Calculus of One Real Variable
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Lecture – 38
Infinite Series (Continued)

So, we are back again talking about infinite series, and we have learnt in the last class that you have sequence of positive terms, and what is the meaning of a sum and all those things suppose just I have an infinite series.

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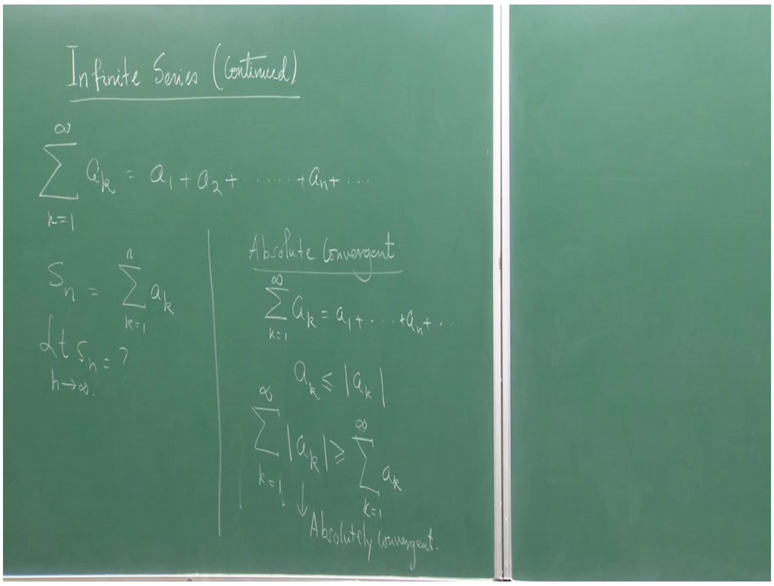

Now ak which is nothing, but a 1 plus a 2 plus an plus and it goes on, and it does not there is no sign on a 1 a 2 and that could be as positive numbers negative numbers anything.

So, in this case how do I find the sum? The answer is same; you again form the nth partial sum the sequence of partial sum. Here again it is the same thing, and then try to see whether this limit exists so it did. So, whatever be your series, it does not matter, whether it have a positive term on negative term, you are going to have the same thing. Now once you have a series like this, which can have positive term and negative term, how can I use my previous discussions that we had heard in the last class, to actually make any good conclusions about this, about the convergence of this.

So, these leads us to the notion of absolute convergence. So, absolute convergence is in the following. So, you have the series as above. Now what we do ask is a following. So, you know that each of ak is less than equal to modulus of ak. So, then we try to look at this particular series. This is a series of non negative terms. Suppose a positive terms if you want to call it, and now we know how to find the sum of that, and we know every, some fact about this one from yesterday class, suppose this converges.

So, hereby. So, for any nth partial sum. So, we had considered this series and of course, this means. So, if this series converges; that is, this is some number as a sum, then this series will also convergent, and will have a sum, because it cannot just blow up right. So, this is very important. So, when some, if once given a series like this, if this series convergence converges, and we say that the series is absolutely convergent.

So, the question would be, is there a series which is convergent, but not absolutely convergent, and we will go back to answering this question. So, is there a series which is convergent, but yet not absolutely convergent. So, we will go back to ask this question, and ask this question and let us see what we can do here, but before we do. So, we will take a small digression, because we are taking a journey. So, we can take a little bit of rest in the restaurant type of thing like, when travel by highways. Nowadays you just sometimes stop and take a cup of tea. So, it is like we are having a cup of tea, now basically.

So, we are going to now say is there any test by which you can know if a series is convergent, how do you know. It is a first series, is convergent, this is something very important. First of all let me tell you there is a simple necessary condition, there is a particular behavior; any series which converges will have a particular behavior right. So, if you take a series of positive terms say. So, take summation ak.

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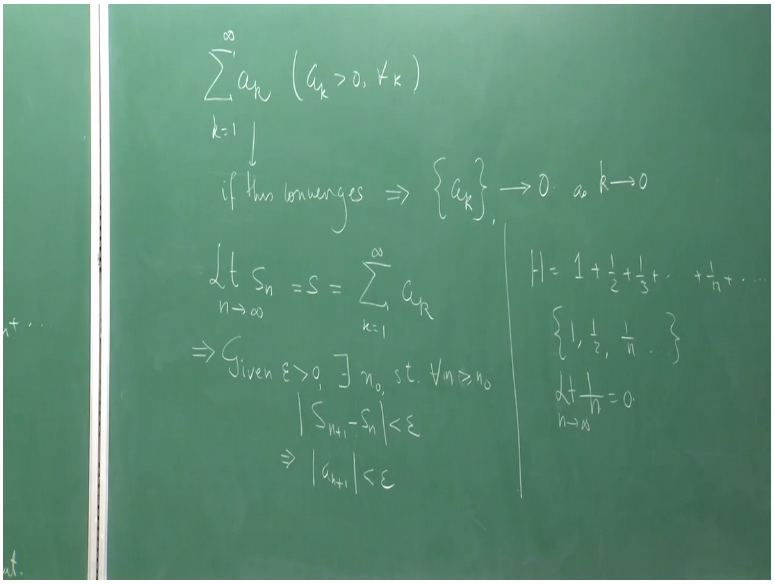

And ak is strictly greater than equal to 0 for all k, and if this converges, then it implies that the sequence ak, this sequence ak will go to 0 will converge 0 as k tends to infinity.

So, proving this is not a very big issue why, because what is the sum. The sum is nothing, but the sum s is nothing, but the limit right, this is the sum. So, you have sequence of positive term, this is going to happen. So, we are trying to find the necessary criteria. So, limit of sn is s [FL]. So, s is a cauchy sequence, sn is a cauchy sequence, what we have learnt earlier. So, this would imply that given epsilon greater than zero, there exists n naught such that for all m bigger than equal to n naught mod of s or or all n bigger than n naught mod of sn plus 1 minus sn, must be strictly less than epsilon.

Because it is just we are adding instead of cauchy sequence, we are adding this particular thing, it will be a cauchy sequence. So, this will also be true. So, which simply means that what is this difference in this is nothing, but an plus 1 which simply shows that the sequence an or the sequence ak actually goes to 0. So, given an epsilon I have found an n such that an minus 1 minus 0 is strictly less than epsilon.

So, this condition is just necessary, it is not sufficient, if you have a series of positive terms whose sequence actually goes to 0 that 1 whose sequence that is the sequence ak goes to zero, that does not mean that a sequence converges, and 1 of the most important example is the harmonic sequence a series. So, you have the harmonic series which is given by this famous 1 summation 1 by n, but if you look at the sequence ak what is this, or what limit of 1 by n is 0, but this sequence we have proved in the last class that this sequence does not really converge ok.

Now, we have to find some other tests of checking when a series is convergent which may be, which may or may not a non negative term general, general series is convergent. So, here will you have 2 types of tests called the ratio test and the root test, and we will see what is the meaning of ratio test, and the root test to introduce what is called the ratio test, and the root test we introduce the 2 notions from sequences 1 is called the limit supremum and 1 is called the limit infimum.

So, limit infimum.

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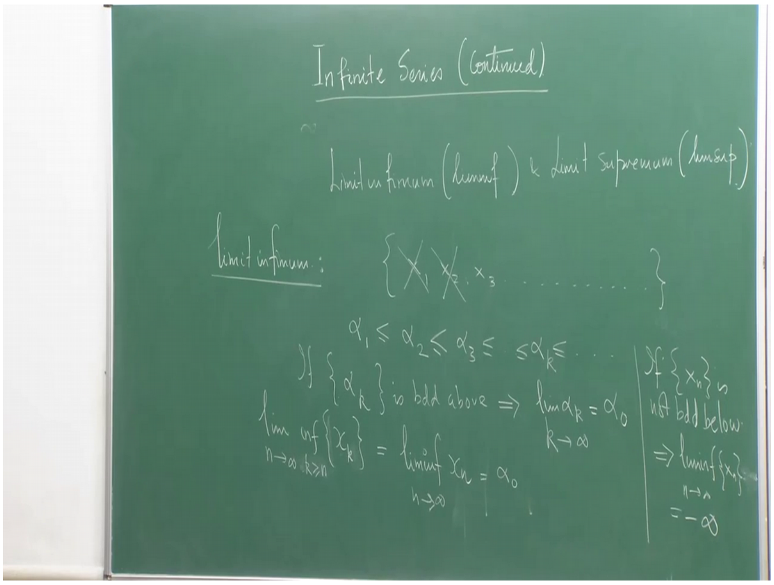

So, it is written as limit inf and limit supremum. So, these are the symbols. So, we will see what does this mean. Now you know given a sequence, it does not always mean that it is convergent, it could a bounded sequence, still does it is not a convergent sequence. Now we know from the Bolzano Weierstrass theorem, which we have already mentioned that the bounded sequence is a convergence of sequence, every bounded sequence has a convergence of sequence.

So, we know they can not need not necessarily be just one subsequence, they can be more than one subsequence, they can infinite number of such countably infinite of course, such sub sequences. Now each of the subsequence converges, converges to some number. Now collect all those numbers which have call the limit points of the set of the sequence, and take the 1, take the supremum of the limit points which is the limit infimum and take the infimum of the limit points, which is the set of limit points which is called a limit infimum.

But there is a particularly interesting way to actually go ahead, and define this thing. Let us see what happens. So, let us see what is the meaning of limit supremum and limit infimum right. So, we will not get into too much of details, but will tell you exactly what way we can work it out. The behavior of limit supremum, and limit infimum is similar to the behavior of supremum infimum. Now so, you take, let me define what is limit infimum. Here let me take this sequence xn, just for the heck of it assume that it is a bounded sequence, just for the heck of it assume that it is a bounded sequence, it is bounded below that the infimum exists.

So, what I do first, I find the infimum over the whole sequence, and I call that alpha 1. Now I cut x 1 from this 1, and I have x 2 x 3 x 4 x 5 x 6. So, now I was technically look like a smaller sequence, but countably is not it is the same as the same cardinality as a normal line; that is exactly what we have learned. Now the sequence remaining part of the sequence x 2 x 3 dot dot dot dot. Now if I want to find the infimum over that, the infimum, is now one element has been thrown in which could have been actually the infimum right.

So, the infimum could either remain the same or it would increase. For example, if we x 1 is not there. So, some bigger value of the. So, x 1 is the infimum alpha 1 was x 1, then some bigger value of. So, among x 2, x 3 somebody should now become infimum, this x 1 is no longer there. So, then that case it will strictly be bigger than alpha 1. So, this is alpha 2, and then I again cut off x 2 and I proceed alpha 3. So, when I cut off x 1 I have alpha 1 when I do not cut off anything I have alpha 1, when I have cut off x 1 I have alpha 2, when the sequence starts from 2. So, when the sequence starts from k; that is I have cut off all the things up to xk minus 1 then as alpha k and so and so forth.

So, what I have done, I have created a sequence of numbers which are themselves increasing or non decreasing. Now suppose this is bounded above. So, its increasing sequence and it is bounded above. So, if the sequence alpha k is bounded above it implies that limit alpha k, is equal to that bound supremum. So, let us call it say alpha naught and then in this case I will say that limit infimum of xn as n tends to infinity is equal to alpha naught.

So, what operation I have actually done, I have done the following operation, what I have done I have taken infimum of x n infimum of xn, where n is greater than or equal to k. So, its starts from k and above. So, you take basically what you do or you start with now you start with here better way to write it for example, sorry I be a mistake some. So, I have the sequence xk exactly k greater than equal to n, because I will a (Refer Time: 15:34) like to write limit n tends to infinity without written intensity. So, what I do. So, k is greater than equal to n. So, these are sequence which starts from xn xn plus 1 xn plus 2 dot dot dot goes off.

So, up to xn minus 1 I have removed. So, I have got infimum of that. So, basically what I am doing starting from n equal to 1. So, k greater than equal to 1. So, x 1 x 2 dot dot dot dot xn. Again k equal to 2 and it goes on. So, my n equal to 2, n equal to 3, n equal to 4. So, you will get all those things start with n equal to 1 then do n equal to 2, then n equal to n n equal to 3. Basically you are finding alpha 1 alpha 2 alpha 3 and then you are taking the limit of course, you are taking the supremum, but you essentially. Now we are taking the limit of this sequence new sequence; that is n tends to infinity. So, these exactly our limit infimum.

Now, suppose this is not bounded below, the infimum is minus infinity, then it does not matter then alpha 1 would be minus infinity alpha 2 would be minus infinity, alpha 3 would be minus infinity, alpha k would be minus infinity. So, if xn is not bounded below I am writing short bdd for bounded below, it would imply. What it would imply that, it would imply that. So, now, everything will be minus infinity minus infinity minus infinity, the limit is minus infinity. So, such a thing cannot be bound, you just cannot say this has a bound it is just all are same. So, limit n tends to liminf n tends to infinity xn is equal to what is equal to minus infinity.

Suppose this is bounded below. So, I am not writing it clearly I should write equal to minus infinity suppose this is bounded below. So, you have a alpha 1 alpha 2 alpha 3 alpha k, but this alpha 1 alpha 2 alpha 3 alpha k is unbounded does not matter, then my limit infimum again would not be finite, it will be plus infinity. So, if alpha n is not bounded above then limit infimum xn n tends to infinity is equal to plus infinity. So, it can take plus infinity minus infinity values, and also it can take finite value. So, all these are allowed in this case see you have to understand this fact there, when I do limit supremum, it is the opposite thing then I will do a same operation. First I will take it for the whole set then I will do it for. So, so alpha 1.

So, if I cut off x 1 then alpha 1 the next alpha 2 would be less than equal to alpha 1, because you are now, we have removed 1 element that could itself be the supremum right and. So, what you get. You will get a decreasing sequence or non increasing sequence and if that is bounded below, it will go and hit the infimum in the limit. So, limit supremum, limit supremum is following. Sorry limit supremum xn is equal to. So, first take the supremum k greater than equal to n xk, and then take the limit as n tends to infinity; that is what is limit supremum.

So, basically you do this fact, maybe you can write this, you know much more you want to more cautious. So, you can describe all these things that I have written about limit infimum for limit supremum. So, this limit infimum and limit supremum business is actually used to really check, whether thing is convergent or divergent right. So, let us just check this out. Let us now write down the ratio test.

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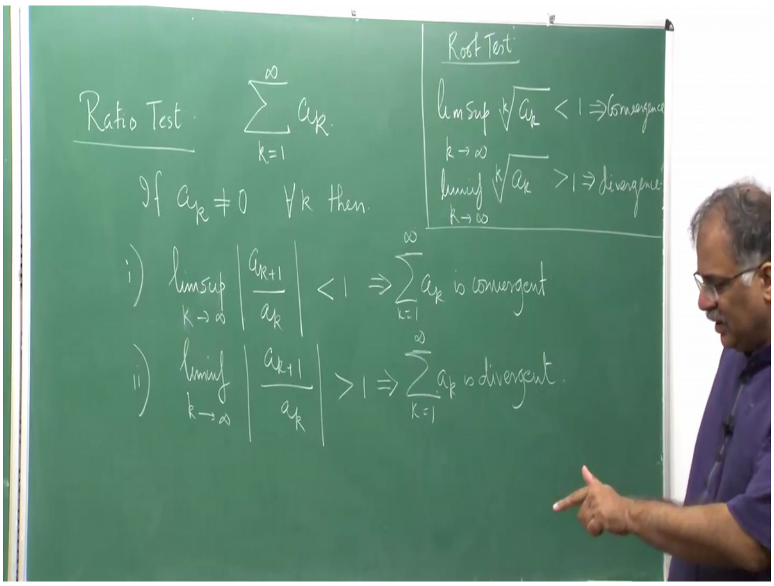

And the root test for checking the convergence and divergence of the series, we are not going to prove how this will actually tell you, what this will just mention you the rules here, because we really do not have time. So, we mentioned the rules how nothing occurs and after that we will end this a very interesting as are called Leibnitz alternating series theorem, and you will see there will be some interesting insights from that.

So, this is the ratio test. So, you have a series right. So, you really have to now check whether this sequence is convergent or divergent. How do you check it. So, what does the ratio test does. Ratio test which is essentially due to cauchy all of these is the following. Now if ak is not equal to 0 of course, it can mean it has to be nonzero if it is a positive thing. of course, it is much more simpler if ak is not equal to 0 for every k, for all k, for all k then number 1 you take the signal lim in for lim sup. I am just give you the information from the book of Peter Duran we, which I have mentioned about in the last class limit infimum of, sorry limit supremum, limit supremum n tends to. Sorry k tends to infinity a mod ak plus 1 by ak. This 2 things will right, and lim inf of this would be strictly greater than 1.

Suppose we have to the these things. So, if lim sup this thing, if this happens, and then it implies that summation ak is convergent. So, it will have a sum and implies in this case that summation akk is equal to 1 to infinity is divergent. Now you can ask a very crucial question, what if that lim sup is equal to limsif. See when lim sup, when limit exists here, when limit exists in a sequence and lim sup must is equal to the limit they are the same.

So, some of limit exists and lim sup equal to limif equal to 1, then what will happen, then. This is slight daisy when n equal to 1 then. We cannot say anything about it; no conclusion. Actually this test would be useless. We cannot do anything about it. Similarly there is a root test. Root test is the following. there are 2 conditions lim sup k th root of ak and ak to the power 1 by k as k goes to infinity and liminf k root of ak k goes to infinity, and this is strictly less than; 1 this is strictly greater than 1. Same story, same story, story is the same.

So, this implies convergence and this implies divergence, but if it is equal to 1 we really do not know this called a Cauchy root test. So, we have mentioned you 2 basic tests which you can check out on several, even just will I lost my some examples of this using the group test and ratio test to put a, put on the, in the your board, question board. I want to end today's discussion with a very interesting result called live needs theorem, on alternating series. Alternating series means all the odd terms it is a 1 minus a 2 plus a 3 minus a 4. So, all the all indices will have a plus in front of them and the even in even subscript will have minus in front of them.

So, here is a very famous result called. So, if you want I just, you just shuts say a example. So, you take the sequence.

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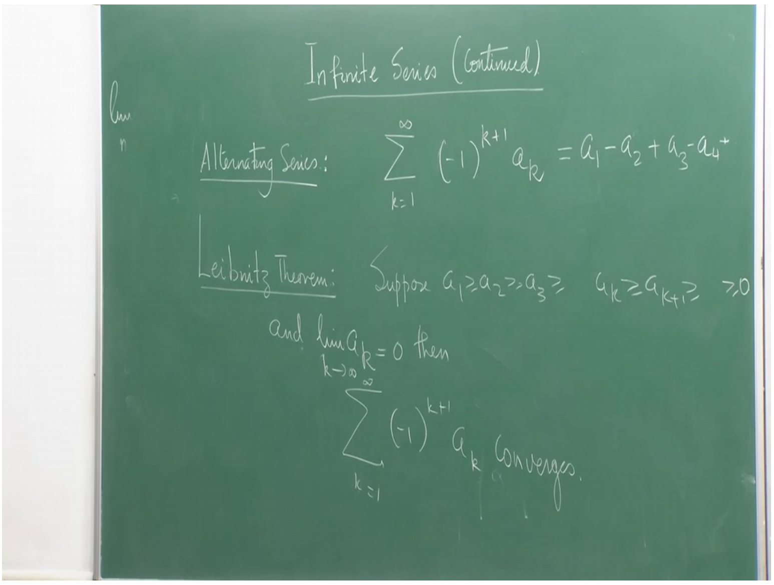

Say minus 1 plus 1 minus 1 plus 1 and so on. So, what is the lim sup or xn here? It is plus 1 and liminf xn n tends to infinity is minus 1. So, lim sup liminf are different, you will immediately know that these are not, this is not a convergent series. So, if limit exists there would be same.

So, let me write down our Leibnitz theorem for the alternating series. So, what is an alternating series? Let me write down alternating series, proof will be on the board, this proof will not be told in the class, we hardly have time now. So, adhering series is series like this summation k is equal to 1 to infinity minus 1 to the power k plus 1 ak. So, it will become a 1 minus a 2 plus a 3 minus a 4 and so and so forth.

Now, what is the Leibnitz theorem; say Leibnitz theorem says suppose a 1 is greater than a 2 a 2 is greater than a 3 and. So, on ak is greater than ak plus 1 and finally, and all the terms are greater than equal to 0. Suppose this happened and also and also that limit of ak and limit of ak as k tends to infinity is zero. Then summation k is equal to 1 to infinity minus 1 to the power k plus 1 ak converges. There is a these sort of things are called conditionally convergent. They may not be absolutely convergent.

Now I will show you for example, now consider the numbers a 1 is say 1. So, this, then half. So, consider this series alternating series which is 1 minus half plus 1 third minus 1 4th plus one fifth. So, now, here we have that sequence 1 minus half plus 1 third minus 1 4th plus 1 is the alternating series just like that, and also it has all the conditions that is required for Leibnitz theorem a 1; that is 1 a 2 half a 3 is 3. So, a 1 is greater than a 2 is greater than a 3 and all the things a 1 a 2 a 3 is all non negative and limit of ak; obviously, limit of 1 by n is going to 0 as n tends to infinity.

So, this sequence is convergent. So, 1 minus half plus 1 third minus 1 4th plus is convergent it has a sum, but it is not absolutely convergent, because if you look take the modulus then the, if you take the modulus of this then the series summation mod ak would become this 1, which is a harmonic series, and you know that this is divergent. What you know. What is the interesting part is, do you know what is the sum of this series. you will be fascinated to know this and there are; obviously, lot more mathematics to be done to actually figure out, what is a sum of this series, sum of the series is logarithm of 2, and with this fascinating result I end my discussion here.

Thank you very much.