Calculus of One Real Variable
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Lecture – 04
Functions

So, welcome once again to the fourth lecture, but you must have observed here I have written already functions. So, functions is something you have already started learning in your school and also in a high school and you know how to handle functions, you know that you put something call x and outcomes or number called y and you can do a lot of things with it play around with it. You heard terms like domain, range you have heard about functions like sin x, cos x, log x, e to the power x. So, all these are quite familiar thing to you, but there are certain (Refer Time: 00:50) even in this concept which you need to get used to.

For today's talk I will conceptually keep myself in the line of a book, from a book called concepts in modern mathematics by Ian Stewart. Ian Stewart is a very big British mathematician and also one of the best writers in mathematics of our time.

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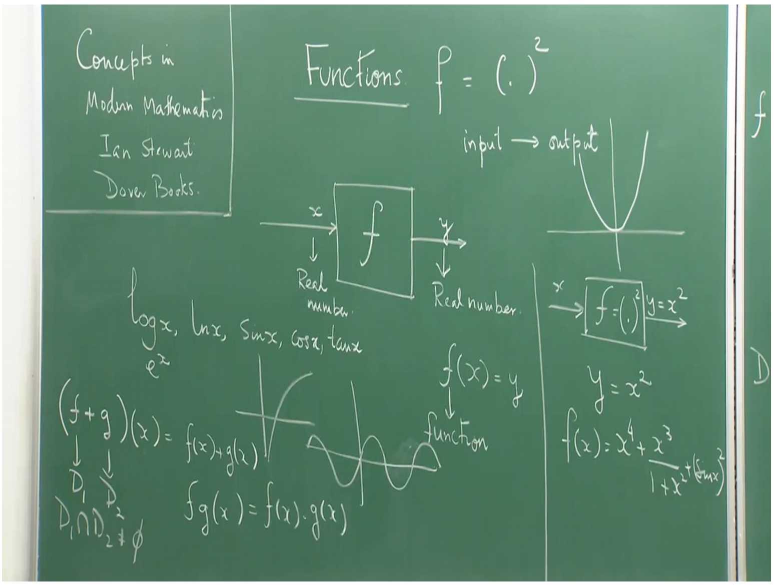

It is published by dover books and (Refer Time: 01:40) it is your school libraries can have it and I think this this is something which most even all students of mathematics should have a look. For anybody who has some interest in mathematics should look at, see a function has always been visualized as a kind of machine. So, here is a machine which we called f and here we given input x and here comes a unique output y. So, every input x there is a unique output y, but if they there is more than 2 outputs then I will never call such machine a function. So, this uniqueness of y is a very very important thing. So, it is a machine where every input has a unique output. So, the key idea. So, assume that this is a mathematical machine where I put some variable some real number in and out comes another real number right. So, a real number in so this is real real number and this is real number out.

You have heard about various kinds of functions, you have seen if you take up a calculus book these are the common functions which you will keep seeing and so and so forth these are the most used functions in calculus for example,. So, what we say is how do we describe this whole procedure. So, here is an input on which the machine f acts and give me a unique output. So, and the machine f acts on x it gives me the output y this machine f is what we call as a function, the question arises that can what sort of machine is this how does it look and whatever x I put in can I get back a y that is the question for example, I say I put in an x and this machine is the squaring machine.

So, whatever number you put in it squares it. So, out comes the output x square. So, this can be also expressed as y equal to x square right. So, that is a very simple way of looking at this whole definition because functions have been used by many many mathematicians over the ages and it took almost a span of 250 years to formalize what is, what we now use as a definition of function. People have used functions understanding what they have to do with them and pretty and mathematics has developed in that way. So, so the development of mathematics has got a definitive link to what we now know to say when we define in a standard class we simply just go and write it f is a function blah blah blah stuff that takes away half the fun of it because you do not tell that this concept this form of formal writing has taken many many informal steps and that is the way we are going to do to it for example.

You already know about the sin function. So, you must have drawn graphs you are known over the log function for example, it looks like this the sin function looks like this sorry it looks like this and so on. So, drawing graphs of functions has been an activity you did in school and I think this activity should be done when you are now in your undergraduate level. So, you should even if a high school kid is listening to this lecture please understand that drawing graphs of functions tells you about the very nature of functions.

Now there can be many many types of formulas, suppose you can have very complicated looking function that if I put an x my function should look like. So, when you have a math book with these sort of things written you think that mathematicians job is to generate formulas and math is some sort of a blind manipulation of formulas, but if you do mathematics in that way you lose half the charm and half the fun of doing mathematics, examples Stewart gives in this book and which he says regularly is a classroom trick for him is to ask the following.

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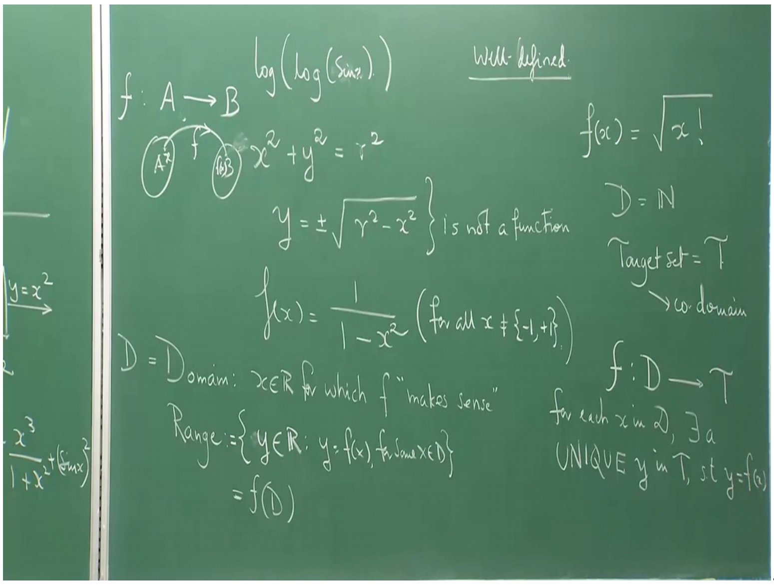

So, those who know something about differentiability already in high school ask them to differentiate this function, he says that most of them do it differentiate them by using chain rule, but hardly many of them realize that this function does not make sense they only realize it when they go to a mathematical software and try to draw the graph. The sin of x need not always take positive values for a given x and log is only defined for positive values right. So, one has to be very careful by the fact that every x need not give me an output. So, there are certain amount of x which gives me output and certain x's which does not give me an output. So, one might say is any formula involving such x and y's is a function for example, your one of the most useful formulas you have learned in school is this, high school is equation of a circle centered at 0 with radius r.

Now, is y a function of x here, here y can be written as. So, you see corresponding to every x there are 2 y's. So, the y is not uniquely determined. So, this is not a function. So, simple square rooting is not a function and even things which look like functions, which does not have this problem may not be defined everywhere even if I take sin x such that take those values of x in this function where sin x is positive say I restrict it to this spot. I go from the I take x between x between 0 and pi by 2 without taking 0 and pi by 2, still this would not be defined right because sin x is less than equal to 1.

So, any log which is less than equal to 1 is negative and again you cannot define this. So, this sort of functions writing something arbitrarily which looks fancy need not be always something well defined. So, this concept of well-defined is a very important notion in mathematics. So, what you define what you say has to make proper sense for example, this one is not a function for example, let me come here I write this formula, is this a function? Looks like, it looks like it will give me only one y not 2 y's, but for every x I cannot define it because if I put x equal to 1 or minus 1 I have a problem of dividing by 0. So, this is defined for all x not equal to minus 1, plus 1.

So, this, so this ideas in those days gave rise to the notion of the fact what we now call domain and range of functions that is given a function every value, every real number x may not be useful for it. So, given every real number x that function may not give me an output, which is another real number. So, if we are defining a function there are 3 critical issues, one is a domain that is the x in element o r for which the function make sense for which f is well defined actually which I am writing more in a crude way for which f makes sense means in the sense that you can actually get an output

Second is a range, which is sometimes useful for example, if you look at the function of squaring whatever x you put in here you will always get a non-negative number. So, the non-negative real numbers is the range of the function, if you draw the graph of this very elegant and very useful function. So, quadratic function it looks like this. So, for every x as you take the square the function value increases over the whole non negative axis is the range. So, range of f is sometimes defined as a set I am assuming that you know what is a set that set of all y in r such that y is equal to f x for some x in d, some from a more compact point of view I can simply write it as f of d if I call d as the domain, if I d call d as the domain. But sometimes getting a range becomes very very problematic, you cannot always say that I have found a range I have found found a range one I can always find the range of a function. Stewart gives a very nice example, that it is always not possible to define the range of a function, he gives a very unique example and I have never seen this example anywhere in any book square root of x factorial.

So, what does square root of x factorial means first of all you cannot have an x factorial unless x is a natural number. So, your domain is immediately restricted to n of course, x factorial is positive, so there is no problem in taking the square root. Now what is the range, how do we describe the range? So, it is, is it the range of all possible factorial values how do you write it down formally you cannot write it down.

So, writing down the range formally is very difficult here it is very easy to write down the range, but what here it is not possible to write down the range that is why it is sometimes more useful to define a function. So, people can say function is from domain to range, it it takes up pics of a value when you put an input comes from the domain and output is in the range, but because range cannot be very clearly specified for many cases it is sometimes useful to talk about something called a target set for a codomain. So, this target set is something is unique to steers Stewart sighting and or sometimes this is also called the co domain here same thing.

So, once you have that this the codomain of this are positive real numbers again. So, whatever square root you take has to be within the domain of positive real numbers. So, the codomain is a positive real number, the interesting feature here is the following that. So, in general when you define a function you can define a function from a subset d of r to another subset t and for which the central idea is the following that for each x in d there exists a unique.

So, I am writing it in capital letters UNIQUE y in t such that y is f x. So, remember when I talked about the function I am talking about this machine. So, f is the function. So, h first of all I am writing the square function. So, I am just writing down the machine I am not writing down what do I put in here, I could put 4, I put 5, I could put 2.5 I could put 2.3 whatever. So, this is, this represents a function, but when I write f of x means I have put x in this place, then I am talking about the value of the function I am talking about the output. So, there is a subtle difference between f and f x those people usually do not care in mathematics that much, but if you go you tend to be too logical then you really have to care about this very little subtle differences. So, now, once we talk about a domain and a range you might get a feeling that we are just talking about 2 sets we need not bother about whether this domain comes from real number set or whether this target comes from the real number set. So, function f in general can be defined as a rule so rule which is assigning the it tells you the way of operating with the given variable which takes an x from a. So, A and B are non-empty sets of course, takes an x from A and puts it in y put puts it in B. So, x and this is your y so or f x.

And f is the rule which is doing so. So. A is again called the co domain, A domain B is called the codomain for example, if you take a thing like a is a set of all circles then I can say that if I keep putting x in this f it will return me the value it will return me is the radius of the circle. So, the set of all circles is not a set in the real line it is a set of you know in the collection is the geometric set in geometry, you collected all possible circles is a class of curves. So, this idea of function can be used in a phase in different settings and. So, this is a very general idea that that is exactly what you are taught in school.

So, usually people would (Refer Time: 19:25) that f is from a to b and this is this, but all this thinking has gone behind bringing up a definition at a higher level we had a higher level of abstraction where we are not keeping our self consigned to real lines. So, when we will study we will remain consigned to real line, but we will have specific study of functions like log x, e to the power x, etcetera etcetera which we do not get in here. Now if there are 2 functions, I know, should know how I you can combine them, the question is why you asked this question because if you are talking about your domain and range as a set of real numbers because real numbers can be combined in various ways you can you want to say where the functions can also be combined in that ways because ultimately the value of a function is a real number.

So, that makes us give certain definitions which might look absolutely obvious to you suppose what is the meaning of 2 functions adding 2 functions if f and g are given 2 functions over 2 different domains. So, suppose this is a domain d 1 and this is a domain d 2 then this f plus g is a function where you have to operate both f and g.

So, then this function is only defined if the intersection between the domains is non-empty, if the intersection between the domains is empty you cannot add these 2 functions. So, even if it and the formula is very simple. So, if I put operate an operator on input x this is nothing, but f of x plus g of x, we might overlook this very simple thing when we try to define, but unless you have this this function is not defined. It goes with any other sort of a whenever you are making combination of functions that domains must have a common intersection then only the function gets well defined for example, if you if you want to define the multiplication that is the way it is defined. So, I will leave you to think about other things like subtraction and division they are on the same thing for division you must know that the numerator must not be 0. Now an important thing that I should note is the following I should not talk a bit about the nature of functions how functions behave so I will take simple diagrams to show you.

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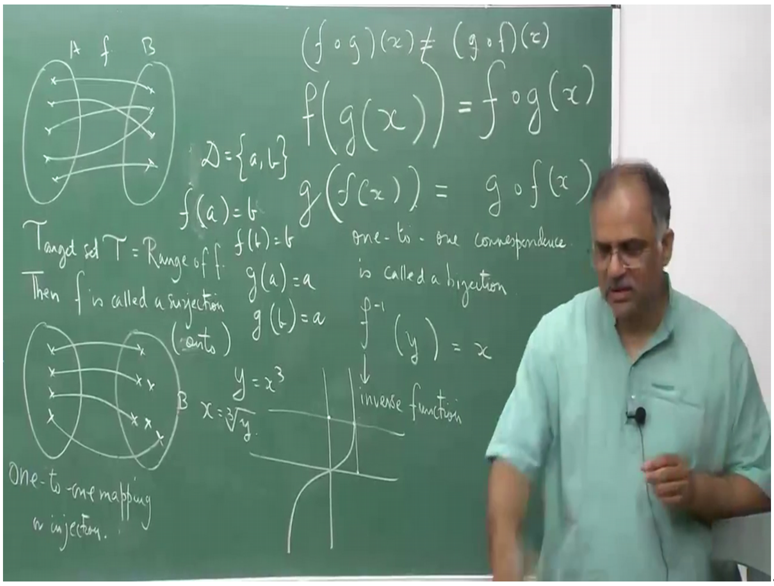

For example a function could be like this. So, here is your a and b here are say 5 objects and here 4 objects. So, what I do I map it here, I map it here, I map it here and map it here. So, what happens here that my codomain b has now become the range of the function. So, so f is the rule here if the target set t is equal to the range of f then f is called a surjection or an on to mapping, in the sense surjection comes from a Greek lattice sorry a Latin word which says which means to throw one thing on the another. So, you are throwing the whole thing on this that is the meaning of surjection then f is called a surjection,n similarly you can have something called an injection or a one to one mapping

So, here you see every, what is happening here this particular x course is the this 2 different inputs are giving the same output remember this this is giving this unique output this is giving this unique output, but the 2 things are giving for example, minus 2 and plus 2 we put in this function will give you the same output 4. So, here you what you see here corresponding to every y there is a unique x to which it is linked, here there is an x which does not is not linked to 2 unique things.

So, in this particular case what you have is the following that you have a one to one mapping, but it is not a surjection because your codomain is not your range your range is the proper subset of the codomain the range consists of these 4 elements. So, this is called a one to one mapping or a one to one mapping or injection, similarly all I have this. So, here it is a one to one mapping as well as a on to mapping surjection, surjection is called on duals sorry when you have such a situation when you have such a situation then what we have is a one to one correspondence, what we have is a one to one correspondence that is I can go if I come from an x to y I can go back from that y to x.

So, 2 way street and so this is called a one to one correspondence or a bijection which is this and this together and there is a very very important criteria when you are looking at finite size this is immediately apparent a function f is a bijection if and only if it these 2 sets a and b has the same cardinality they must have same number of elements, right this is something very very important. Now once I have done that, once this has been done this basic ideas are there this one this bijection talks to me tells me rather that I should be able to get some more idea about the function that this allows me to construct something called the inverse function, that is f inverse which acts on y and gives me back an x this inverse function is possible if you have a bijection

For example here the inverse function is possible, but now you have to change your, your codomain is no longer the domain of the inverse function it is the range which is the domain of the inverse function here of course, there cannot be inverse function because the reverse function, if you look at this this y you are would have 2 outputs. So, it cannot be a function. So, this is called a. So, it gives rise to something called inverse functions, does every function has the inverse y equal to x square does not really have a inverse for example, if I take take the level 4, I have 2 objects 2 and minus 2 to which it corresponds. So, it does not have any inverse, but you take the function y equal to x cube say then it is a inverse to every y that you take this is your y, these is your x y this is the corresponding x. So, the inverse in this case is x is equal to cube root of y. So, you have written the inverse function inverse itself has a function. So, these are something which we have spoken about, what we have not spoken about here is how to compose 2.

We will speak about them when we were talking about derivatives and other things, but composition of 2 functions is a very simple thing that how do we operate 2 functions one after another on the same input. So, what we do is the following just rubbing this off. So, here I have an x and on which I operate the function g. So, it gives me a value g x now g x is in the range of or in the codomain of f. So, the domain of f must now contain the codomain when domain of f must now be contained in the codomain of g. So, if I operate g on f on this function then we call this as f compose g of x. So, I would allow you to take a little bit of time and think how to sort the co domain co domain matter for this similarly it could be the opposite one, that I first operate f on x assuming that x must now be suppose I want to do it like that. So, that x is now in the domain of f then I can do it like this.

Suppose first a operate f and then I operate g, then I get I write this as g compose f on x sorting out the domain codomain issues, but then is is this 2 same, it looks like as if you are multiplying 2 function, but they are not same I would just give an example say you take a function with domain as 2 points a b and say f of let me say f of a b. So, f of a is b and f of b is b. So, and the g of a is a and that they have the same domain d f and g and g of b is also a.

Now you try out yourself at home doing the composition and check whether this is whether they are equal or not in general or you can try to figure out your own examples and that would be little bit of fun, I think it should have some time and patience when you do mathematics that you really need to have some fun. So, you really need to go through this in this particular way. So, I would stop here because I have given you a considerable idea about how we have defined the modern way of thinking about functions. So, this very basic idea of input, output if you think I do not much need to bother about these things and just bother about this thing you can still go and understand your calculus course. So, with this I end and we are now from the next class going to talk about a subtle issue called limit of a function I would let you know that you would understand calculus much better if at the very outset you know that the fancy terms called derivate the integrals that you have been hearing or have even learned in your high school are nothing, but names given to special limits. So, understanding this concept of limit is absolutely central to your understanding calculus. So, these function is central and how functions behave when we look at its successive behavior as we keep on changing the input values if I change the input values of the function how the function behaves that is change, that is how we are recording the change of a function and calculus is all about change it records the rate of change and that is exactly what we are going to study.

Thank you very much.