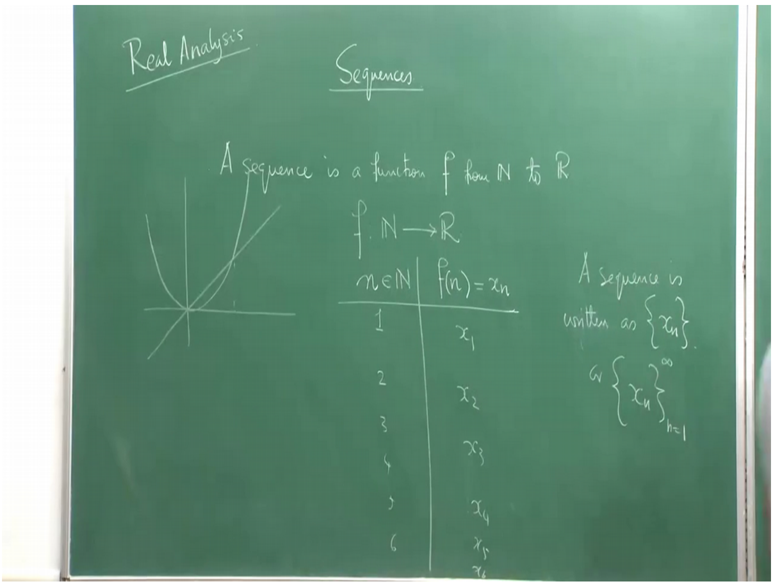
Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

Lecture – 35

Sequences

So, we are now in the borderline of jumping into advanced mathematics in the advanced analysis are the calculus have also advanced mathematics, but we are advancing further. So, what we are getting into is advanced version of calculus where much more detailed analysis is been done with the problems.

(Refer Slide Time: 00:35)



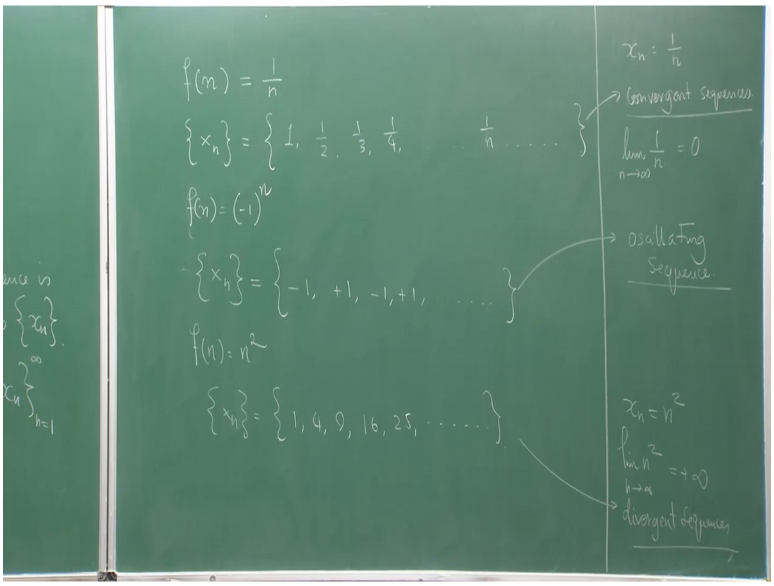
And this leads to a subject called real analysis. Real analysis is one of the most beautiful one of the most fantastically crafted subjects of mathematics. And it is effect or and it is immense influence on mathematics is cleared there is hardly a subject specially if an if you talk of applications. So, there is hardly a subject which can do without real analysis whether it is economics whether it is physics whether it is optimization, whatever you want to say mathematical analysis plays the key role and one of the most fundamental concepts of mathematics. In fact, I would say not just mathematical analysis is the notion of sequences.

A sequence is sequence is a function f from yours natural number set N to R the real number set. So, f to N to R. So, what do you how do you represent the sequence? So, sequence is represented in a following way. So, it will do it in a tabular form. So, here is n which is belonging to capital N which is nothing but 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 just go on and this is f of n which traditionally world over the symbol has been taken as xn. I of course, do not know the reason why they have answer. So, it is used. So, it is very difficult to say reasons there is a I think there is a some book on history of mathematical symbols.

So, I have I can tell you later on what I do not remember it right away. Now So, put n is 1, 2, 3, 4, 5, 6. So, here you have x 1 x 2 x 3 x 4 x 5 x 6 and so on. So, all these some of these values can be repeated. So, it is us when you write a sequence it is actually a complete tabulation of the function values right. So, usually a sequence is given as or written as or written as not given as written as, written as xn again some people also write like this. Or So, n equal to 1 it is x 1 n equal to 2 it is x 2. So, your shorthand of writing.

So, it is actually complete enumeration right. It is it is not it is not the standard way you write functions please understand, but you want to write the range of the functions you take away the repeated values. So, it is a complete enumeration, complete enumeration means complete enumeration means exactly doing this.

(Refer Slide Time: 04:31)



So, writing down it in the tabular form. So, for example, if I take the sequence f of n as 1 by n. So, the sequence xn now. So, we will not write n equal to 1 to infinity will just write this is obvious is 1 where n is 1 is half n is 2 is one third one 4^{th}. Sorry, one nth you just goes on without n.

So, this is the complete enumeration right. For example, let me tell you another sequence. So, in this case what is my sequence xn, sequence xn is the following. When n is one is minus 1 where n is 2 it is 1. So, n is 3 it is minus 1. So, it is minus 1 plus 1 minus 1 plus 1 and just goes on. So, you cannot write that this sequence is minus 1 plus 1 the range of the sequence is minus 1 plus 1 the second sequence what the sequence always has to be represented in a completely this completely this particular tabular form.

Now let me put for example, this sequence. Here you also you see there is no repetition, here there is repetition. Here also you for example, this sequence has no repetition. So, if I say. So, basically with we are just squaring the numbers 1 4 in case of 3 it is 9 in case of 4 it is 16 25 and so on. So, forth every sequence, the cardinality every sequence if you go back and think about what we have spoken about the count ability and unaccountability of sets. Every sequence is a countably infinite set every sequence has a cardinality exactly same as a set of natural numbers which is call the aleph naught a number given by introduced by georg cantor.

So now why have I chosen these 3 particular sequences? I could have chosen something else. The idea to choose this particularly it is 3 sequences has obviously, a reason. So, let us to look at the nature of each of this sequence. So, let us look at the nature of this sequence. Where each xn can be now written as 1 by n. So, what is the nature of this sequence the nature of this sequence is that, as I increase the value of n the value of 1 by n is going down. It keeps on going down keeps on going on, but it continues to given positive.

So, 0 is a lower bound of this right. In fact, it you can show as 0 is infimum. Now you see a values are going down and down and down and down, and it goes towards 0. So, this kind of sequences is called convergence sequences. These kind of sequences are called convergence sequences. So, here So, there is we can write this following language the limit of 1 by n as n tends to infinity is equal to 0. You understand this very well. That as my value goes down, but what about this sequence? Let us it looks a little pretty strange minus 1 plus 1 minus 1. Let us look at this sequence.

So here my xn is equal to n square. So, what would happen? Do I, can I say something about this? Where does the function value goes? What is the value of the sequence? Where does the value tending towards? So, here it is cured that it is tending towards 0 is going down and down and down. Here it is going up and up and up and up and up. The more n you increase the more n square becomes bigger and bigger and bigger. And n squared starts going faster than n, if you look at the function y equal to x and if you look at the function y equal to x square and if we just draw the graph of these 2, that after x equal to 1 x square starts going faster than x.

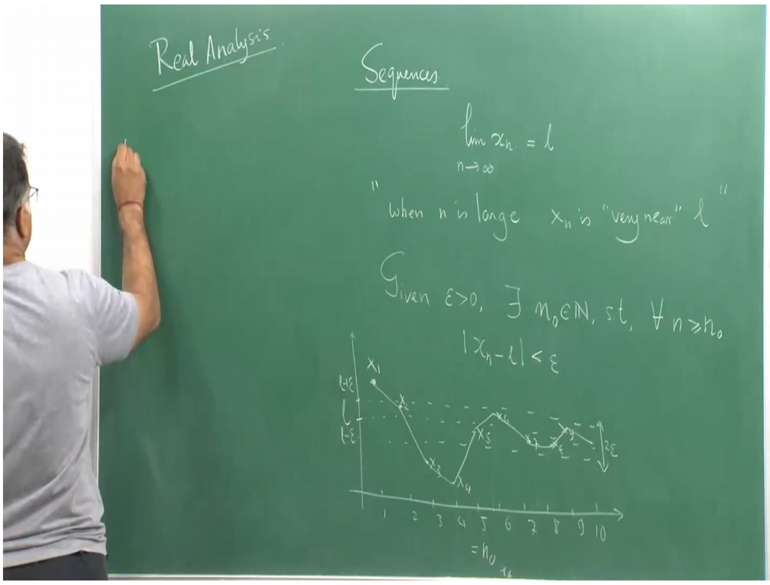
So, the which means this goes to actually becomes it blows up and you can write that this goes to plus infinity. So, if the limit of a sequence is plus infinity such a sequence is called divergent. So, this is an example of a divergent sequence. You might thinked convergence sequence is a better are and divergence sequence is are not So good idea, let me tell you the famous mathematician GH Hardy who he had actually not (Refer Time: 10:31) ramanujan in Cambridge. And actually spread the world the word across the world of what a genius srinivasa ramanujan was our own srinivasa ramanujan, he had a whole book called divergence series.

So, do not think that mathematicians can play around with things which suppose it will looks strange. Now what is this series doing divergence. So, these are divergence sequence. So, so we have a convergence sequence not series sorry this is called a divergence sequence is a convergence sequence. So, we understand it is intuitively we have the feel that I go to would some numerical value. So, I converge to that value if I just blow up I do not converge, if I blow up in the plus infinity sense or minus infinity sense because I do not have any finites limit then of course, I say it is divergent. But what about this sequence? Is very strange it is also divergence sequence, what about different type. So, divergence sequence sort of 2 types one is the one which blows up either goes to plus infinity or minus infinity another one suppose you have put minus n square then it would have blown to our other side.

So, this sequence is called an oscillating sequence is diverging it is not converging anything which does not converge must diverged is called divergent. So, it is called oscillating sequence. So, what does this great oscillating sequence does? It just goes between plus 1 and minus 1 plus 1 and minus 1 it is not going out to minus infinity of going down to minus infinity of going up to plus infinity, nor under any circumstances it is moving towards the fixed number. So, such a sequence is called oscillating sequence.

So, these are the 3 these are the 3 basic nature as over sequence. Now once this is known 3 basic natures, let us try to make a more better definition of a convergence sequence. That let those examples be there though they would they we need them very soon.

(Refer Slide Time: 13:13)



So, what I am what I mean by this statement? Limit xn n goes to infinity is equal to sum l what is the meaning of this. So, that is n goes to infinity you of; obviously, we were just keep on keeping on increasing the value of the natural numbers what does it mean? Again we will go to the epsilon formulation there is no delta here by the way. What it says that when n is very large xn is actually very near l that is the meaning of this.

So, the English statement is this when n is large xn is very near of course, you have to quantify what is very near very near l as the English meaning of the statement intuitively this is the meaning. So, when I have this meaning how do I translate it into a mathematical language. So, I said for what n I give you a distance will I say choose a distance between xn and l. So, let the for what n you will have xn and l is there a distance less than a given epsilon which I give it to you. So, the questions is So, so natural in that I will tell you can you find an n for which xn and l is actually very near. So, why I will give you what that for me very near means it should be less than some epsilon. So, can you find such an n.

So, if you can do that for every epsilon, then of course, you are talking about a sequence been convergent. So, given epsilon greater than 0 there exists n naught element of n such that for all n bigger than equal to n naught xn minus l this distance the absolute value is less than epsilon strictly less than epsilon that is exactly the meaning for any given epsilon this must be true. So, what it says that So, you come up from x 1 x 2 dot dot dot x n naught after that whatever xn n you take xn the distance is less than epsilon. So now, if you change the epsilon make it more smaller then the n naught would be changed maybe you have to come much more.

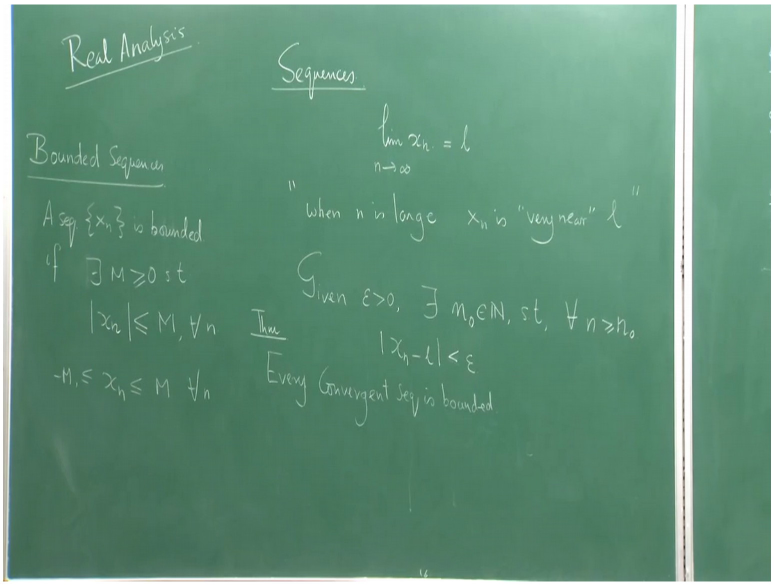
So, what it says? N bigger than n naught be is up to n naught there only finite number of objects x 1 x 2 dot, dot, dot, 1, 2, 3, 4, 5, 6, 7, 8 n naught. And after that for all the ns which is the called a tail of the sequence after n naught it is within that limit, xn minus l we just I mean draw it and show you. So, what it simply says is the following it tells you that except a finite number of if you give me any epsilon I can show you except a finite number of elements of the sequence members, all the other members are within that limit.

So, for example, let me draw. So, suppose sequence is said to go to a l. So, this is one this is 2 this is 3 this is 4 this is 5 this is 6 this is 7 this So, on So, this is l. And then these are l plus epsilon and l minus epsilon bands. So, there is a 2 epsilon band around l right. So, this is this point is l plus epsilon, and this is l minus epsilon that is that is what it is, and now suppose you have x 1 is here is x 1 then x 2 is here within this band and then x 3 is here, x 4 is here and then x 5 is here x 6 is here x 7 is here x 8 is here x 9 is here and so on.

So, what is happening? So, for now our n naught here would be equal to 5. So, after 5 and from 5 and then onwards all the extensors within this 2 epsilon band. So, this is the 2 epsilon band. Now will give first mathematical result about the nature of a sequence, but by discussing first what is called a bounded sequence is bounded sequences plays a very major role in many many things. So, it is a diagrammatic representation. So, if you look at this draw the diagram like just you connect the sequences by a line will immediately see the nature.

So, after x 5 everything is within that 2 epsilon band x 5. And so, that is that a very nice way of looking at the behavior of sequences, as if you are walking down the c alley.

(Refer Slide Time: 18:56)



There is something called bounded sequences. So, a sequence from now on in short seq, sequence xn is bounded which I also write in short bdd if So, if maybe I should not do this you become. So, sequence is bounded if there exists m greater than 0 such that could be 0 greater than equal to 0. Does not matter xn could 0, 0, 0, 0 there is a meaning which simply means that xn must live within m and minus m for all n that it that is what it simply means.

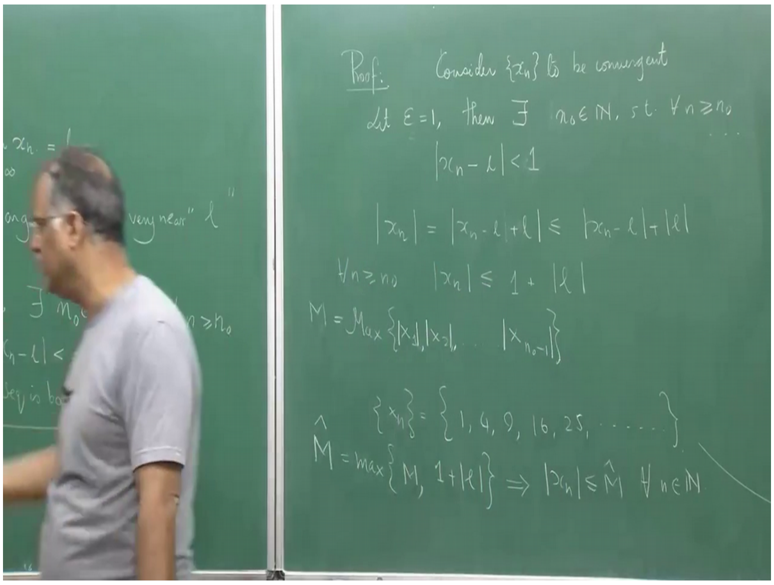
Now there is a beautiful connection between convergence sequence and bounded sequence. For example, this sequence which does not converge is bounded. So, every bounded sequence need not be converge when you talk about bounded sequence you immediately think that possibly these are bounded by some numbers fixed numbers, you know the function has to be convergence somewhere it is limit cannot get out of certain things of course, the limit of xn would always if there is a limit should lie between m and minus m, but who said that there will be a limit that is the whole point.

So, every convergence sequence will become bounded, but every bounded sequence is not convergence. So, we will first start a way of proving this fact that every convergence sequence is bounded this is one of the earliest prove that you see in mathematical analysis that real analysis that every convergence sequence is bounded. So, how do you prove that example that a bounded sequence is not convergence is here, right? At your knows on the. So, here you see. So, that is exactly why I kept on holding to this sequences. So, how do you prove this factor? So, this is not anybody any mathematician what did sort would call of theorem.

Now there is a very, very important result. So, let us start doing the proof we will start hum removing some parts from here. So, we remove the first part. So, let me do the proof. So, you have already have the answer of the other part, let me do the proof. So, how do I prove this statement, mathematician would formally write the word proof. And you see I want to just let you know that you could say that are you do calculus pretty well without even knowing all this, but does not matter to know something higher So, that you can actually do a lot with it will learn and do lot more with it.

Now after tending you these are tell you something interesting something associated without continuous functions, whether that you know the idea of continuity can be put into the language of a sequence. Now proof is the following. So, xn is bounded So right.

(Refer Slide Time: 23:00)



Now, consider extend to be convergent. Let epsilon equal to 1, then there exists n not element of n such that for all n bigger than this n naught mod xn minus l is less than epsilon less than 1, this is all right.

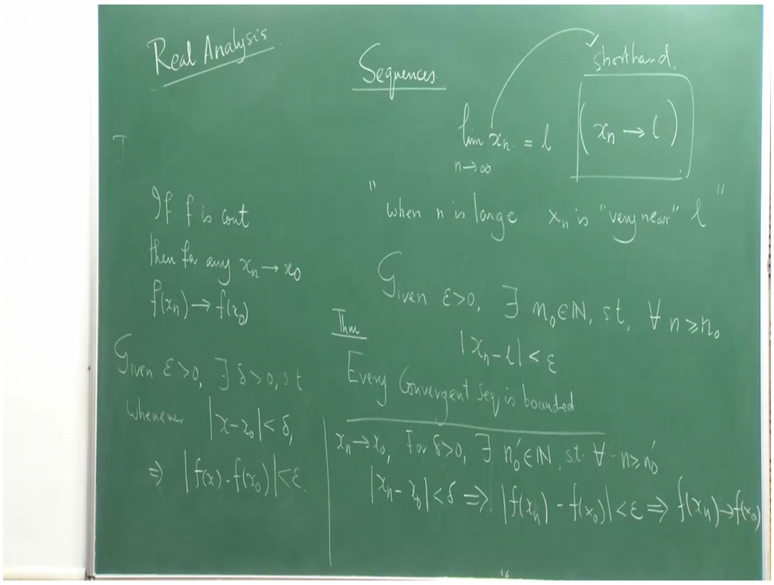
So now, take this sequence xn just write xn. And this can be written as xn minus l plus xn minus l plus l which can be written as, now for all n bigger than equal to n naught from this in equal this fact this inequality you would have xn less than equal to this is less than 1, that is why this equation 1 plus mod l. Now take now take now take the maximum value of x 1 x 2 x of n naught minus 1. So, they would let this be M. So, from n naught onwards this is true, it is bounded by 1 plus mod l because l is a fixed number this a this convergent you know the limit this is l.

So, this bounded by this number. Now there only finite number of them are there. So, you take those values the final is a find finite number of parts of the sequence which you have not considered, because everything beyond that goes inside the 2 epsilon or l you know 2 band right, the band of length 2. Now among these remaining part the finite you have you take the maximum one. So, for all of these are now less than equal to M. So now, find m hat which is equal to maximum between this one this capital M and 1 plus mod l just look at the thinking.

So, when n is bigger than n naught I know that the function value is less than equal to this. So, the hence the function value would also be less than equal to m hat because m hat is bigger than 1 plus l. And here I know that the function value, anyway for the remaining finite part the function value is less than every the sequence value is less than m mod of this or rather. So, I should rather take like this I forgot sorry I just made a mistake. So, the mod of x 1 x 2 xn all of these is less than m, and that m is also less than m hat. So, what I have concluded from here? I have concluded that mod of xn is less than equal to m hat right. Because m is non negative and this is nonnegative. So, m hat is nonnegative for all n element of m and low.

So, we have proved that it is bounded by the given definition. Now what is the link between sequences and continuity? This is some important link that you should really know at this very out stage one of a very, oh is this. Now let me tell you something out of the link between sequences and continuity. So, if a function is if f is continuous say from rn to R, just not bother about the.

(Refer Slide Time: 28:06)



Then for any xn going to x fxn must go to fx. So, this symbolism, this symbolism you might certainly see why I am writing this symbolism, this is a equivalent symbolism for this way. Instead of writing x limit of xn is this you should you can always write like this, this is a shorthand of this shorthand I like shorthands it is better.

So, this is what if f is continuous this will happen. In fact, opposite also have been function f is continuous if and only for every sequence xn this is happening. So, let us just prove this part let us try to you think about the other part. So, what happens if f is continuous? Now you take any xn minus x [FL]. So, what is the meaning given epsilon greater than 0? Over the continuity of f what are y given epsilon greater than 0, there exists delta greater than 0 such that whenever anyway. So, x is tending to so, maybe I should write x naught. So, when x is tending to x naught that is an x minus x naught is less than delta. We have fx it implies that fx minus fx naught is less than epsilon [FL].

Now So, given because I know that f is continuous at x naught. So, given any epsilon this is true. Now I also know that xn is going towards x naught. Now for delta for delta greater than equal to 0, there exists an n naught dash element of n of course, depending on the choice of this delta, such that for all n which is bigger than n naught dash I would have xn minus x naught to be less than delta. And by this then and for all such points it will imply again by the definition of continuity that f of xn is minus fx f of x naught is less than epsilon.

So, given the epsilon greater than 0 I have shown that there exists a n naught dash element of n such that for all n become bigger than this n naught dash fxn minus fx naught is less than epsilon, what does it imply? It simply implies that fxn goes to fx naught. You can retrace it back and come back and show that f is continuous. So, that part you can do and this is a very basic introduction about sequences. So, tomorrow what we are going to do is we are going to learn about monotonic sequences on the behavior increasing sequence and decre in sequence. What is their behavior? Once we learn about these 2 important features of sequence we are going to learn an important concept called limit supremum and limit infimum. And then once that is learned we will introduce to you what is called a series right, and that will be very important.

So, tomorrow we are going to look into this facts and which will be the starting our 8th week and the last week our syllabus. And once we will be doing that we will talk more about taylors theorem and associated things and are as we go on. So, I hope we are a very basic introduction. So, so let us see that that what you have learned today is a way of doing proofs in mathematics. Way of doing proofs in real analysis this is very important. So, also sequential definition of continuity f is continuous if and only for any xn going to x naught fxn goes to fx naught. So, we have shown that if f is continuous then this happens. So, you can also retrace the logic back and try to see whether you can yourself try and prove that f is continuous if this happens for any sequence. So, you have to understand the word any sequence is a word any is not for some fixed sequence. So, any sequence this will happen. Then only the function is continuous it is slightly tricky to go back, but just try it out.

Thank you.