Calculus of One Real Variable

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Lecture - 37
Infinite Series

So, you see nothing is more better than a course ending. It is always very nice when your course ends and you know we are also happier students that course is ending, but when the course ends we are in a much more advanced territory. See here we are going to speak about infinite series and I will be concentrating myself in studying a series of this kind, whatever and all of these numbers a k a positive numbers, always a k is element of nk is 1, 2, 3, 4, 5, 6 and so on now you might ask me why the hell I should study this.

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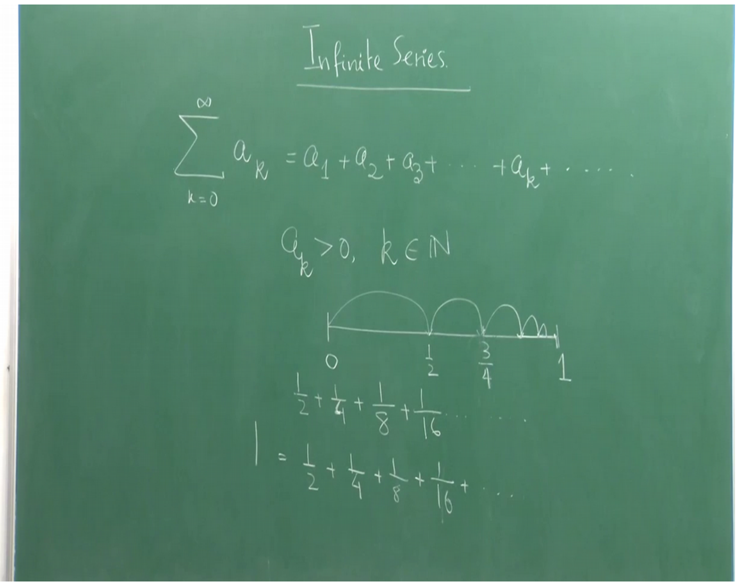

How can we get a series what was this sort of strange things are that you add infinite quantities, can you really add infinite quantities can you go on adding up to infinite times it is not possible. It is looks very strange, but I tell you that it is possible and it comes from the very basic notion of movement. So, suppose I start from 0 and 1 this interval 0 1. So, there is a road which is 1 kilometer or 1 meter say in this room. So, I am going from here to there. So, this is how Greeks proved that motion is impossible what do you know that motion is possible. So, if you want to go from here to there do you do you first go half of the road. So, you first go half of the road, exactly in the middle.

Then the half is remaining. So, your next strategy would be go to go to the next half of the road and the next half is remaining. So, you go to the next half. So, between one and one and half, half and one, you go to basically three fourth, and then between this you have another half right go to this half and again this part you have a half and again this part you have a half. So, you make the this you are approaching this point one. So, what is the distance now travelled by you? Its half plus one fourth sorry not half this this is oneth ha sorry its 3 by 4. So, you have covered half.

So, then you covered sorry then you covered half to three fourth. So, you have covered a distance of one fourth kilometre, then you covered a distance of one eighth kilometer from here to here half of this and so on. So, you keep on doing it keep on doing it keep on doing it. So, so here is the end. So, there is half of this then you go then half of this. So, distance between your current position and the final point becomes smaller and smaller, but you never reach that final point. So, you never reach that final point. So, now, what I have done. So, you have kept on adding it one by sixteenth and so on.

So, basically what is this? This is half plus one fourth, plus one eighth, plus 1 by 16 plus now the. So, what the Greeks said now come on this. So, motion is impossible you will never reach from that point 0 to 1, but motion is possible because I can always cover infinite steps in finite time and what does this tell me that there is. So, you can tell me does this sum add up to anything. What I tell you that if you can really go in finite times this sum should add up to 1, this must be very surprising infinite number of terms though they are smaller and smaller they add up to 1, I see is they add up to 1 because I am going from 0 to one and that is exactly what I did.

So, if I can move on go on infinite times I will finally, reach there that is a tricky thing to tell that if I have gone infinite times and I will reach there I will never go infinite times. So, what does its meaning of this one as a sum of this? So, what is the meaning of sum of a series or. So, you see this very basic very ancient days this whole idea came. Now infinite series occur in many many situations in science it is a very very natural thing and of course, in mathematics now what I mean by the sum of a series. So, what I mean by sum of a series.

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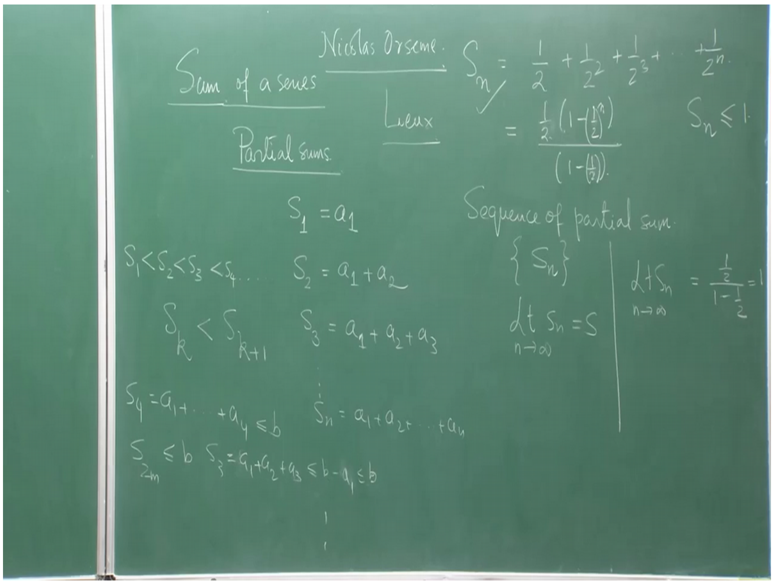

Now, it is very important to understand that we have just learned about sum of a sequence sorry is a limit of a sequence, we have not learnt about a sum of a series. So, our route to anything that we do in analysis a advance calculus if we want to say is through sequences. As I told you that this last week's lecture is actually taking you off from basic calculus into more advance into the advanced real more analysis. So, some of our series actually is done in following way. So, I will define what is called partial sums. So, partial sums is done in the following way. So, S 1 is equal to a 1, S 2 is a 1 plus a 2, S 3 is a 1 plus a 2 plus a 3 similarly Sn is a 1 plus a 2 plus a n. Now here we I have got a sequence of partial sums Sn sequence of partial sum and Sn is a Sn now suppose limit of Sn as n tends to infinity exists and it is equal to S.

And this S is called the sum of this series, the smallest that I have is called the sum of this series similarly you have to observe one thing because these are all positive terms S1 is strictly less than S2 is strictly less than S3 is strictly less than S4 and so and so forth. So, what you get is that S of k is strictly less than S of k plus 1. So, it is a non-decreasing series. In fact, it is a strictly increasing series when you have a strictly increasing series, we have just learned yesterday in the last class that if this is bounded above right if this is bounded above, then it will converge and it will converge to the supremum. So, that is exactly what we have to say we have just talks spoken about a bounded sequence, but bounded sequence is was bounded above and below. So, here anyway it is bounded because it is always bounded by strictly bigger than 0 there is a lower bound. So, is if it is bounded above also. So, then it will converge and converts to the supremum.

So, if this forms a bounded sequence, then I actually can speak about convergence for example, if I take this you see Sn here, which is one 2 plus 1 by 2 square plus 1 by 2 cube plus 1 by 2 n this is your standard high school geometric series gp series. So, if you are looking at the sum of those these are gp series which you know from high school we have this. So, you have the common ratio to be half and then this is your sum. So, now, what would happen if you take in this particular case, if you take the limit of Sn as n tends to infinity, you know that half as n tends to infinity. So, this will become 1 by 2 to the power n. So, as n becomes infinity to 2 the power n grows very large. So, this part vanishes and the limit.

So, the limit ultimately becomes half into 1 minus half which is equal to 1. So, limit is ultimately 1. Now you might be answering me that could you have observed that this is bounded I would say yes I could observe that all of these Sns are actually less than equal to 1 because of this very diagram we will immediately see from the geometry that s Sn is less than equal to 1 . So, if you do not even bother about this you know that this sequence actually converges because it is again a increasing series which is bounded above and here we can actually tell what the limit is. So, the limit is 1 and so, this writing which actually conforms with our intuition makes sense. So, mathematics always comes from intuition that is a something that one really has to keep in mind before we do anything.

Any sequence any series which does not converge, it could have that the sequence is unbounded above and it does not converge. And then what sort of thing would happen when can I have a. So, then it will not converge and hence then we do not have any sum of ours series, and then we call such series as divergent. So, let me give you a very interesting example of a divergent series which is quite the standard example which is given, but there is a little story behind it, but is a very tricky and very nicely done stuff this is called the harmonic series hn.

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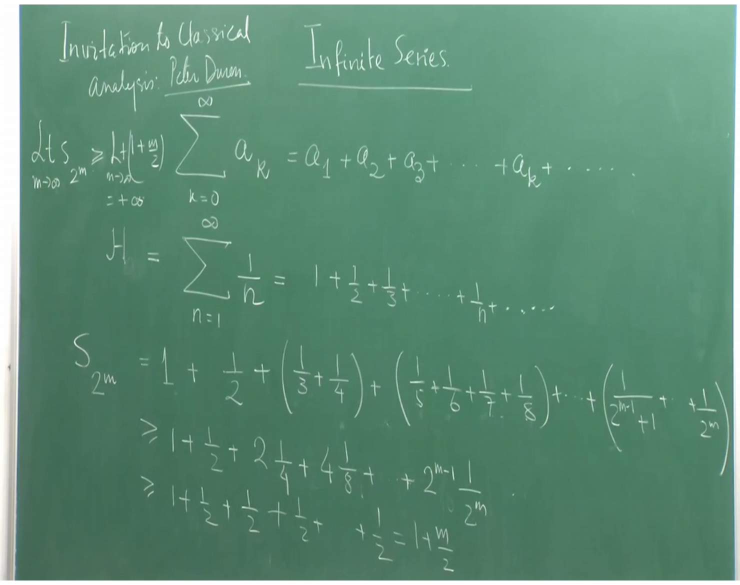

Sorry harmonics series h you know if I just take the sequence one half one third one by n converges the converge, as to 0 that is what we have decided in the last class, but here it does not right this is a very strange thing that, when you sum them up it does not converge.

It just does not converge it blows up. So, how did how does one prove that? It is proof is very non-traditional, it is not trying to show that. So, how it said it needs a little bit of trick. So, idea is to show that my limit here Sn limit of Sn, does not go to a finite number it blows up it goes to plus infinity that is what will be shown here. 1859 if I am not wrong and tell you how the hist little bit history in a beautiful city in France called leaux le this their head priest of the charge local charge Nikolas Orseme died, but nobody knew that Nikolas Orseme was actually a mathematician and one of his most enduring apart from any other things one of his many enduring contributions to mathematics is to actually show that this series is divergent.

So, it was only discover after his death that here actually proved among his manuscripts in this factor. So, what he said here observe that suppose I take a partial sum with odd number even number of terms, and I prove that this is bounded by sum b, then S1 plus S2 plus S3 sorry this is also actually bounded. This is bounded by also bonded by b your because these are all positive terms and I am sorry a one a two I am not writing S 1 S 2 I make a mistake sorry a 1 a 2. So, what I am trying to say is that if I take some positives I mean, I have take forward only even positive terms and show that it is bounded then I can show that all the terms all the partial sequence S4.

So, if I say that S 2 m is bounded is bounded by b, then all the whole sequence is bounded by b right the whole sequence can be shown to be bounded, that in the same way the way we have declared because if I say I take first four S4 is this is b and what is S3? What S3 is S 1 plus S 2 plus S3 which is less than equal to b minus S4 which is less than equal to b.

Student: (Refer Time: 16:24).

Sorry b minus a 1 sorry I made a mistake b minus a 1 which is less than equal to b.

Student: (Refer Time: 16:32).

I thing I am just (Refer Time: 16:36) .So, again writing mistakes. So, a 1 plus a 2 say S3 is also bounded by b. So, if I just show that this is blowing a is bounded then I can show the sequence is bounded, if I show that this is blowing up this particular subsequence is blowing up then the sequence is actually blowing up .

Student: (Refer Time: 17:03).

It still be a 4 b minus a 4, b minus a 4 is positive. So, it is less than b now. So, what was Nikolas Orsemes proof? So, let us write down then Nikolas Orsemes proof this is very very important. So, how did he prove? So, he took S 2 to the power m and even thing. So, is you see first he has taken left the first term one he has forgotten, then he joined these two terms one third and one fourth there is there is slight you might get why he is he doing so, I am giving you the proof from a book called invitation to classical analysis by Peter Durand. So, I am writing what this proof is very common you can find it in many book. I am just writing the book which I am using to get this proof, but this is so classical and you can just do it. So, you have this series and what do you do. So, you will write this as one plus half. So, you know one third is also bigger than one fourth. So, it is 2 into one fourth.

One fifth is bigger than one eighth, one sixth is bigger than one eighth, one seventh is bigger than one eighth. So, it is greater than equal to 4 into one eighth and so and so forth. So, it is 2 to the power m minus 1 times, 1 by 2 to the power m this last part is greater than that see here when you see when to cube it is two cube here m is 3. So, here it is 4 plus 1, 2 to the power m 3 minus 1 by 2 square 2 square plus 13. So, this is the structure general structure now they may sum it up. So, what do I get? 1 plus half plus 2 to two square. So, it is 2 this is 1 by two right this is 2 square, two cube this is half and this is half. So, how many times I have? The half m times 1 plus half.

So, there m minus 1term. So, this is half of this is m terms here and this is one. So, m times I am adding the halfs. So, basically I will get 1 plus m by 2. Now limit of S 2, 2 be power m as m tends to infinity is equal to limit of 1 plus m by 2 as m tends to infinity is infinity.

Student: (Refer Time: 20:59).

No sorry not equal to greater than equals 2. So, what is happening that this sum this sequence is always bounded below by some quantity, and that quantity actually blows up is bounded below by some series of this form, and this series actually grows up the sum goes up is unbounded. So, what I am showing here that this sequence when you have s when I take a sequence, when I when I basically take a sequence this sequence is unbounded. So, my sequence is bigger than a sequence which is unbounded. So, this cannot be bounded when a unbounded below. So, this cannot be bounded below. So, its limit of this has to be infinity this limit also has to be infinity. So, which means that this series does not converge because the limit ultimately we will say that is the it is a sum means it is the finite sum, but here there is a magic, I will change this game and write down to you the famous Euler series.

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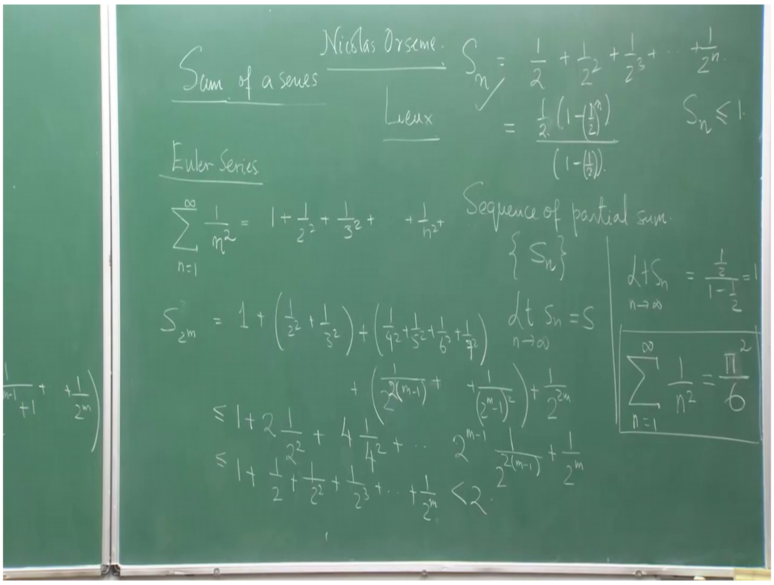

So, Euler dealt with the series which is slightly different than this, what about this series is it convergent, this is a fascinating thing radius convergent it has it has a sum which Euler find out found out, but all as proof is nowadays people laugh over it oh it is not so rigorous. So, we do not understand what is rigor by ought. So, now as I told you earlier would not be rigor after 200 years. So, what all as inside was always right. So, inside is again I tell you is more important in mathematics and other things.

So, let us see how this is been proved. So, here I have to prove that it is less than equal to sum thus the S 2 m is bounded by sum number. So, the whole thing is bounded by some number, it is a positive sequence. So, it is the increasing sequence bound as a upper bound and hence it will by our previous learning it will converge and converge to their upper one converge to the supremum need not be the upper bound, but to the supremum sorry again S 2 m.

So, how does S 2 m look like here? Same technique using we are gathering this terms the same ideas of Nicolas Orseme is also used here 7, I am just using the book. So, that I do not. So, I am proven to making mistakes, this classical analysis though it is a pretty interesting can become a little clumsy when you write. So, you have to be very careful. So, this is what about your stuff. So, you know 1 by 3 square is also less than 1 by 2 square because 4 is less than 9.

So, what I get this is less than 1 plus 2 into 1 by 2 square am I right? Plus here 4 into 1 by 4 square plus go it goes on 2 to the power m minus 1 because that 2 2 of half two of once a half square four or one four square because all of them is less than one four square , 2 to the power m minus 1 plus 1 by 2 m . So, what I get. So, if you look at this again it is less than equal to 1 plus.

Student: (Refer Time: 26:19).

Half this is less this is.

Student: (Refer Time: 26:24).

Two square exactly 1 by 2 square this is less than 1 by 2 cube and so on to 1 by 2 to the power m, but we have already studied the geometric series the earlier just this walking series. So, and that we know that this part has to be always less than 1. So, this total part has to be less than 2. So, this sequence always an upper bound 2 and be in a positive series we will converge straight to its supremum. Now 2 m is not its supremum, what is its supremum? Euler give a fantastic answer to this and you should know this this is one of the most brilliant results in all mathematics one of the most beautiful results.

Sometimes you do not always ask how can mathematics be applied how can mathematics be applied. A life is not interesting if it is just only bothered with what is necessary and what is practical. Life is important life is interesting when it also is associated with something beautiful. So, here is a result which speaks about the beauty of mathematics, and you believe it or not its supremum or its limit is a irrational number it is pi square pi our favourite pi square by 6, this is one of the most important results in all mathematics. So, pi square is irrational. So, won and proved that actually. So, this is what happens. So, you see today we had a very nice idea with this two important class of series. So, what is convergence and what is divergence, and how we have used the idea of a sequence how we have used the results of a sequence about sequences who talk about series.

And make you meaning of the our term the sum of an infinite series of positive terms. So, tomorrow we will speak more about the series in the last two classes we would be dedicated to Taylor’s theorem last one and half class, and maybe some history I will tell you at the late at the end of the last class.

So, thank you very much I hope that you are enjoying it.