take the minimum of both of them then it will work for this then that delta would work for this.

(Refer Slide Time: 16:49)



So, you can choose a delta which is minimum delta 1 and delta 2 mean less than both delta 1 and delta 2. So, then for that delta this will work because then the x that you choose is both within the delta out 2 neighborhood. And within the delta 1 neighborhood right. So, then you can work with both.

So, both definitions would work. Now I will particularly choose epsilon to be l minus m and this is strictly bigger than 0 because l l m n are not same. So now, let me write down. So now So, for any epsilon greater than 0 there would exist now common delta which is minimum of this such that f x minus lambda l should be less than l minus m by 2. Because now I have chosen epsilon 2 l minus m. So, what that l minus m there is some delta 1 again for the next case for m l minus m is another delta 2.

So, again I take the minimum and it works for both. Now what I would do is I will come in to a contradictionary in a simple way. So, this I will write as and this is a well known result that mod of a plus b mod a plus mod b. Now here I have chosen particularly epsilon equal to l minus m. So, I know that for all x. So, for all x with this delta such now for every x with x lying between delta plus a and delta minus a plus delta and a minus delta with this particular delta, I would have these to be less than epsilon by 2 and this 2 epsilon by 2, but for this particular choice.

So, which means this is strictly less than l minus m. Now somebody might suddenly get confused as I have suddenly chosen epsilon is equal to l minus m. Observe the very statement here given any epsilon if you want to be more precise I should write here given any. So, given any epsilon whatever be my epsilon it I do not care as long as it is greater than 0 no matter how small there would always exist a delta 1. So, if I put epsilon as mod of l minus m there exist a delta 1 for which this will happen with l by n here I will replace is this by l minus m by 2.

The same thing will happen here if I put epsilon equal to mod l minus m there will be a delta 2. Now the delta that will work for both of them would be the common delta the minimum one delta 1 and delta 2. And for all such x's this is true. So, what I am finally, getting the l minus m is strictly less than l minus m which is a contradiction, and hence my original assumption that l is not equal to m is wrong. So, this proof is a proof by contradiction it is one of the greatest tools in the hands of a mathematician. We have already used this proof by contradiction specially in the cantor's diagonal method the proof by contradiction is a key thing mean that side you take something take the wrong hype take the thing which is opposite of the conclusion and show that there will be a contradiction.

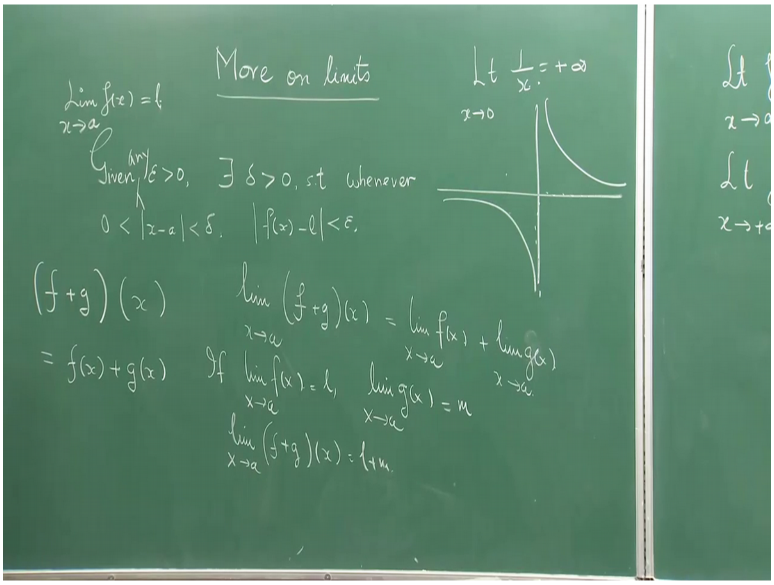
Now, this is one method of proof and people might say ok oh this is for mathematician I do not care, if you just give me the information that is fine, but let me tell you in mathematics just having an information does not make you have a true knowledge about mathematics. Because mathematics is all about structures it is not just about numbers. So, here what you see is a structure of mathematics how mathematician things how a proof has a logical validity.

Well unless you demonstrate that if you say oh I can really drive my car at 200 kilometers per hour then you should really demonstrate that you can drive it 200 kilometers per hour. You cannot just say and you cannot just say that oh I can drive it 200 kilometers per hour anybody should believe it. See a mathematical statement if I say oh the limits must be limits it is like a statement same as I can drive at 200 kilometers per hour, the question is why should you believe me the only thing. If you believe me because I can make a demonstration that what I am telling is true.

So, this mathematical demonstration is just like any other demonstration in life. So, if you want to be a school teacher for example, you need to give explanation when your interview comes you have to teach things. So, that people know whether you're a good teacher even you go for an interview process for a job, why you have an interview? Because you have to demonstrate your capability. So, demonstrating anything when you make a statement that I am this or I am that or this happens. And it is very important to demonstrate if you cannot demonstrate that does not make any sense.

So, math in mathematics the demonstration the methodology of demonstration is done by the rules of logic. And this is the way one does it. I do not think it is once you start getting habituated to it makes some much of a problem it just, you will see the use of this rule again I will use a result which you know very well. And which you might think this is very fine very simple result why unnecessarily to make my life unhappy by trying to prove this.

(Refer Slide Time: 23:23)



So, there might be questions by some students that why are you trying to touch, a nose in a roundabout way for example, this statement everybody knows it assuming that all of you are largely from the undergraduate first year first year students. So, what do you mean by this? What is happening? So, first of all limit of the sum is sum of the limits. Oh we know these from this high school (Refer Time: 24:00) it is as we are doing all these things, but for a mathematician such an answer is not acceptable. So, there is a famous story of an astronomer physicist and a mathematician and were roaming in the scottish islands and they saw some ship, far away which was black possibly.

So, the astronomer say oh all ships in scotland are black. And the physicist says now come on you cannot make such a sweeping generalization. You can say black ships are found in scotland. Then of course, that on to the that what these are mathematicians say sorry, I disagree with you both. To me the animal over there because it was slightly far off. So, it was not exactly clear that there was a sheep or not what appeared more or less like a sheep or to a mathematician, the regal look at the rigger he says that the animal over there appears to be black on the side facing us. So, if I am going to really look in to this formula the first step of rigor as a mathematician would be to define what I mean by this new function f plus g it is like a summing of 2 machines. I try to explain to you functions as some sort of machine. And so, here we are summing up 2 machines.

So, what do you mean by this summing up? So, if now it operates on input what does it say. So, we have to first define that this is same as first operating f and g individually and then adding out adding the outputs. That that is the meaning. So, this would be this is the definition of the function f plus g. In terms of the function f and g which are already defined. Once you do that you can define this. Now this is only true if the limit of g and limit of f exists.

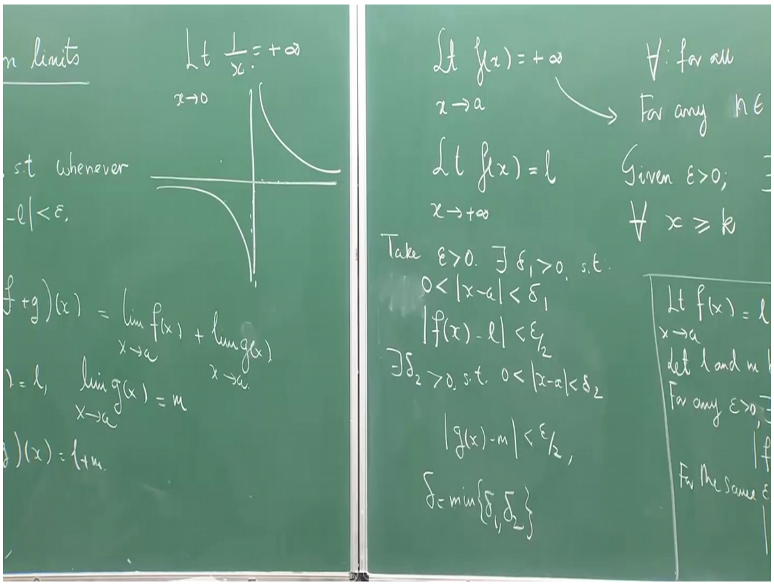
Now if limit of f x extend to a is l and limit of g x extends to a is m, then it is oh you know that of course, see here the interesting factor is that we are not sure whether the limit would exist or not. Because the right limit left limit issue is there. So, in this case if you just know that the limit of these 2 exist separately then the limit of this exists and it is l plus n. Now limit might not always exists for example, I have quite often seen in standard local calculus books they write a statement like this, but this is grossly wrong. Because this limit does not exist as x tends to 0, because when you go if you look at the graph of the parabola see that is why it is very important that you draw graphs of functions. You should you should actually sit down may be with computers, now with mathematical software try to look at graph of the functions, which you tell you the real nature of the function.

So, an x goes towards 0 the function must blows up towards plus infinity. When it is coming from the right side, but when it goes from the left side for the function f x equal to 1 by x it goes down to minus infinity. So, there are 2 limits left and right are not same. So, you cannot write such statements. So, limit does not exist in that case in a in the theory in a theoretical sense we often admit limits as plus infinity. Here it means it does not matter which way I am going towards a like x square. So, is tending to 0 I will have this when you have f x equal to x square you just have to flip this graph. Because it will always not take the positive value. So, it will it will be a symmetric flipping it will be a mirror image, and then your limit would exist. So, this example is a good example of the trick. Now you might ask how do you prove this? You would say what is there to prove nothing are nothing to prove.

But no, this needs a proof. How do you? Because if you can show that l plus m is the limit of this then you are actually proving 2 things. In one goal that you are proving that the limit also exists because they are concretely showing that it is l plus m. So, that is the things or these. So, mathematical rigor is not just a cup of tea which you might just not want it is a cup of [FL] juice possibly you have to drink for many so, but it is fun once you start your had been able to do proofs it is just a matter of time for those who have little bit of liking for mathematics would start liking it. Of course, there will be this cross fight between algebra and analysis would be more hand waving to many people who like algebra.

But very famous economists recently told me that when he was up when he learned mathematics at his undergraduate level. He always thought that algebra was the king he solved one of most famous books mean algebra by herstein at his undergrad level completely. An analysis was all topsy turvy, but later while he was doing researchers and economists he found that these approximations etcetera in analysis was the key to everything. Now the analysis is largely about change in this calculus is about change and is about approximations. How well we can record a change that is that is the whole thing.

(Refer Slide Time: 30:35)



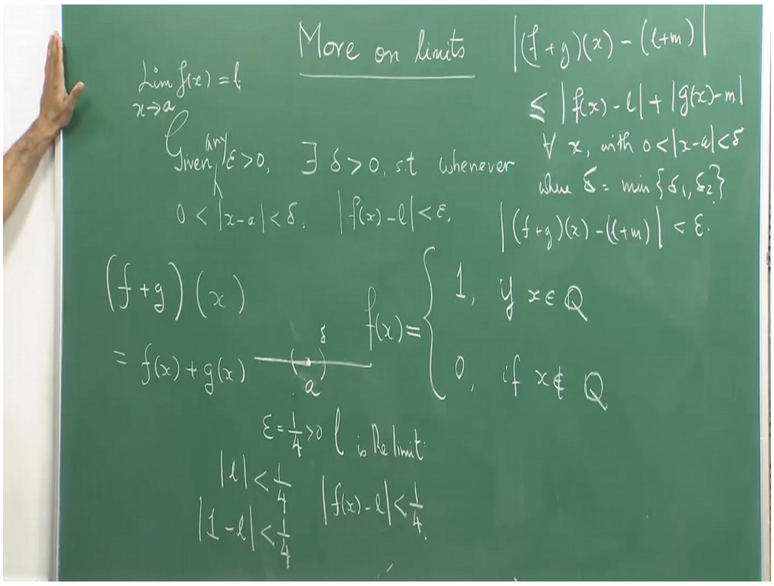
Now, for example, if I have to prove this. So, I have to prove this thing. So, take epsilon greater than 0 right. Second what would happen? So, there would exist delta 1 greater than 0 such that such that oh for.

Student: Right.

Forget to write I am. So, habituated doing all these things and I just skips my mind. Similar, similarly there would exist delta 2 such that whenever x is here. Now I can use a common delta. Now I can use though. So, if I take the minimum of delta 1 and delta 2, both these would hold at the same time. So now, your very simple job left to do means just basically some manual algebraic manipulation.

So now, we have x minus x a when x is lying between a plus delta and a minus delta where delta is this one.

(Refer Slide Time: 32:04)



So, for all such delta my a s my all such x's let us see what happens. So, given the epsilon is this delta the corresponding delta for this. So, you just f x put with. So, you use this definition f plus f x plus g x and then make f x minus l plus g x minus m, and then use mod of a plus b is less than mod a plus mod b.

So now, for all x with delta where delta is the minimum of delta 1 delta 2, I know that this is strictly less than epsilon by 2 these is strictly less than epsilon by 2. So, I will have f plus g x minus l plus m is strictly less than epsilon. So, whatever epsilon I choose I this is my delta for which this is happening. So, l plus m is the limit of f plus g if l and m are the individual limits. Now you might be wondering that you are showing the existence of limits and etcetera by or by calculating epsilon and delta's, what about showing the non existence of limits using epsilon and delta's.

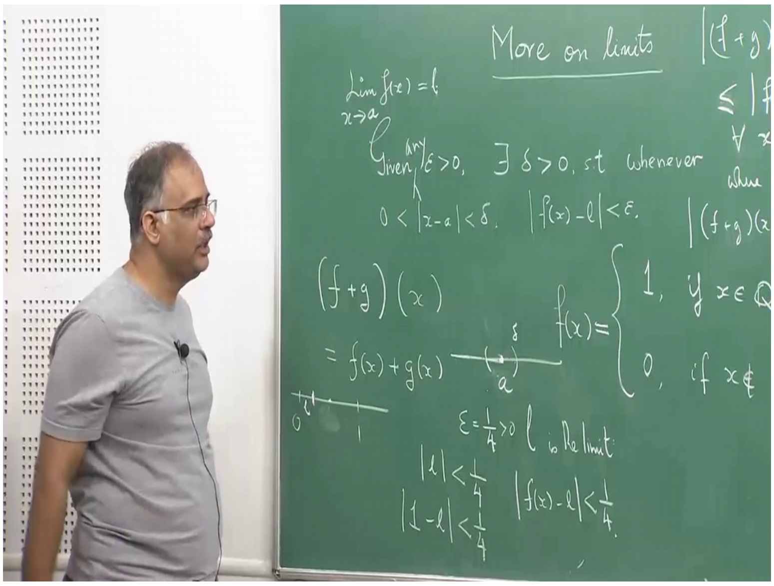
So, with that we with such an example I will end today's discussion. I am not going to work with it this sort of a hospital school etcetera at this moment that I will not doing those I will not doing this kids stuff of doing calculating limits and doing examples very basic examples. See as I have already told you in instruction this is calculus for people who are genuinely interested to learn the structure of the result and not just giving very particular example. Some particular examples would be given in several cases, but not that every time you just give someone 2 particular example which will be coming in your examination possibly, and that is that is I am please understand that these courses are not just to make you effective for an examination and then you forget it.

These courses are for you to remember them because you can see them whenever you want. Now let me take a function which is a very famous function in mathematics call the dirichlet function for the famous German mathematician those (Refer Time: 35:22) dirichlet. If x is rational. So, x is in q is equal to 0 if x is irrational. What should be the limit? This function does not have any limit. Because if you take any rational number say it I take x to be a rational number. And I say there is a say a take a and a to be a rational number. And now if I x is approaching a from both sides. Now among these x's there will be all in there will be rationals as well as irrationals.

So, at the rational points it will take. So, what is happening is that, I can now just think of taking the only rational numbers. So, then my value would be one, but if I take a value from this side, just take the irrational numbers that also coincides to a and the value would be 0, but if I take only the rational numbers then to be 1. Now, but here I have to move x as near to a. So, I take any limit as l I take l as a limit right some l. Now what I want to say is that you cannot reach any limit here. Because if you look at the if you just think about it intuitively this is just oscillating very fast. 0 1 because I have infinite in fact, countable rationals and in uncountable irrationals. So, they are oscillating 1 0 1 0 1 0. So, it is very difficult to say it is you can imagine in your mind that; obviously, it is not going towards any limit. Now how would you handle that scenario?

So, can I For example if I take epsilon equal to 1 by 4th. And say l is the limit, some limit l suppose it has a limit and given this epsilon I should be able to find a delta. Such that this should happen now; however, small I make that delta it does not matter whatever small I make the delta. Model would be less than 1 one-fourth. So, for some x it will be 0. So, your mod l your some x’s will be 0. So, for whatever delta; however, small I take it does not matter. So, I find some delta then for which there will be some irrationals for all those irrational points in any irrational point we will immediately give you this if this is true, but there will be some rationals also giving these 2 quantities. Do you think these 2 quantities are possible at the same time these 2 are not possible at the same time.

(Refer Slide Time: 39:17)



So, be because. So, here is l assume l is positive just for the moment here is 0 here is 1. So, what is happening? So, l is less than one-fourth. So, means l has to be here. So, 1 minus l is this part. So, it has to be bigger than three-fourth mean is this is one-fourth. So, 1 minus l has to be three-fourth or bigger. So, how can 1 minus l be strictly less than one-fourth. So, immediately there is a contradiction, hence you see I cannot find it does not matter what delta I take these equations will always be there. So, I cannot.

So, I am using the epsilon delta thing to show that given give me any I have given you an epsilon and I have taken an epsilon can I find a can I, I cannot find a corresponding delta or which this will hold because that is leading to a contradiction. So, with this I end my talk on limits and we are going to talk about continuity and few hard results about the behavior of continuous functions the intermediate value theorems and though those will come in the next classes. So, tomorrow's lecture would be on continuous functions.

Thank you very much.