Calculus of One Real Variable

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Lecture - 03
Example of Irrational Number

Welcome to the third lecture of this series of lectures on one variable calculus, in the last 2 lectures we had learned quite a bit about numbers, we have spoken about rational numbers and irrational numbers. We have spoken about the reunion which we call real numbers, we have also shown that the rational numbers are countable and real numbers are uncountable and hence these set of irrational numbers are uncountable, but it will be interesting too because they are so abundant irrational numbers will be interesting to see that we should be able to show that some number is irrational.

Rational numbers are easy to detect because if you have a number in the form p by q it is rational, but how do you know which number is irrational, irrational numbers are not in the form p by q is not so easy to detect. So, the detecting numbers are irrational it is an interesting exercise and I think it is not very bad and it will bad idea and it will show the inner connection between 2 different parts of mathematics, it will show how theory of equations are used to actually tell you which number is irrational. We have already mentioned in the first lecture about numbers that 1 of the most well-known irrational number is that.

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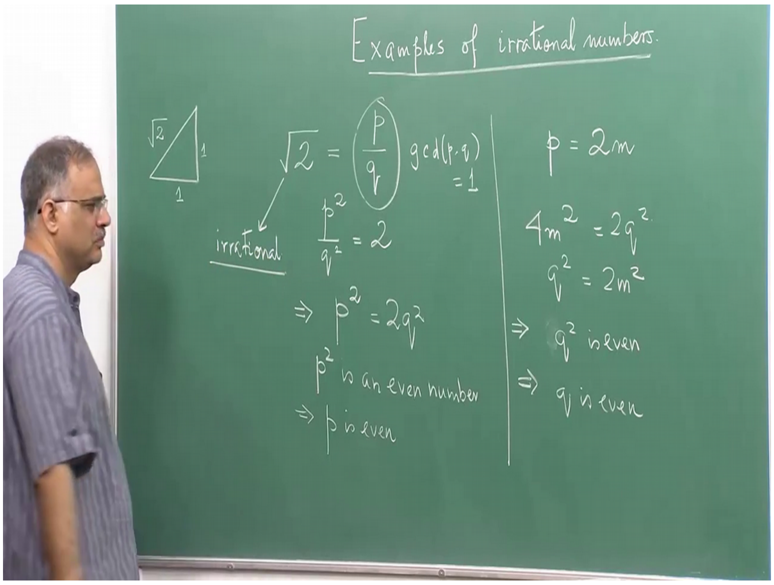

Called root 2 and this is the example that is given in your classes in higher secondary also that this comes out of a geometrical fact we simply comes out of the Pythagorean theorem, that if you have a right angle triangle with say 1 centimeter side each then this, 2 must this should be also the hypotenuse should be of length root 2 and this cannot be expressed as a rational number these are called incommensurable quantities by the Greeks.

So, now when you look in to this how do you prove it? The proof is again using the method of contradiction, in the sense that you if I want to prove that it is irrational I consider that it is rational and I take it in the form p by q and then show there is som, I goofed up somewhere there is some contradiction. So, this is very proof by contradiction is 1 of the biggest tools the mathematician has and he uses it repeatedly everywhere. So, what we do now, next we square it and that would simply mean you imply that p square is 2 q square.

So, 2 q square is the number divisible by 2 any number even number is divisible by 2. So, what we do is the following we immediately understand that p square is an even number. So, what does it imply, it implies that p is even see what if square over number is even then the number must also even. So, if p is even we can write p as twice of some quantity m. So, it was any even number can be written as 2 over m, where m is some number whereas, p is or even number is always divided by 2. So, p must be expressed as twice of some number. So, that is m is a quotient when you divide p by 2, now what you do is you put back here, see 1 important assumption that we take when we express the thing in the form p by q that we are expressing in the most reduced term, that is here in this case you have the greatest common divisor between p and q as 1, a relatively prime actually. That is you cannot make any further cancellations to reduce their levels when it is not 2 by 4, if it is 2 we should write it as we should never write as 2 by 4 it is write it as half it is called the most reduced form of a rational number.

So, there is no common factor between them other than 1. So, p if p is 2 m I will put here in this equation p square equal to c square to get 4 m square is equal to 2q square, we simply would give me that q square is 2 m square which would imply that, sorry q square is even. So, it implies that q is even so, if p and q are both even numbers which means 2 is a common factor means 2 divides both p and q right. So, it cannot happen because here I have considered p and q in the reduced forms. So, they do not have any other common factor other than 1. So, here we reach a contradiction and hence we conclude that this is irrational. Now I will ask you the following question and nobody asks and this is given in the book of beautiful book of Niven.

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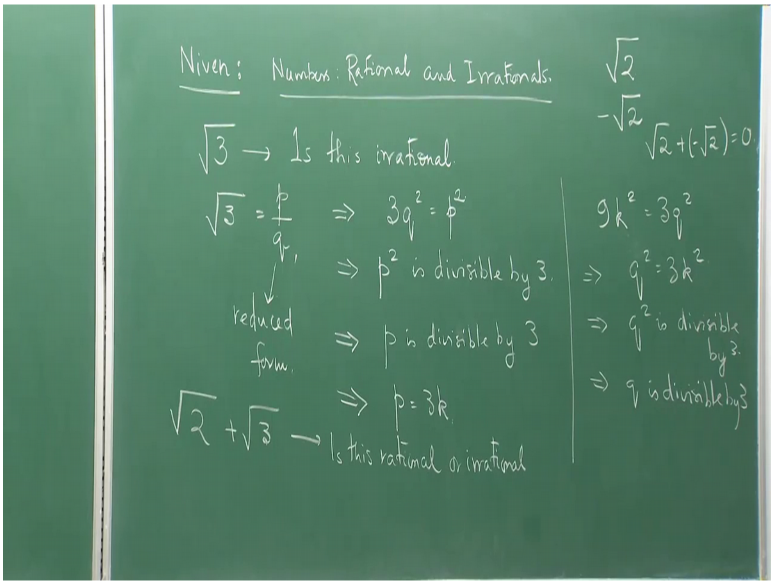

Niven has this beautiful book which I think is not available nowadays or almost always available on the net it is called numbers rational and irrational, it is published by the mathematical association of America. Even, even was old Canadian American mathematician in the fifties, I think he was very active any some lovely books which every what he should read actually anybody interested in mathematics, because he shows how beautiful mathematics can be. So, we will follow his chapter on irrational numbers for this lecture.

So, he asks a question what about root 3, is it rational or irrational is this irrational that is the question. So, in your school you are never ask this question, even in your standard calculus courses you never ask the question why should root 3 be irrational, now let me again try in the same way. So, I again assume that root 3 is equal to p by q you will be given the notes, the notes should be uploaded and then you can see it in detail. So, I am not writing everything every word in detail or the way I have written in notes.

So, what does it mean? So, I am expressing this again p by q must be in the reduced form that it is you will always assume that this g c d the greatest common divisor between p and q is 1. So, this would imply that 3 q square is equal to p square. So, what it simply means that p is divisible by 3, p square is divisible by 3 I am sorry p. So, it simply means p square is divisible by 3 p square is divisible by 3, but interesting fact that if the square of a number is divisible by 3 the number itself was also has to be divisible by 3.

This is a very number theory consists of all these fun and tricky facts and you really have to use it whenever require so it implies p is divisible by 3. So, that implies that p can be expressed as 3 times some k say that is divisible by 3, 3 is the quotient when you divide p by 3. So, if I put this in this equation. So, I you will have 9 k square is 3 q square. So, that would imply that q square is 3 k square, which implies that q square is divisible by 3 and hence it would imply that q is divisible by 3.

So, now, 3 becomes a common factor between p and q which is not possible because we have said that p and q the only common factor of p and q has is 1. So, again just if you want to keep in mind this is in a reduced form, now even ask you another question what is what would happen if I add these 2, will they be rational or irrational because you might say oh you are adding 2 irrational numbers you will get a irrational number p, where adding 2 irrational numbers need not always give me any rational numbers. Because if you have root 2 then without much difficulty using the same procedure we you can prove that minus root 2 is also irrational so, but then root 2 plus minus root 2 would give you 0 which is the irrational number. If you take 2 rational numbers and add them or if you take 2 rational numbers and multiply them you will always get a rational number.

But you will never you cannot guarantee that such a thing would happen when you have 2 irrational numbers. So, the question is, is this rational or irrational. So, shall we go here the policy of doing this, when you add 2 numbers the policy of doing check of rationality or irrationality in this form is not always helpful, that is you cannot say and put root 3 by saying as p by q and do it, it will mess up. Policy of doing this as I will show you on this side of the board is the following now later if I assume that it to be rational. So, let

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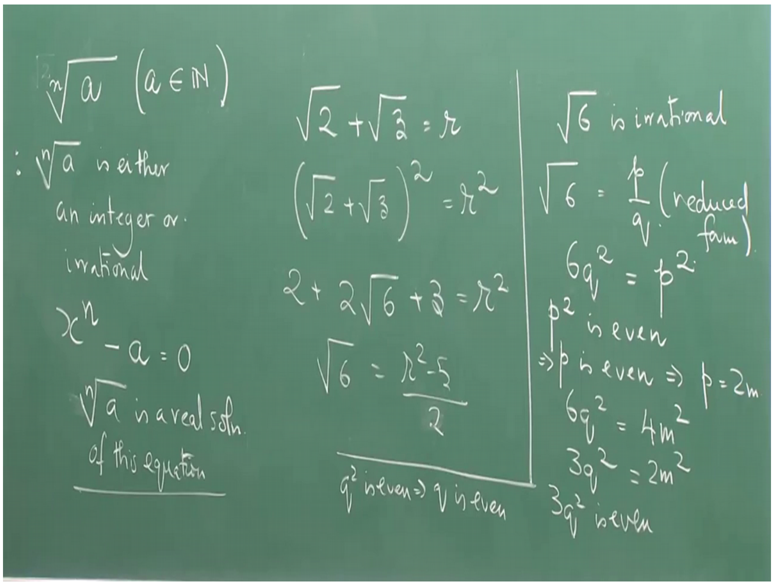

Root 2 plus root 3 be equal to r rational number if I square it. So, this is what you get or in another way you at route 6 is r square minus 5 by 6. So, you add 2 plus 3 and that is taken to the other side and divide by sorry by 2, make if I divide when you say. So, r if it is a rational number r square minus 5 is a rational number, now if we divide 2 rational numbers they will give me another rational number, then the question would arise is root 6 a rational number root 6 the rational number then what I am writing is true.

So, if you want to prove it irrational you have to prove that root 6 is irrational and that is exactly what I want you to prove you can keep on thinking about this. So, how do you, how do you prove that root 6 is irrational. So, if you want to so when you are proving just a single quantity then you have to go in this p y q method with p y q in the reduced form. So, just for your practice I just will just do this example.

So, again you will go because if you prove the root 6 irrational then this is a meaningless equation right a rational cannot be equal to irrational. So, then you will immediately conclude that 2, 3 plus root 2 plus root 3 is not rational you, somebody say I know that he be rational because it is nothing, but root 2 in to root 3, but remember multiplying 2 irrational numbers need not be again a irrational number because again you have root root and exactly in a similar procedure you can prove that 1 by root 2 is irrational.

Only the p r role of p and q switches right. So, then when you take a product of this you will get 1 which is rational. So, these are certain things have to keep in mind while doing mathematics. So, again if you do in the form p by q so you will have 6 q square is equal to p square, which means p square is even naturally because it is 2 in to 3 q square. So, p is divisible by 2 so p square is even so it implies p is even. So, now, p is even my, I can write p s 2 of m and write p s p equal to 2 of m. So, this is clear now because that is the type of reasoning that we are taking.

So, once you do that you write 6 q square is 4, 4 sorry 4 m square. So, you are getting 3 q square is 2 m square. So, what does it means, it means that 3 q square is even. Now 3 q square is even q square itself must be even because 3 is odd if you want to multiply an odd number with some number and get an even number. So, if you multiply an odd number with odd number you would never get a even number, to multiply an odd number with a even number you will get an even number. So, so you will immediately have from here as that q square is even and which will imply q is even.

So, both p and q being even now would have a common factor of 2, which is again a contradiction whereas, you know that p and q has been chosen in the reduced form, all these things that we have done this these are the parts you will learn in your 11 standard in the in our in our classrooms they are called (Refer Time: 16:29), go to any class 11 book you will be hearing this word (Refer Time: 16:36) So, in general the question that I would be interested to ask is the following. Is it always easy to show such numbers to irrational it is not always easy these examples are though it is not also a traditional these examples are not traditionally given in the classrooms, but for example, I am asking the question.

When can this n th root of a when is this number irrational? The answer is surprisingly interesting, answer is the following answer is that n th root of a is either an integer or irrational of course, a is a positive number naturally we are not talking about root of a negative number here. So, a possibly is an integer. So, when is this rush irrational number. So, the interesting thing is that it is this is either an integer or irrational, it is never in the fractional form.

So, that is a very interesting question, but how do you actually prove that, a can be integer means how square root of n th 3, n th root, but yeah you can take integer because if you take the square root then there will be problem, because you cannot take the you take the 4th root there will be problem. So, you because you want have real numbers we are generating real numbers we are not generating anything else 2 minus 8 (Refer Time: 18:47) 1 by 3 (Refer Time: 18:49) exactly. Now, observe that observe that if I look at this number I can look at this someone and say this is something you have done in your complex number theory that x n minus a is equal to 0 has 1 real solution at least. So, n th root of a is a real solution of this polynomial equation.

So, the question would be will this solution will this, will this polynomial equation, where I can write it as a solution real solution of this equation can it give me some hint about the irrationality of the number or rationality of the number or or been integer. So, to do that we need to get some, know something about the properties of numbers, properties of solutions of polynomial equations and whatever we do will not do the proof here it will be there in the notes everything is based on a very single thing, unique factorization theorem for natural numbers.

That is the key to all these things that we are discussing and that shows the unity of mathematics in some sense. So, now, look at this thing, consider a polynomial equation. So, what I am going to write down is the polynomial equation in you have seen all these things when you talk about the theory of equations and many of you must be knowing, but this is what is called a polynomial in a single real variable.

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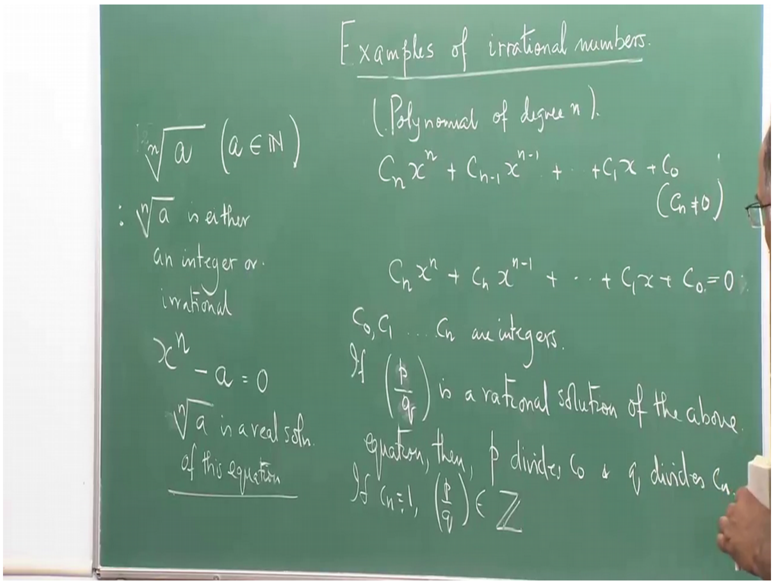

So, polynomial of degree n so so the power progressively goes down to c 1 x to the power 1 plus c 0 and c n is not equal to 0 because if c n is 0 then it becomes a polynomial of degree n minus 1. It is a polynomial of degree n look at the following polynomial equation and these numbers c 1 c n minus 1, c 1, c 0 these are called coefficients these coefficients we for the time being take them to integers. You can always replace integers by rational numbers I will ask you to have fun by checking why you can do.

So, it will be really fun just if you go through it, but the interesting fact is that we can now look at certain properties of rational solutions of this, suppose we have a polynomial equation that is and you knows that such an equation will have at most, it will have exactly n solution or at most it will have exactly n solutions right. So, these c 0, c 1, c n are integers I am not giving you the proof I am just stating a result, the result says that if a by b here I am just talking about rational numbers a and b could be negative positive does not matter or if you want to say p by q.

If p by q is a rational solution our of the above equation, equation then p divides c 0 and q divides c n, but this all comes from very simple result this all comes from prime factorization, p divides c 0 and q divides c n. So, from which we can make a very interesting conclusion that if c n is equal to 1, then q must also be 1 which means because otherwise if it is 1 there is no other number other than 1, no other integer which can divide 1.

So, 1 or minus 1 it does not matter. So, sub assuming the power sign of p q in p then you will have p by 1 and so this is integer. So, if c n is equal to 1 p by q is element of the set of integers. So, this is a very very important result, to prove this view you need some stuff which is in the notes which I will not get it to you in the class because it is just because we are; obviously, constant over with for time. So, we have some if 5 10 minutes left. So, here is something you have a information, that if it is rational then there is a behavior p behaves in some way and q behaves in sub way if you have a rational solution, Now let me ask you this question.

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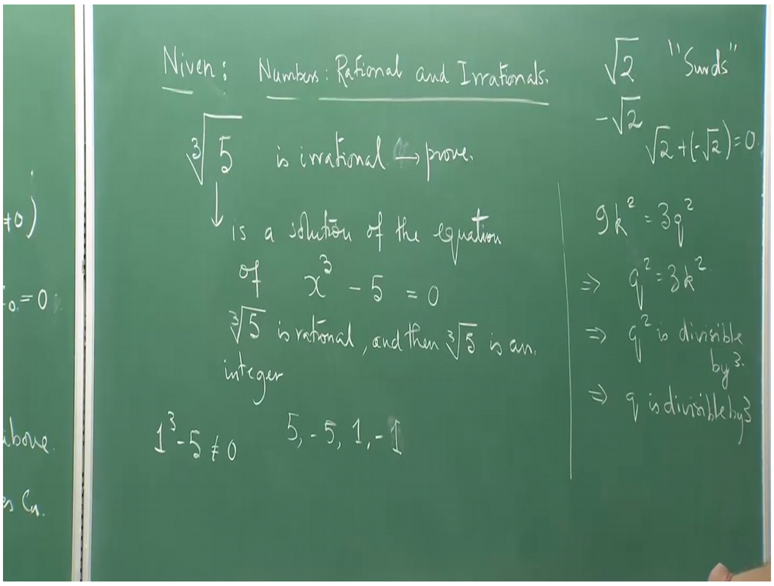

Cube root of 5 is irrational can you prove it and you will see how beautifully we can use this idea, this simple idea can be beautiful use here because I can now write that this is a solution is a solution of the equation, of the polynomial equation of degree 3 x cube minus 5 is equal to 0. So, in this case c 0 is 5 and c 1 is c n is 1 minus c 3 is 1, rest of the coefficients are 0. So, suppose root 5 is cube root of 5 which is a real solution of this.

So, cube root of 5 is rational then, it must be an integer because c n is 1, c 3 is 1 the highest power has coefficient 1. So, it must be integer. So, actually at and then cube root of 5 is an integer then what could be the integers, but that integer now should divide c 0 which is 5 because p is dividing c 0 and q is dividing c n, see how beautiful mathematics is and then you know the only chances here are 5, minus 5, 1 and minus 1. So, now, I put x as say 1, say 1 cube minus 5 it is; obviously, not 0 it is minus 4 similarly you look for each and everything here every solution if you put x as 5 or minus I 1 and minus 1 you will not get 0.

So, cube root of 5 cannot be a rational number. So, you see how the theory of equations are linked finding rational number mean irrationality of numbers of this form. So, once you have so you can use the same language here that either it is a integer, fine if it is not an integer there is no other choice, but it has to be rational it cannot be of the form p by q. Now you might ask that only this square root type a cube root or n th group type things are irrational nothing else.

So, there are famous numbers in mathematics famous constants in mathematics which are irrational, the most famous constant in mathematics is pi the ratio between the circumference and the diameter or circle is irrational not only that pi square is irrational. So, this is irrational and there is another important number called e and the square of e the exponential that it is actually 1 plus 1 by 1 factorial plus 1 by 2, 1 plus 1 by 2 factorial plus 1 by 3 factorial and so and so. It is a infinite series if you are not so much in to it do not bother much about it.

So, in my notes I will have proofs of the irrationality of these numbers and say something more about them, but these are for somebody who has already got some knowledge. So, the note should be so that people can choose and read whatever parts they want to read. So, before we finish today with last 5 minutes let me give you another important class of numbers logarithms, many other logarithms are irrational actually. Now I will not talk about base e logarithms which is called natural log, we will call the straight the class 10 standard logarithm which is as follows.

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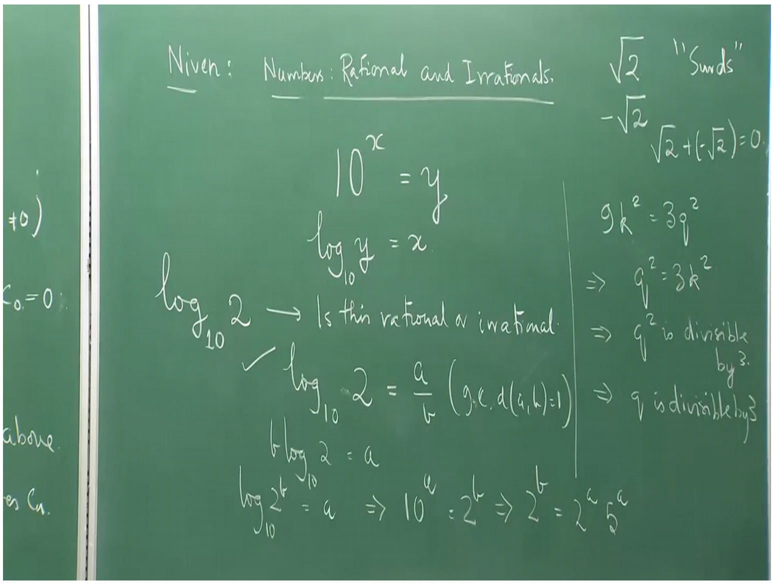

So, you know that if 10 to the power x is equal to y, if I ask the question whatever what is the power of x what is the to which power 10 must be raised to give me the number y. So, log is just the inverse of this operation inverse of exponentiation is the log of log operation. So, you say log to base 10 of y is your x. Now I ask you a very simple question if you look at Napier's logged logarithmic table they will give you point something, something, but that will be just a approximation no I am asking is this rational or irrational.

Let me assume that it is rational as before and the whole thing is in the reduced form say, a by b in the reduced form g c d is 1 greatest common divisor between a and b is 1 that is you cannot divide them further no common factor other than 1. So, it means b to the power log 10 of 2 is a it simply means log 10 of 2 b is equal to a and that simply means 10 to the power a is 2 to the power b and hence it implies that 2 to the power b is 2 to the power a in to 5 to the power a.

So, 2 to the power b is some number and this looks like this is a prime factorization then of 2 to the power b which has got to be unique and each of the prime numbers in the prime factorization must be a factor of this number 2 to the power b right each of these numbers here must divide this number 2, to the power b or 2 to the power b cannot be divided by 5. So, here comes the contradiction and hence this is an irrational number. So, with this I end this talk I hope you enjoyed this with some nice examples and we would you see more in the notes specially those parts are for those who are more initiated calculus for the ambitious this is a which name of a book recently by T W Corner calculus for the ambitious. So, those are more ambitious can go and have a look there when I in the notes, but otherwise those who just want to have a basic idea what, what how to go and hit irrational numbers this amount of is knowledge is pretty fine to develop a working knowledge of the calculus.

Thank you, thank you very much.