Calculus of One Real Variable
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Lecture - 40
Taylor’s Theorem, Other issues and end of the course – II

So, now let us see what Taylor did. So, we had last lecture of the course and our aim is to tell you about the Taylor’s theorem give certain examples and finish off after that you can take up the study further I would rather suggest that you really look at good text rather than a text which I am trying to tell you rather than arbitrarily looking at internet resources and many of the times those resources tell you things which can be wrong. So, now, I write down a polynomial function polynomial is something which you know which is very simple which is taught at a very basic level.

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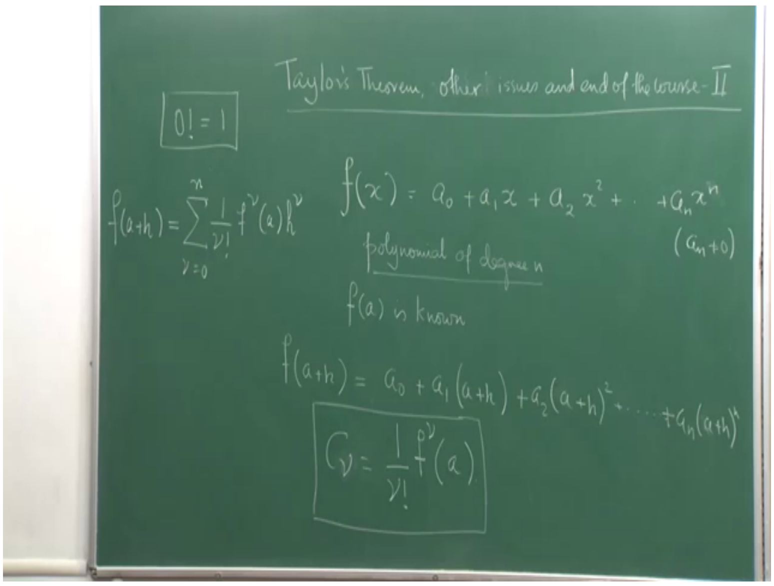

So, it is a truncated power series actually. So, this polynomial is called a n th degree polynomial x n if n if an is not equal to 0, then it is called a polynomial of degree n. Now suppose I know the function value at a f a is known to me where f x is known to me some x does not matter, then what is f of a plus h that is that is the question. So, if I know; what is a then what is f of a plus h. So, when I write x is a plus h my a is fixed. So, what I do; I write a 0 plus a 1 a plus h plus a 2 and so on up to you can say I can do binomial expressions and I could not do something with it.

Then I will write able to write down the coefficients the new ones Taylor found a very simple way to do that. So, what would happen if I do a 0, then a 1 plus a 1 into a plus a 1 into h. So, now, it will become a function of h. So, it will become power series in h write a 0 plus a 1 into a. So, this is the constant plus a 1 into h plus a 2 into a square plus 2 h all these things. So, h is a coefficient of h has to be; now it will change. So, how are you going to find it you are really going to do all this expansion do binomial expansion and then going to sum them up it will be cruel absolutely?

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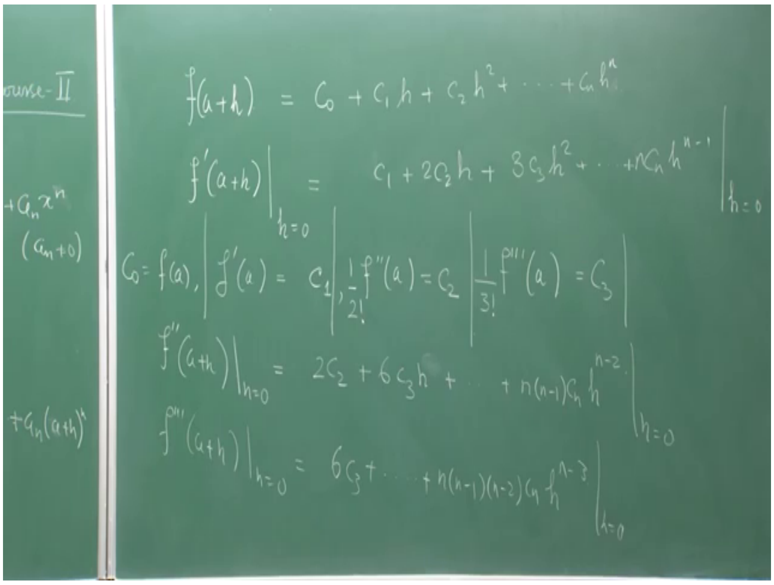

So, what would you do? So, Taylor said ultimately this would happen it will become a because a is fixed it is going to become a function in h a polynomial in h. So, it is c 0 plus c 1 h plus c 2 h square plus c n h n is it let is differentiate f with respect to h and let us and let us see this c what happens and calculate the c 0 c 1 all those things. So, f dash of a plus h; he wanted to calculate at h equal to 0. So, you calculate the derivative of this. So, it will become c 1 plus c 2 c 2 h plus 3 c 3 h square plus so and so forth n c n h n minus 1.

Now, I am doing it at h equal to 0. So, I am computing it at h equal to 0. So, say f dash at h equal to 0 means f dash a is equal to c 1 what about c 0 oh just put h equal to 0 here if you put h equal to 0 f a is equal to c 0. So, your c 0 finally, is f a. So, your c 0 is equal to f a and f and now how to find c 2. So, we take the derivative again f double dash a plus h at h tends to 0 h at h equal to 0. So, see all these derivatives are assumed to be is not sorry; not is a continuous because a because a polynomial we will have a degree of depend a derivative to up to infinite order right up to n th order anyway, it will have an in inventors going on it will just go on will give you 0 derivative.

So, it is always differential. So, all these functions are continuous; all the derivatives are continuous. Now what is happening is that here if I do the second derivative means I take a derivative of this. So, it will become 2 c 2 plus 6 c 3 h square c 3 h plus n; n minus 1 c n h n minus 2. Now you calculate this at h equal to 0; once you do that you will immediately see that f double dash a by 2 which will for some reason, we will now write factorial 2 it is equal of all this as 1 f a is equal to this is equal to c 2.

Similarly, you can try out with the third derivative. So, what will happen? It will be 6 c 3 plus I am not writing the other terms n into n minus 1 into n minus 2 into c n into h n minus 3 all evaluated at h equal to 0. So, what will be this? So, it will become separating them. So, now, I will have f triple dash a 1 by 6 which is 1 by 3 factorial that is why we have just taken this factorial writing and that is beautiful, it will always come like that is equal to c 3 and so on.

So, what is the general form of this c 0 c 1? So, in the general case c n; right or not c n on any c nu say c nu is 1 by nu factorial f the number of times is differentiable f nu I am not writing dash, dash, dash f nu means the number of times is differentiable at a. So, finally, that f of a plus h can be expressed as summations nu equal to 0 to n 1 by f nu a into h nu and you can ask; what about when nu is 0. So, there is this standard convention in mathematics, I told you once at lot of mathematics depends on convention 0 factorial is 1.

So, this is what Taylor achieved. So, if you know the value at one point; right then you simply know the value at any if you know the value at a, then you simply know the value at any other point. So, and it is just a finite series it is just finite its very simple. So, you just calculate the derivatives and you see it. Now it is always not easy to calculate the values of polynomials also because it involves lot of power. So, this will immediately do it. So, you just do not have to bother about the power you have to just know; what is the difference between the 2 values and take the power of that just take the power of a number and take the power of sum of 2 things and I have expressed binomial thing; it is not always easy.

So, the intuition that Newton gave to Taylor was the following that if you do not have a polynomial function, then what would be the situation; if you do not have a polynomial function, then suppose f is differentiable or and continuous up to n times and now it is differentiable n plus 1 th time whatever. So, intuition was this; the intuition is that you know if you know a, then f of a plus h can be approximated at a by a polynomial up to certain degree and then there will be error.

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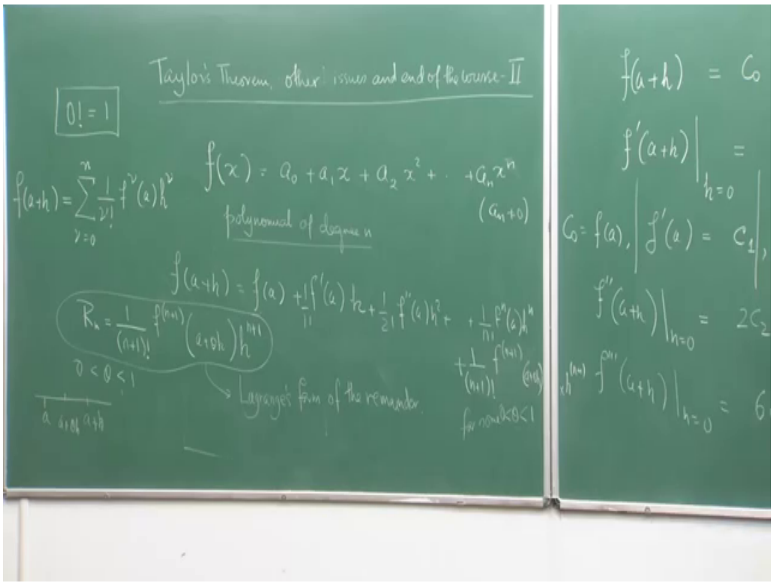

So, what you are trying to say is that given any continuous and differentiable function which has. So, many derivate continue derivative say n th a derivative up to n th order which is continuous can be approximated as a polynomial. So, is it because you know if a function is as we have been telling when you are talking about Taylor Milligan mean value theorem that a function; if a h is very small it can be approximated by a linear function f a plus f dash a into h; the derivative if it is better if you approximate it is a better approximation; if you take a quadratic function where the second term comes, it is of much better approximation if you take the cubic term comes.

So, if we increase. So, those are the; so, you get another on the polynomial plus an error. So, basically what you are trying to do is given a point where the derivatives can be calculated you are trying to approximate the function value by using polynomials because polynomials use those basic arithmetic operations to compute a polynomial you use basic arithmetic operations multiplication division even taking a power of an object power of a number is also nothing, but repeated multiplication. So, is it you have this one factorial plus this part plus R n plus a remainder term.

So, this remainder term this is an idea that we have to take from the last lecture that a function which is not a polynomial one by one minus x is not a polynomial, but that it could be approximated by the polynomial S n plus R n a remainder term which vanishes as n goes to infinity that is the key idea. So, that idea which Newton already had because he had figured out the binomial expression that idea he told to Taylor that the; this is the idea. Now you have to follow that for polynomial term when it is a polynomial you do not have R n because n plus 1 is all derivatives are 0, but if you have a function where the n plus 1 is or a derivative and rest of the day up to n th order derivative is a continuous, then you can have a polynomial expression and the R n term could be possibly written in terms of the R n plus one at derivative.

So, this is the key idea that that Taylor gave to Newton and then Taylor figured that out. So, the first part this part is called a Taylor's polynomial; this part is called the Taylor's polynomial and this is the remainder term. Now what is the remainder term; how does the remainder term look like to prove; how does the remainder term look like; it needs a lot of a work to be done which we are not going to get into and because you know in the last class we do not want to get into too much details.

So, Lagrange figured out that if there is a if there is the n th plus 1 derivative which need not be continuous, I can write the remainder term in terms of that this that is f of a plus h minus this part can be written in terms of the n plus 1 th derivative and how can it be written. So, Lagrange suggested that the remainder term can be written as follows and this is quite suggestive. So, let us just look at the intuition of it theta is lying between 1 and 0. So, basically a plus theta h is a point between a and a plus h; suppose, I assuming a is positive or negative does not matter is some point between a and a plus h.

So, a plus theta h is much; see what is the intuition between such a writing of the remainder term this is a Lagrange's form of the remainder this is called the Lagrange's form of the remainder, you can see derivations from the books and many other places, but I just want to give you the key idea he says that suppose this was the n plus 1 and degree polynomial; what would have happened what Taylor was writing; it would be nothing, but 1 by n plus 1 factorial f n plus 1 a into h n plus 1 look at the key idea; here if it was a polynomial R n would be if it is the n plus 1 degree polynomial f R n would be nothing, but this with n plus 1 that would be R n it will be one by n plus 1 factorial f of n plus 1 a by h n plus 1 that would be R n.

But it is not a polynomial, but suppose in my mind, I want to think that it is like a polynomial, then I would say that I am able to do an exact manipulation an exact expression, but not with f nu over f n plus 1 of a, but fn plus 1 of some form like this a plus theta h. So, basically he was telling that I am somehow. So, this is the n th order expression, right Taylor polynomial p n. So, Taylor's polynomial is this, right. So, he is telling that essentially because it is not a polynomial I would have instead of just a I would have not just a, but a shift from a see I cannot go beyond a plus h; you are not have to remain within that.

So, I just if I move away from a; I will capture that error. So, it cannot be f n a because it is not a polynomial, but it could be something a little bit different that will actually pull in there are make an exact representation. So, I can have it as a plus theta h into h n minus in into h n plus 1. So, that is the sorry here I should have also h n plus 1; sorry. So, this is this is the Lagrange form of the remainder and this is what do not forget the h n plus 1. So, this is this is the basic intuition; I am not going to the detail map because detailed map would require detail map on a detailed discussion and that we have no time for it and in many cases you have to use possibly the mean value theorem integrals and all which we have not taught.

So, here let us just have the intuition; it is much better to have the intuition clear than anything else. So, what is the intuition; once again if we do the polynomial, I will have a; this is thing with just theta is 0. So, theta is something which is capturing the error that is the key idea; theta is something which is capturing the error here and so, he is telling that it because it is not a polynomial; I might not have this; why I might not if f n a, but I will have f n a plus theta h see if I calculate at f n a; it will not be an exact representation, but I will calculate it little bit off from f n a; little bit off from a, then it might give me.

So, there is some theta for some theta. So, I cannot pinpoint the theta; it depends on the choice of a and h and all those things. So, theta depends on the choice of a, choice of h and all those things. So, this is the key idea. Now what would happen is that when is R n x going to 0.

Student: Generally for any object (Refer Time: 20:11)

No, no, I am taking non polynomial function. For polynomial function, theta is 0, please understand; I am making sure; I am telling that these for function is not a polynomial one. (Refer Time: 20:22)

Student: (Refer Time: 20:23)

It cannot be 1 also, right because 1; how can it be? You are trying to compute the value of a plus h. This is actually higher level application of a Lagrange mean value theorem. So, which we are not doing it; so, what would happen; now how do I know that this sequence; is there any condition under which this remainder term would actually go to 0.

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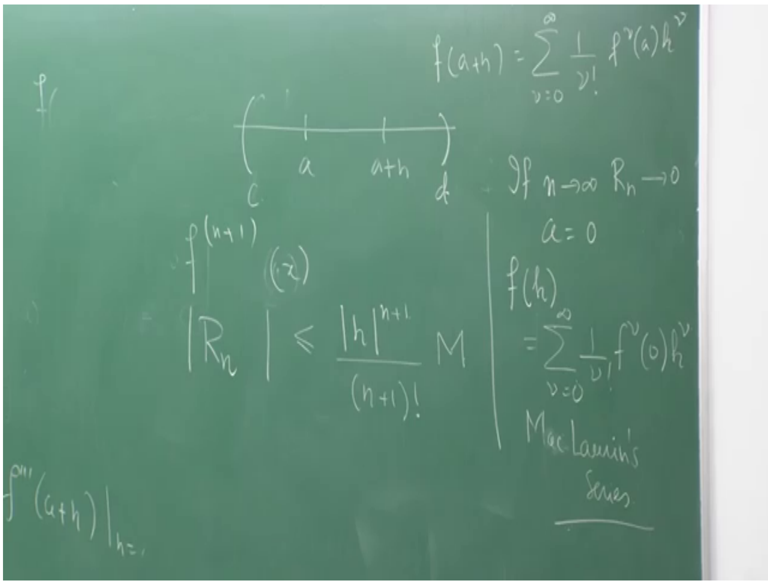

Now, assume all these a and a plus h; a and a plus h is contained in some bigger open set say c and d and in that open set, an open interval not open set, I am very sorry; open interval all; the functions are all the derivatives are continuous and all those things. Now assume that f of n plus 1; this is also continuous function in that interval c d and then what would happen is that in the closed interval a to a plus h. So, if I take this and if I take if the function f n a x; in the closed interval a to a plus h, then, oh this is a bounded function because it is continuous.

Then your R n mod of R n. So, this would be less than some m this one becomes less than equal to mod of h n plus 1; mod of n plus 1 factorial into some m which is the bound of this function. Now the question is that when can be show. So, now, as n tends to infinity; what happens to this? So, there are several cases where we can show that n tends to infinity. Now if n tends to infinity, R n goes to 0, then what we get at; what we get is that f x can be expressed as a Taylor's power series that is f of a plus h, then becomes summation nu equal to 0 to infinity one by nu factorial f nu a h nu.

So, it becomes a convergent series provided R n goes to 0 and n tends to infinity, it is not that every time R n will go to 0. It will go for 0 for certain examples and what we should be looking at. So, now, if a is 0 suppose, I put a is equal to 0 and suppose now; then in that case you will have f of h is equal to summation nu equal to 0 to infinity 1 by nu factorial f nu 0 h nu. So, this when I have rewritten in terms of 0 I had done the expansion in terms of 0 that is a is 0 and this is called the McLaren's series, I have forgotten the spelling McLaren's series.

Now, the question is in what case? I will be able to and what in let us take an example where we can actually show that we can we are able to make R n tend to 0 as n tends to infinity; let us show in first one example, then we will say that we can express it in a power series; having these does not say that it will go to 0; only we can say that mod R n is bounded. So, as n goes to 0; so, for what I am creating is that error is bounded above by some quantity, but that is depending on h and n plus 1 factorial also blows up or whether it goes faster than h n plus 1 to 0 or not that we do not know that depends on the specific example specific situations basically.

So, only we have proved that R n can be made to be a sequence bounded by a function of h that is the only thing that we can show provided this is there otherwise we cannot even show that.

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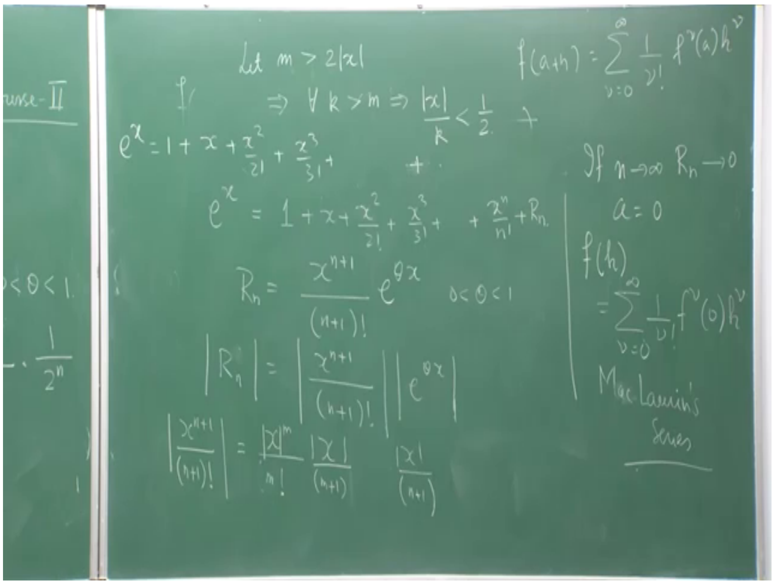

So, let us take an example; a very important example the exponential series e to the power x, I am drawing all these things off if I take e to the power x e to the power 0 is 1 1 plus x plus x square by factorial 2 plus x cube by factorial 3 plus the last term x n by factorial n plus R n x plus R n. So, R n x if you want to call it because we choosing that particular x.

So, what is this R n? R n is nothing, but. So, what expansion we are doing here. So, we are doing an expansion of the McLaren type this is your age. So, your f dash, so this is you are taking the first derivative second derivative e is nothing, but e to the power x and you when you put 0 they are just one. So, is the McLaren type expansion; let us see what is R n; R n is x to the power n plus 1; sorry, n plus 1 factorial e to the power a is 0 0 plus h is h is actually h x. Now theta x where theta is between 0 and 1.

So, let us look at this; sorry, mod R n is equal to mod of x n plus 1 n plus 1 factorial. So, mod e to the power theta x; so, let us look at this quantity first; let us see what is this. So, I can write here as x m by m plus and maybe x m by m factorial x by m plus 1 factorial sorry m plus 1 into n plus 1 is up to m factorial you write that with powers and then you separate them.

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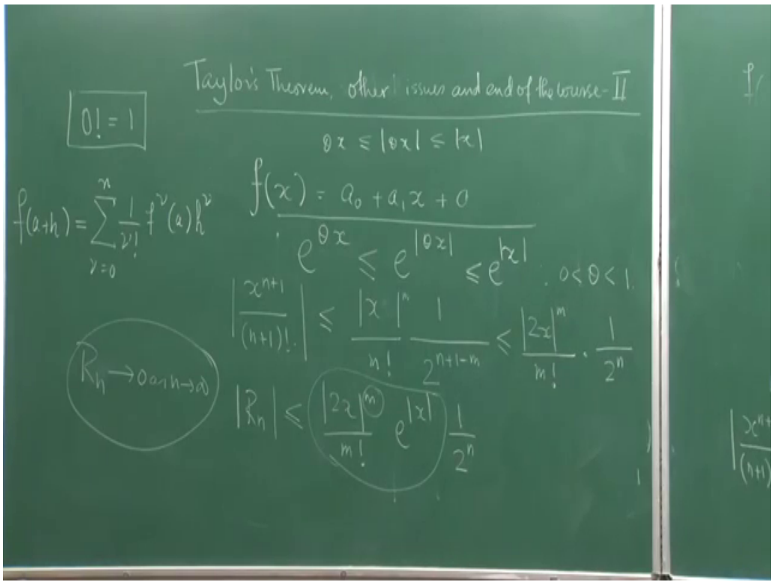

Now you see we have broken off x n plus 1 by n plus 1 factor into these things you can understand, right. So, how many terms are here? Here we have n plus 1 minus m terms, right; m terms have been taken off and let us now take a m such that that value of m is strictly bigger than 2 of x. So, we have fixed up in x and we are just looking for that particular x and because our x chosen will be arbitrary. So, this game will operate for all x. So, whatever be the x is does not matter you always choose m take a m such that m is strictly greater than 2 x that you can choose.

And you can always choose your n to be bigger; much bigger than that m because you know this is infinitely differentiable is e to the power x. So, then for any k bigger than m mod x by k would be strictly less than half; so, you just bring it like this here m bigger than k. So, mod x by k is less than half. So, basically its means this is less than half this is less than of up to this less than half. So, here this simply means this is x to the power m by m factorial each of them is less than half and I told you there are n plus 1 minus m quantity. So, there is 1 half into n plus 1 minus m to the power n plus 1 minus m.

Now, this is less than; so, this is a positive thing. So, 2 of x; obviously, is bigger than x. So, mod 2 x to the power m is bigger than this it is less than this. So, what would happen once you write this; this 2 to the power m will cancel up with this one and we will leave it as this; right. So, now what is R n? R n is less than equal to this. Now what is e to the power theta x? Theta x is less than mod of theta x that is theta x when any x is less than mod of x mod of x. So, because it is the increasing function e to the power theta x or non decreasing e to the power theta x is less than equal to e to the power theta mod x and since mod theta is less than 1. So, this is less than mod x. So, this is again less than e to the power mod x. So, you will get this 1 into 1 by 2.

And now as n tends to infinity this is a fixed number m; we have chosen some number m which is fixed. So, this part an x is fixed anyway this is independent of n. So, as n tends to infinity 1 by 2 to the power n goes to 0. So, from here we have proved that R n goes to 0 as n goes to infinity now this thing is done for every x. So, I have chosen 1 x and took taken 1 m and proved it for that whatever x you do; I will prove it; this is the same story. So, which means now e to the power x can be written as a power series which you know very well this is something you have possibly taken in to the memory.

There are several other important functions like sin x cos x cos hyperbolic x which have these sort of series which are holding for all x for example, 1 minus a 1 by 1 minus x you saw; it does not hold for all x. So, here it holds for all x and that. So, these are actually Taylor's representation, there is an important question which I result which I have not stated, but I will ask primitive has to put in the board is that every function may not have McLaren expansion; there could be a function which has you can compute derivatives to any order at 0.

But you would not be able to get the McLaren's expansion, but let me keep that example and then end it. So, I will here; I will give an example where even though you can compute the derivative at every point 0 because at every point every level of derivative a computed 0s; see what happens just by computing at 1.0 need not give you the expansion the idea says that a function has to be differentiable in a neighborhood right containing 0 and h in a neighborhood content 0 and h is the function f should be differentiable all the derivatives have to be continuous up to a mod n plus 1 th order then only this is true.

Now, I will give you a very important example; this many times asked in interviews or any other thing this is a very very very standard example which I want to give where it where the McLaren series cannot be handled that is you can you can you have the function.

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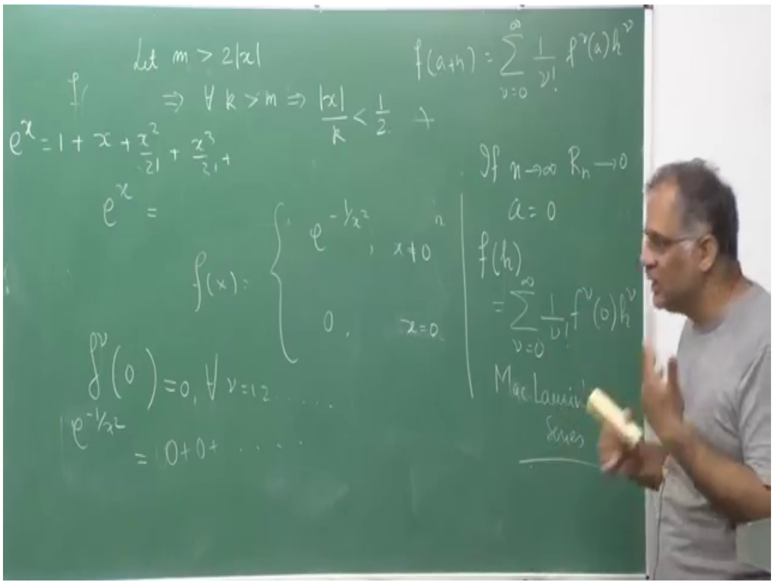

That is an x is equal to 0 this has derivatives of all order actually at 0. So, f nu equal to first look at this function f x equal to e to the power minus one by x square x is not equal to 0 equal to 0 and x is equal to 0. So, very strange, but f nu 0 is equal to 0 for all nu 1 2 matter for any order of any value of the f nu 0 is equal to 0 it exists, but take any h or any x, it is e to the power minus one by x square. So, that cannot be expressed through this expression because f of 0 is also 0. So, this one side is just 0 if you do a McLaren's expansion. So, McLaren's expansion is 0, but here for x non equal to 0 e to the power minus one by x square is not equal to 0.

So, just by having the derivative at one point the point 0 does not tell you that McLaren's expansion holds because you have to have the derivative existing in continuous at all points in a neighborhood containing 0 and h and that is what is not true here. So, McLaren's expansion always need not hold. So, this is some important example which is there and with this I end the course and hope that you have liked it and hope that this would help you to investigate further I will write tell my ts to write down more series a about sin x cos x cos hyperbolic x in the portal and so that you can learn it and thank.

Thanks to everyone for giving a concentration and hope; this will be useful in whatever you do in the future and all the best for all the students who are listening to this; best for them and all those who are possibly elder have done something else already have accomplished what they want in their life; what want to have our journey into the calculus all the best for them also there are so many texts you can go and enjoy yourself.

Thank you very much, thank you once again.