Calculus of One Real Variable

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Lecture - 39
Taylor's Theorem, Other issues and end of the course – I

So, the last 2 lectures whenever a course ends people gets happy, I used to get happy too. It seems as if some sort of a brick is getting out of your head, but those who are just seeing the course just for seeing it. And you can for preparing the exam for the exams need not worries not a brick in the head what it is some sort of you know some sort of strength that you acquire when you have learned the calculus. Do not worry to what about exams, doing good in exams or bad in exams does not tell effectively anything about you. It is possibly one of the stupidest way to check whether somebody has learned something or not.

There is a famous quotation by james Watson one of the inventors of the structure of the DNA along with Watson and crick that, except for mothers of scientists and some media personality most scientists are outright dumb. So, do not worry if you think that you are not understanding things here, as I have not understood too much. And I wish that anybody who wants to take a career in science, should read the book the double helix by james Watson and you could see how much a human endeavor science is it is not just sit down and it is done things are done.

Today I am going to talk about taylors theorem, but I might just be more freewheeling this whole thing taylors theorem is one of the culminations where you actually step in to do analysis. So, here once the Taylor theorem ends actually advanced calculus also in my opinion, hence then what you do is analysis after that what wherever you use taylors theorem it is analysis and it is applications.

One of the key areas where it is used is for example, optimization and which has huge applications. So, it is not just about just a result, but it is about how effective a result is. So, we publish a lot of papers in science you would observe that there is a huge amount of publication which takes place. There too many scientific journals I think they are there outstripping the number of novels published in a year.

So, the issue is that how much of that knowledge we really need. It is very difficult to say how much of that knowledge we really need, but the question is how much of the knowledge is effective. Now what is in this edge where we are so much you know deciding about what is important what is practical. And to do science and think about it in it is own practical ways, in the sense that 2 something which is immediately applicable to a needs of society.

As it is important, but one has to remember that science and humanity has been most though useful science is important, I completely agree, but one has to also admit that among the most important advancement of humankind is possibly done through useless science. Rather useless science or not I can not say useless when I will give that is useful I can not uses, but rather science which is based on curiosity. So, somebody has some idea and he pursues it. Whether it was Newton's thinking about gravity, whether it was the invention of the calculus whether it was invention of many other things in mathematics. Many of them or even Einstein's theory of relativity many of them are just curiosity. That you can use your GPS today it is because of Einstein's theory of relativity. That rockets are been launched and our satellites are launched put into satellite is a combination of special theory of relativity and Newton's mechanics which depend on calculus.

So, these were things nobody knew how effective it would be even number theory was studied nobody thought that it would be. So, important coding which is used in more and many other secret issues. So, every endeavor of humankind in science is actually effective, but it is ultimately it is this research out of curiosity which becomes truly effective. And this is what we have discussed all along in this possibly 38 lectures is essentially a resaw science which was developed, partly out of curiosity partly out of need and partly out of some questions posed by someone. And the way of doing such science was would has vanished.

For example, many people would not know what I want to bring it I didn't use this particular book, but I want to show this book to the viewers and I wish that if anybody has any interest in mathematics or even engineering or whatever. You should really read this book introduction to calculus and analysis by Richard courant and fritz john. Richard courant fritz john are one of the leading mathematicians of the twentieth century. And the there is a courant need for a mathematical sciences in New York his name and fritz john is has a huge contribution in optimization, he made one contribution and that is a huge contribution which is the subject. And here is a huge contribution in the understanding of partial differential equation, and the interesting fact is that fritz john actually worked in the courant institution of mathematical sciences.

Courant is very famous for writing with Harold Robbins of book called what is mathematics. So, when people do mathematics or people think about mathematics, most of us who are aspiring to be engineers or something doctor or some something else, are mostly people expect into engineer or the doctors which are in our country the most coveted of all it is a pro probation possibly. People think that mathematics what you done in class 12 and what is talk to you in the first year at the your universities in the engineering courses is that is what mathematics.

It is about some bizarre guys forming bizarre problems and you just have to solve it, no. Mathematics is about ideas. If people had not thought about progressing mathematical ideas, you wouldn't have at the calculus just people are contented to do plus minus and doing daily course with mathematics. But people were more curious, people were more curious about how planets move. You might think that why should I bother about planets are planets moving or not, but I should just look out for my own daily life.

The question is that you are really bothered about your own place in the world in the universe and you want to explain the world around you. And the need for doing So, had actually made mathematics progress. And mathematics is progressing till that day, I want you to get out get out of this course with the idea that mathematics does not stop in your first year of engineering. Or you know first year of in a past course mathematics does not stop there that is exactly what I want to tell you.

Students in my who get into mathematics would realize that soon, but those who would not be getting into mathematics who go for other professions where mathematics is used. I want to tell them that that is very important to keep in mind that you should you should really I mean go with the idea you should never go with the idea that mathematics finishes with your first year courses. So, that would be a big mistake and a big dishonor you will make to yourself not only to mathematics.

So now I will start with the idea which I had told I will tell you when I teach you sequences, but I forgot to tell you when I was teaching you sequences. So, I remembered while coming here that I need to tell you. Remember that we had this important result that If I have a continuous function.

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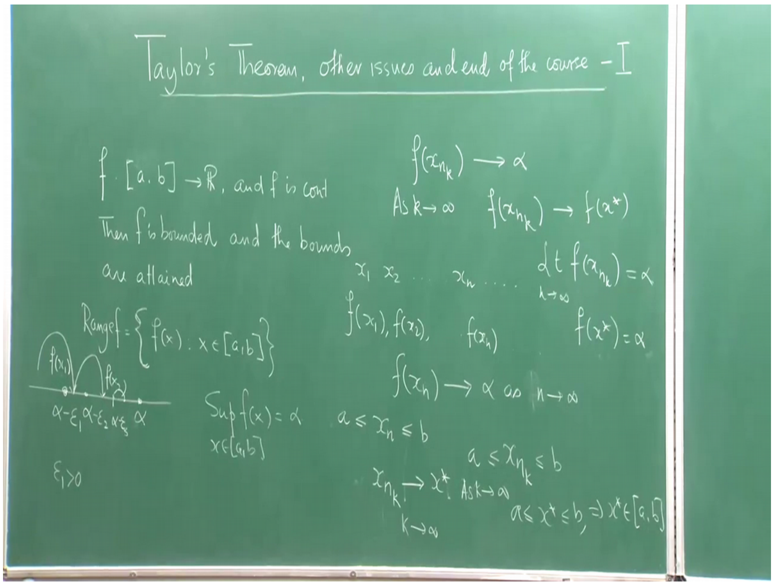

If this is the function a b to R and f is continuous, then f is bounded and f is bounded. And the bound is attained that is there is f is bounded above and below then there is a x such that f of x is the minimum value there is a y such that f of y is maximum value and the bounds are attained. So, you know this result we have proved up to this part that is bounded, but and the bounds are attained is not true, was not proved and left it to discuss when I would talk about sequences.

So, so what is what is bounded? So, the range of f which is a set of all f x x element of av this is bounded. This set is a bounded set. So, these are bounded set of real numbers, and this bounded set of real numbers has a upper bound and lower bound and hence it would have a supremum. So, let us write supremum of f x equal to alpha.

So, what happens is the following. So now, consider see how much mathematics is involved here in a very little thing. So, consider this to be alpha. So now, take any epsilon 1 if take epsilon 1 greater than 0. So, take alpha minus epsilon 1 alpha minus epsilon 1, but alpha minus epsilon 1, epsilon 1 is greater than 0. So, alpha minus epsilon 1 cannot be the supremum. So, there is an right. So, there is a x 1 such that f x 1 is here. So, there is a x 1. So, there is a x 1 in this such that f x 1 is here. Similarly if you take alpha minus epsilon by 2 right, but epsilon by 2 is less than epsilon by 1 when you are decreasing epsilon to you are moving towards alpha. So, you have some x 2 such that f x 2 is strictly bigger than alpha minus epsilon by 2, this is a definition of supremum that is all.

So, what do you generate? You generate a sequence x 1 x 2 x n. And you see you as I make epsilon value smaller and smaller I keep on having point's f x 1 f x 2 f x 3 near the value alpha. So, I generate a sequence f x 1 f x 2 f x n and this sequence f x n actually goes to in alpha as, as if now as n goes to infinity. Now add a very important thing to note what about these sets x 1 x 2 x n? So, these x 1 x 2 x n all are lying between a and b ok.

Now, this is a bounded sequence. So, bounded sequence has a convergent subsequence. So, let that subsequence be xnk. So, xnk is going to some x star. Now xnk for each k because it is just a sub sequence xnk is also lying between b and also lying between a. Now in the limit as k goes to infinity here; obviously, this is k going to infinity. So, here we have applied the Bolzano weierstrass theorem, just see how many step what steps are involved. So, here the Bolzano theorem weierstrass theorem is applied that you have bounded sequence it is the convergence of sequence it goes to x star. So, all of these convergent sequences are within that all the xnk that also bounded between a and b because they are part of part of the sequence x n.

So now, as k goes to infinity again what would happen? What would is that a would be less than equal to x star and less than equal to b, which would immediately imply that x star is element of ab. So, once you have done that they you have thought it thought it out in this way. So, what does it mean that? Now I consider a sequence f of x f of xnk the subsequence the functional value with the sub sequences. That anyway goes to alpha because of whole sequence is going to alpha, that anyway you will go to alpha once a function once a sequence goes to a limit. Whatever subsequence will take you take will go to the same limit because limsup who could be a finally, equal to liminf which I have spoken in the last class.

So now as n as k tends to infinity, what do we have? Because f is a continuous function and xnk goes to x star we have showed that in that case fxnk goes to fxnk goes to f x star. So, you have limit of fxnk k going to infinity equal to alpha. And this limit this limit is nothing but f x star. So, f x star is equal to alpha. So, there is a x star in a b for which f x star is alpha which is the nothing but the supremum of the function values. So, here is a proof. So, you see how many ideas are required to actually finally, come and prove this fact. So, this is one way of doing it maybe there are many other ways of doing, but this is the most elegant way of doing it.

So, we will start talking about taylors theorem, not Taylor or actually a student of Newton. Newton actually had started talking about calculus, but he never, but most of his proofs in his famous book principia mathematical. The mathematical principles of natural philosophy it is possibly one of the greatest scientific books still, not only by it is rigor, but also by the level of level of the way things are done. And he had used tremendous amount of geometry there.

Now Subrahmanyan Chandrasekhar one of the most famous astrophysicist of our times Nobel laureate is a Indian American, but whereas, when he was an American citizen when you got his Nobel prize. And he in 1983 he in the last 10 years of years life decided to do something, which is unique. He decided to read the principia, Newton writing. And he wanted to take Newton's theorems In fact, people wouldn't know that Newton has given his principle and terms propositions theorems and lemmas and all those things v not theorem they didn't called theorems are the term proposition when a proposition means I propose this.

We ultimately they dealt all with natural phenomena you know. So, people can change assumptions and things can change. So, what happened was that he said I decided not to look into the solutions, but use the solution to the solution using modern techniques. So, in most cases he found that when he had actually covered a result when you prove the result by walking through 2 3 pages, he found that Newton had already solved did you when you went when you went back to Newton solution it was run in half a page the Newton is to use geometry. So, this word Newton for calculus he used to he So, at the time they were only talking about mechanics. So, every everything was taken every function was a function of time right.

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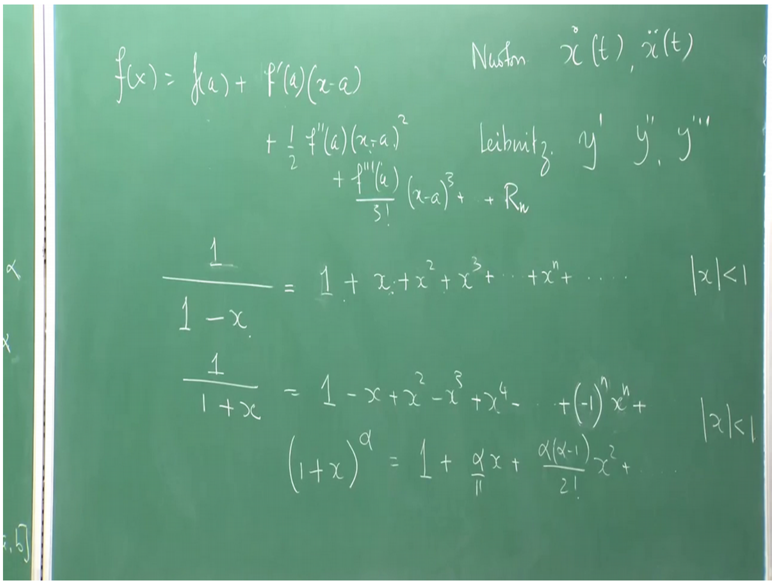

So, Newton needs to use this symbols x dot t. So, x dot t x double dot t all these things. While le Leibniz is it was. So, called rival in Germany would use these signs y y dash y double dash y triple dash and so on. So, these are the this is Newton's this is Newton and this is a Leibnitz. There was a great rival reacts to who had first got to the calculus. Calculus is all about change actually calculus is about how you study change, where everything in nature is changing. So, see there is a lot of this is very important to understand lot of mathematics has come trying to understand nature natural science. So, that is trying to explain the world around us.

You would I would those who are more scientifically curious here should read a very nice book, written by a Nobel laureate physicists called to explain the world. They are he starts from Greek science and how it went and comes up to Newton and you will see how the ideas have evolved. So, scientific ideas take too much time to you all. Now this idea of taylors theorem which you might have seen in books, if people write like this if you look when I calculus book and go to the cha taylors theorem chapter you would see something writing like this.

Derivative f dash and x minus a plus. So, if you know the function value at a you can calculate the function value at f x something and it just possibly goes on like this or n n somewhere with some term. So, which term I will call say R n said ended at they in they in a after they in a derivative. So, So, something you will look, look at something like this. Now you might be wondering as to as to what is all this what does it mean? So, should you know the function value at a you are saying that I can calculate for any other x I can calculate the function value f x. Why takes so much trouble? When I was undergrad student I have worried about unable to explain myself why should I do this.

So, one day realized has figured out I thought that suppose somebody gives me to calculate the function sine 41 degrees how would I calculate it. I know sine 30 degree I know sine 45 degree, but I do not know what is sine 41 degree? Then I realized that the interesting part of the taylors theorem is that for these sort of functions were calculating function values are not always easy for every x, you convert that calculation. For example, with for trigonometric function or transcendental functions like e to the power x and all, or logarithmic functions you convert the calculation calculating a function value at a given point once a particular simple at a simple reference point function evaluate is known you can convert the calculating the function value at any point into the simple basic arithmetic operations.

Multiplication division addition subtraction, 4 basic operation. But these ideas didn't just come from the air, they all came from a single theorem which you all studied at high school, is the binomial theorem. And that was what Newton was teaching to students when Taylor caught on to the idea. And Taylor first thought it out for polynomials we need Newton all the interesting polynomials.

So, why they were interested in polynomials? They were interested in polynomials because at this time they will have lot of astronomical datas. So, you will have t values and then you will have o a x values. And you will have y values, but you do not have a function. So, we will make looking at the natural phenomena will make some assumption about the relation between y and x is the continuous type of nice looking thing, and then you will try to fit a polynomial into it which is called interpolation of Newton's interpolation. You will start learning when you start learning or engineering or whatever you want to study.

So, Newton were also talking over emphasizing on these interpolation techniques, how to really figure out the shape of a function or estimate it roughly. As I told you the (Refer Time: 23:55) of an said their exact sciences are more to do with approximations. So, then he caught out Taylor that you can actually use this to calculate the exactly the function value of polynomial. And then one went to other things. Then Newton caught it that and Newton told him that even actually also go to the other cases by this method. Many times may be important result like taylors polynomial is beyond the threshold of publicity city for a person like Newton when I for publication he didn't bother he simply told Taylor that you can just fine. It is good for this polynomial it is done, but you can actually do it for non polynomials also and this is the way to do it, which is now known as taylors theorem.

For example many of you who use a l hospitals [FL] speak French the result was known actually right due to Bernoulli Johann Bernoulli, it was not due to l hospital. L hospital he was a tutor to l hospital you l l hospital wrote a book on calculus in which he wrote about that technique. And he became to be known as a l hospital. So, and now we will keep talk about this, this story of Taylor expansion from the book of Quran 10 john why I talk about Quran 10 john, and why I tell you to read it from here. I am not marketing Quran 10 john right. Both Quran 10 john are not alive the idea is that in there. So, historically rich and you would know why such things were developed. There are always reasons why things are developed. Because just after this he talks about numerical solutions equations he talks about interpolation, just after taylors theorem because that is exactly the reason why all these things were developed.

Because this is something you know in your school or high school. And I was also taught when I was in high school. In Kolkata there was a small book called darshan mukarjees calculus, and we used to take a great pride in able to solve the integrals there. So, this is if I am not wrong I have my ts sitting here. So, here in just it is always good to speak to people when lecture right they can correct me if I am wrong. So, makes a 4 x cube what is I do not know I just have a check, it just goes on basically. So, this is this is called it is an infinite series, but it is not fixed numbers like a one a 2 to a 2 it is with x s. So, for every x it need not converged, it only converts for x strictly lying between plus 1 and minus 1 that is mod x strictly less than 1. This was already known to Newton and you similarly had this for 1 plus x which was 1 minus x plus x square minus x cube. And the x 1 means x to the power 0 basically and so on.

These also true for mod x, if n is odd it is 1 minus 1 a plus 1, then a minus 1 the n is sorry, n is 8 n is x no I am making a mistake something like that I have forwarded. When basically when n is odd you will have minus when n is even you will have plus. So, you can.

Student: Minus 1 to the power n.

Minus 1 to the power n right. Because n, n is even you will have plus 1 when n is odd is plus 1 then. So, this is this is similarly you can write for 1 plus x square also 1 by 1 plus x square. Now you might ask me the why what is a big deal I can calculate it, but if I give you a very bad x, and 1 plus x is a very bad number. And 1 by 1 plus x that division is not easy to carry out very fast. Why you a x is say 20 lakhs? Carry out that 1 by 1 plus 20 lakhs plus 1. It is very difficult to and you have 3 minutes time up 2 minutes time approximate it, but you can immediately plug this in to 3 values and immediately give a rough approximation.

So, you see that you be able to immediately do a lot of approximations, if you know that I can express functions in this sort of c way. So, what is this value 1? It is the value at x equal to 0. So, it is the same x minus a here, here only x minus 0. Since similarly in So, this is what this is what is essentially that generalized a rule the binomial rule, which says that if you have 1 plus x to the power alpha then you have 1 plus alpha x plus I just have to I do not remember the com combination alpha, when I alpha by one factorial then alpha into alpha minus 1 by alpha c 2 basically, but we cannot write alpha c 2 because we need not be integers from it positive integers x s square and so on so.

This is a general rule it is a general binomial theorem. So, these are all this general binomial theorem is not ending. It is just it is just going on it is a infinite series, when you have positive terms 1 plus x to the power some square then it ends end becomes of polynomial So tay. Now what we will learn that many, many good examples of this, but let us just see what is the meaning of this and then we will see what Taylor had done. So, in this first class let us just take what is the meaning of all this, and then go to what Taylor had done, and write down some important results. See how do you actually sum and how do you know what is the meaning of this right.

So, basically we are interested now in power series. This is called a powered series which looks like this.

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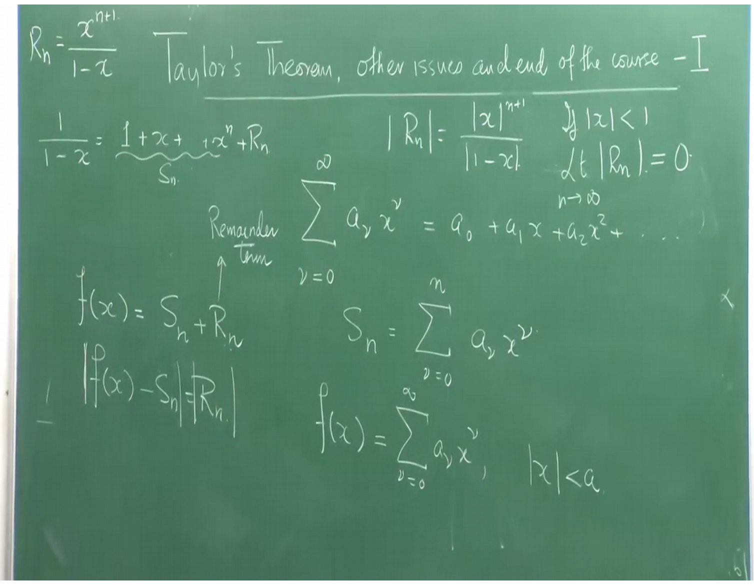

What you have just stood for 1 by 1 minus x plus 1 plus 1 a 1 plus x to the power minus 1 plus 1 minus x to the power minus 1, but ah they look like this, but just we will have putting in some more coordinates. So, how do you sum such a series? Again again you sum it through just your standard way of taking partial sums. So, suppose I write partial sum is this. So, if I write down the nth partial sum as this what do I mean? These are n n plus 1 objects were. So, components, but now suppose I write down this formula f x. And when I say f x is equal to summation nu is equal to 0 to infinity a nu x to the power nu. When mod x is say strictly less than sum a. Here it is strictly less than 1. So, what does this mean?

So, in this class we will try to understand first what does this mean, and then you see how effective they are. So, understanding of taylors theorem would be for you. Now this is my nth partial sum and this is my limit when x is this. So, what it means that f x is not equal to S n. So, f x mine is equal to S n plus some additional term which I call the remainder term remainder term. So, so what is remainder, the remainder term is actually nothing but the error term it is the error. It tells you how much you are left what is the gap between f x and S n. Now So, this term is called the remainder after the at the at the level of the nth partial term, it is called nth n remainder after the nth term.

So, similarly if I take this one. So, I can write this as 1 by 1 minus x as 1 plus x plus x to the power n plus R n. So, this is my S n. So, if you calculate R n now just taking their taking this part and subtracting and doing this simple fraction thing. Then in this case this particular case R n would become takes n plus 1 by 1 minus x. So, when mod of x. So, what I want to do? So, when mod of x is less than 1. So, if I now a take mod of R n. So, when mod of x is less than 1 I am bringing it here. So, mod of R n is equal to mod of x n plus 1 by mod of 1 minus x. When mod x what happened? Mod of R n this one I am just bringing this one here.

So now what is the idea behind this? The idea is the following idea is that if n becomes larger and larger and larger as n becomes larger and larger I am having a better and better approximation of f x by S n, that is this term should go to 0 as n goes towards infinity, that is the meaning of the remainder term the error term error must vanish and infinity right that is the clear idea. So, when will this go to infinity, when mod x is less than equal to 1. So, when if mod x is strictly less than 1 and limit of mod R n intends to infinity goes to 0.

Because basically I do not know whether it is positive or negative because I want these distance should be made as small as possible that is all. So, which means that that this f x you can when a which means that this is meaningful at least this is meaningful or that is meaningful when mod x is less than 1. So, this is the key idea behind the fact that I can always use a power series to approximate a function. So, that is the idea behind taylors theorem use the power series to approximate the function, but for the taylors theorem this a nus has a particular form involving these derivatives, these are your a nus sort of thing.

That is what we are going to discuss right in the next class, which is essentially the last class of the whole syllabus. So, we will come back in few minutes and discuss. Now that this is actually a 2 classes I am taking continuation together. So, this class and the next class one after another, but we will just stop here. And then we will discuss after some time which is which I should now mark as taylors theorem other issues and end of the course 2. Other issues of course, you might ask for this is the other issue one issue is that sequence issue I just told you, and the other issues are the stories that I am viewing inside the narrative. And that is also important some stories has to be told.

Thank you very much.