Calculus of One Real Variable

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Lecture – 29
Applications of Definite Integral – I

So, we have been on this course of the calculus for quite some time calculus of one variables. We had tried to be very different from the traditional courses, but some traditional things have to be learned they cannot be thrown away. So, easily and as a result of which I would like to tell you the traditional thing that we do specifically computing areas, computing volumes, etcetera, using the definite integral and that is exactly what this section this part of the talk.

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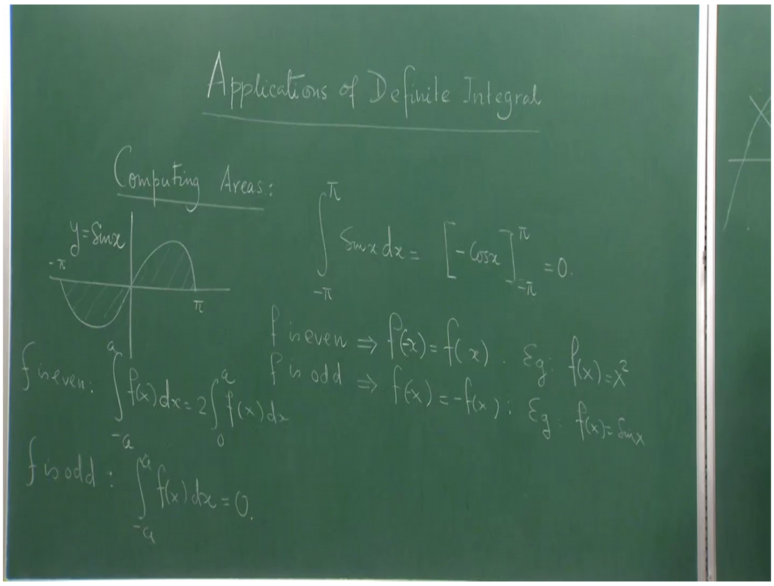

See there was if I look back at the course, we made it with some thought, but as the course progresses as the course evolves in my mind things change. I think that sometimes I think this must be told and this must not be told. So, in that way you have certain things which followed the course certain things which does not didn't follow the course, but overall I tried to do something which would be more novel than a traditional course it is not just funny to just to talk about traditional things which you can find in any book any course you will just open the YouTube and find such things. To tell things which are not really there you cannot find it anywhere that is what makes the difference.

For example, when we will learn Taylor series, we will see how that can be your McLaurin series we will see that how that can be applied. So, nicely to show for example, that pi is irrational. So, those sort of things are not done in a traditional course and that is exactly what we are going to do. For example, and show that pi actually differs from 22 by 7. 22 point 22 by 7 actually bigger than pi.

So, strictly bigger. So, that is exactly certain interesting stuff that which is not done in traditional courses would be done here. So, when we you learn to compute areas one of the first things that question arises and that was also taught in many standard courses what is the area under this curve. That is y equal to sin x. When so, this area you see if I walking on the counterclockwise direction the area is on the always on the left this area would be actually positive well. This area when taking the counterclockwise direction, it is on the right that area would be negative and there would be equal and they will cancel.

So, basically if I want to, I want to tell you just compute the integral simply writing that this is nothing, but sorry be 0 yes see that this will be0, but these actually has a more general structure. This this result happens because of a more general structure the 2 classes of functions called odd and even. So, f is even if you have f of x is same of f of minus x example for example, fx is x square.

So, another is f is odd, is it f of x f of minus x is minus of fx. Here f of you are write it both side minus. Here f of minus x is fx here f of minus x is minus fx. So, this is a function which has symmetry on both sides. And here it does not in this case it does not have the symmetry sort of anti-symmetry execute, for example, So, when you put minus x instead of x will be minus x cube.

Another example is sin x. Cos x is an even function a sin x is odd function. For even function it is very important that for you to know at the very outset that if my area of my range of integration is symmetric, there is minus a to plus a right 0 in between then this is nothing, but once it is even. So, this is nothing, but twice of twice of, but if f is odd like the one in sin x then you will simplify find the that is exactly what you see here.

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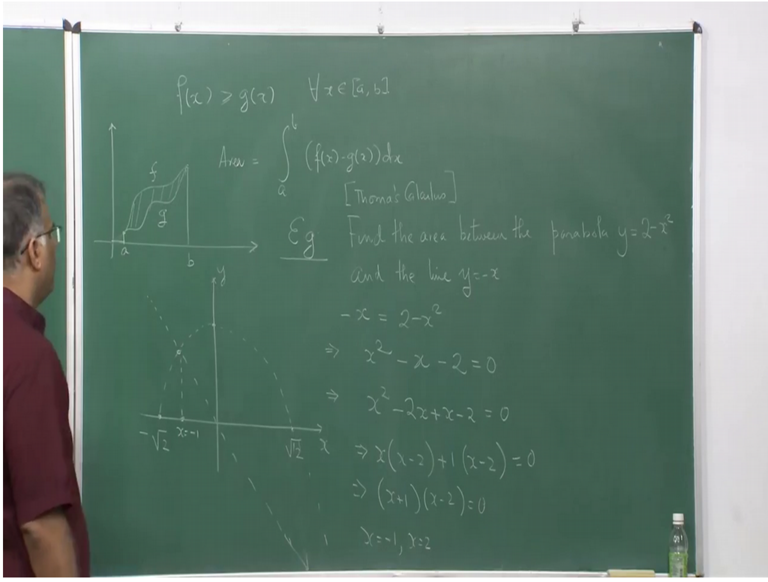

Another very typical type of situation arises, when you really have to look into the area between 2 curves. For example, if you have 2 curves fx and gx with the relation that fx, fx is greater than gx at least on the given interval of integration right. Then at least for all x in abc. So, what is the area between the 2 curves between a and b. So, it is a situation like this. So, maybe this is maybe this is my a, a and this is b. So, this is f and this is g.

So, I am essentially trying to know what is the area between. Of course, you understand I compute the area under whole of under g and then subtract that 2 areas and that that is what I will get. So, of course, you will have in. So, the area in this case is a to b, fx minus gx dx. So, from Thomas calculus will take some examples to tell you what how to handle such a situation, right.

So, as an example let us see. So, the first example will say that find the area between the parabola and the line y equal to minus x. So, this example is from Thomas calculus. A new name of the famous book Thomas calculus an analytic geometry. These are all standard takes all over the world for basic calculus courses at the undergrad level. So, here what am I supposed to do, let me see. So, I will try to graph this parabola first, insist that you always try to draw graphs of function.

So, if this is x and this is y. So, if you have 2 minus x square here. So, if x is 0 y is 2. So, the graph is like this. So, this is parabola it should be something like this. So, and when y is 0 x is root over 2. Then y is 0 in the x square is 2. So, x, x is plus minus root over 2. I am just drawing it keep dot be the exact root 2. Now I need to know at which points the curve is intersecting right. At which point this curve y equal to minus x is intersecting. So, y equal to minus x is the line this, passing through 0. So, y is equal to minus x. Where does it hit it? So, it where does it cut the parabola it cut the parabola somewhere here, right.

So, what are those cases. So, so put y equal to minus x. That will imply x square minus x minus 2 is equal to 0 x square minus 2 x plus x minus 2 is equal to 0 which gives me what does it give me x into x minus 2 plus 1 into x minus 2 equal to 0. So, it gives me x is equal to x plus 1 into x minus 2 is equal to 0. So, the points where the parabola intersects this line is when x is equal to 1 sorry x equal to minus 1. So, here I have 2 solutions x equal to minus 1 and x is equal to 2. So, here it intersects; obviously, not here it just goes down like this and intersects somewhere there to somewhere here possibly.

So, if I want to draw the graph more clearly instead of drawing it. So, big I would rather now draw this graph. So, now, I have that in these 2 parts minus 1 and between x equal to minus 1 and x equal to 2, the function value of the parabola is always about the function value of y equal to minus x.

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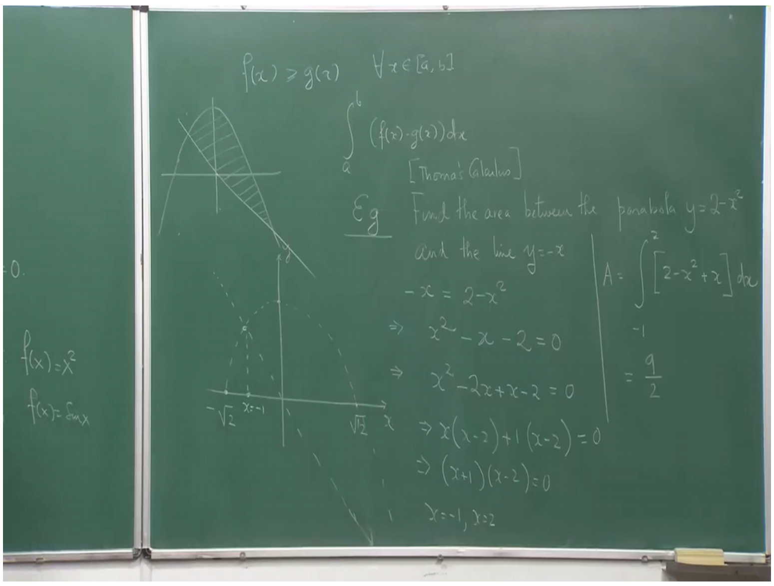

So, if I draw this graph. So, it will be something like this. So, here is your parabola symmetrical. So, y equal to x it is cutting at one. It will like this this is what we have to figure out this area.

So, what we you will now do. So, our tool is very simple. So, f is 2 minus x square and y is minus x. So, it is. So, it is 2 minus x square minus x. So, the area is area a is integral from minus 1 to 2, 2 minus x square minus a minus or minus x is plus x dx. So, that is that is the range. So, now, if you integrate out I am not going to integrate out I will rather ask you to check the answer it is 9 by 2.

So, another slightly more difficult example which will be again from Thomas and calculus. I will show you 2 ways of working with such things with that particular example at least, but. So, let us look at another application which says find the area of the region in the first quadrant bounded above by the curve y equal to root x and below y x axis and the line y equal to x minus 2.

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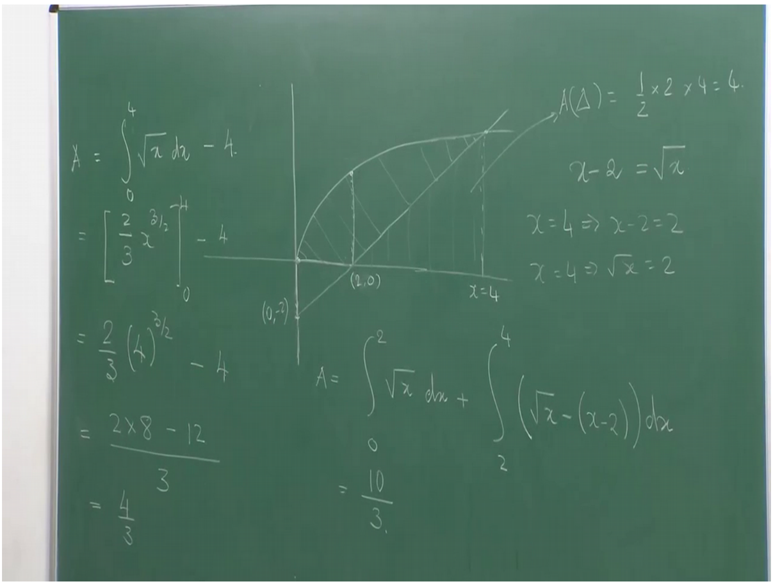

So, I am supposed to find. So, this is your y equal to root x. So, when it is 2 this is root 2. So, this is 2 and root 2 the line y equal to x minus 2 passes. Because when x is 0 when x is 0 y is minus 2. And when y is 0 x is 2. So, the line definitely passes through 0 minus 2 line that line y equal to x minus 2 definitely passes through I mean here it passes through or this is 0 2. And this is sorry this 2 0 and 0 minus 2.

So, this line gets passes like this, right. So, it must be intersecting the curve somewhere. Where does it intersect where does this line y equal to x minus 2 intersect the curve? So, when is x minus 2 equal to root of x. What is that value of x. Answer a very simple you do not have to solve it is 4? Because if you have a x equal to 4 and it implies x minus 2 is

2. And if you have a x equal to 4 it implies that root x is also equal to 2. So, that is it root x means positive root x only we are not looking into a negative plus we are not writing minus root x because you write that then root x does not remain a function, right.

So, basically now my job is the following. So, find the area under the curve I am supposed to now, this point here this point here, but now I am supposed to find the area only in the first quadrant. So, this part is forgotten. So, I just have to find this area which means I am finding the area from 0 to x equal to 4, but people will make a division. What division they will make? See we will divide like this first they will say just take the area of root x find the area take the area of root x from 0 to 2 and then at this area, but here from 2 to 4 root x is bigger than x minus 2 y equal to x minus 2.

So, between 2 to 4 you will do root x minus x minus 2 that will that would be the way to compute it. So, I will just write the answer I will not go to write down the details and so answer here is 10 by 3 because we do not have. So, much time now I you can do it in slightly different way also in this particular case, because if you observe it very carefully what will you have? This is a triangle actually. This triangle has length 2 and height 4. This particular triangle this triangle what is the area. So, area of this triangle is equal to half of base which is of length 2 into height which is 4. So, it is 4, 4 units.

So, what I will do to find this part of the area, I will compute the area under the curve from 0 to 4. And then subtract this triangle. And I will I will get the same area right. So, what I will do here is basically then I can do 0 to 4, root x dx minus 4. So, let us see whether we get the same answer. So, if I do root x means it will become 2 into 2 third. 2 third adding the correct thing 0 to 4 means 2 third x to the power 2 third. So, it will become. So, 2 third. So, area is equal to 2 third x to the power it is half plus 1 right 3 by 2.

Student: 3 by 2.

3 by 2 2 third 3 by 2 from 0 to 4 minus 4. So, 2 third into 4 to the power 3 by 2 minus 4. So, what is happening 2 into 4 to the power 3 by 2 means 4 to the power half which is 2 2 to the power 6 right. No 2 will give is 8 minus 3 into 4 12 by 3. So, the answer comes out to be 4 by 3. I do not know why does the answer does not match after should match. Is any mistake here? With this hum because that is a natural thing because I will just

have a triangle here. So, I take this triangle out from this whole area under the curve and then only this part is remaining right that is what we require.

So, this area is 2 2 4 which is 2 the length of the base and other part is a is what 4 4 into 2 8 half base into areas 4. So, subtracting that from 0 to 4. So, integrating 0 to 4 under root x is x to the power 2 0 is 0 2 third into 4 to the power 3 by 2 2 square. So, it will become 2 cube which is 8, so 3 into 4 12. So, 16 minus 12 is 4 by 3 where is the mistake then how do how do we handle this can we make a small checkup of this.

Student: Sir, area can my 3 (Refer Time: 23:09).

Hm

Student: (Refer Time: 23:11).

Hm.

Student: (Refer Time: 23:12) 10 by 3 (Refer Time: 23:14).

No, area is 10 by 3 according to the book, but here it appears to be 4 by 3 if you go by that method, but they should match right. So, here it is let us make a check.

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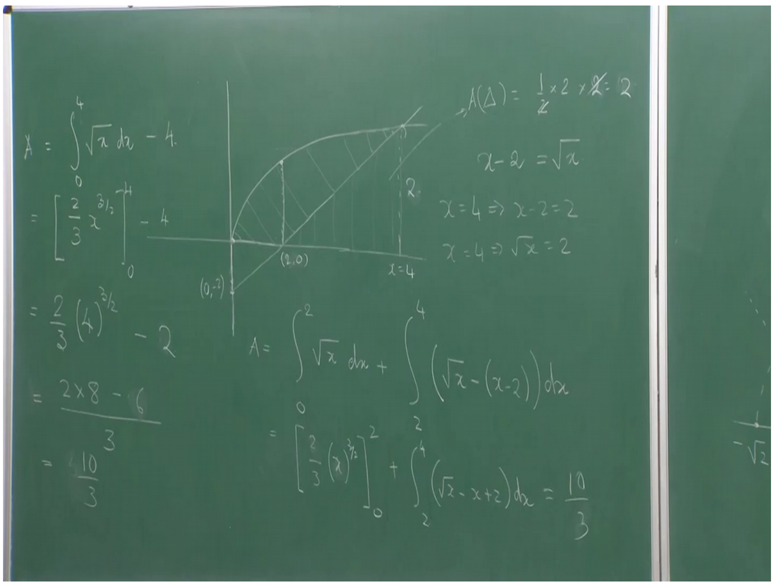

So, here it will become what does it become 3 by 2. Sorry 2 by 3 x to the power 3 by 2 from 0 to 2 plus, 2 to 4 root x minus x plus 2. So, let us do the computation. So, if I do the computation here. Let me see what I get. So, I think this should match because I do not see any logic why it should not match right.

So, these are the trials. So, in mathematics that is the way you do maybe I am making a mistake, but that is what I see this is the whole area under the curve. This is the full area. And I have just taken this triangle out which was which is me the remaining area and that is exactly this simplest way to look at it, knowing that the area of the triangle is does a mistake the answer is correct answer is correct this height is not 4 this height is actually 2. Because it is root of x know. So, this height is 2. So, this height is 2. So, this area is 2. So, it is 2 is the area. So, if I have 2 here. So, if I have 2 here. So, I told you some calculation mistake 2 into it 16 minus 3 into 2 6 16 minus 6 is 10 10 by 3.

So, you see once you are sure that the mathematics that you are doing is correct answer has to match. So, the answer a you match. So, both are both is ten. So, this is also equal to 10 by 3 sorry there was a mistake with this 4 this high it is same as the height of the function value at x equal to 4 which is 2, where the function is fx equal to root x. So, here is an example how you 2 ways to do the same problem; obviously, the one we had done is a much simpler one by looking at the geometry immediately, but here in the book they have followed standard ways of doing things what does not matter though this area, which I just I did it I find that at the end of this chapter they have also done the same thing, but that they. So, 2 different ways to do the same problem. I would like you try for example; this proof it is sometimes fun to do proof.

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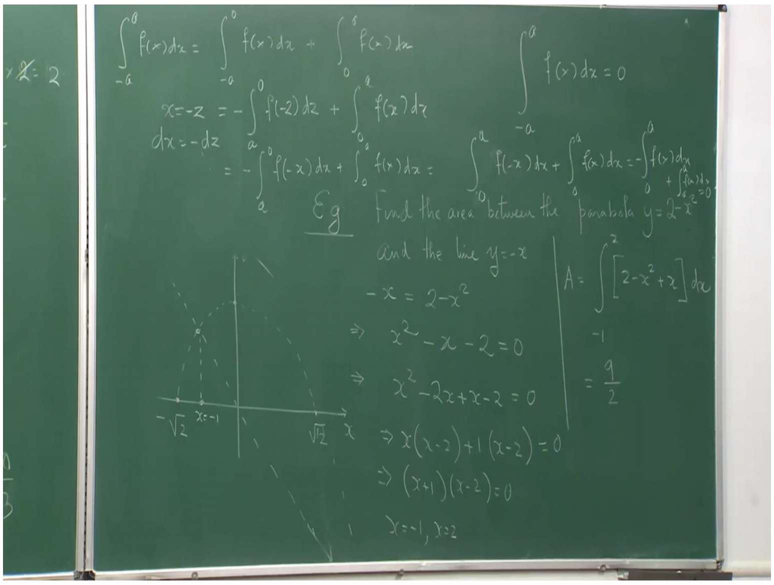

So, f, f is odd function how would you show that it is 0. So, one approach could be that you look at fx dx as from minus a to 0 fx dx plus 0 to a fx dx. So, I am not doing the other part which is already done for. So, this is an exercise asking in Thomas and Thomas calculus into prove this. So, what you do you substitute minus x, x equal to minus x basically or x equal to minus z possibly x equal to minus z. So, dx equal to minus dz.

So x equal to minus z. So, if. So, this becomes f of minus z minus dz. So, when x equal to minus a this is a when x equal to 0 it is 0. So, a 2 0 plus 0 to a instead of z, I can also write fx dx. Now you can say y z and next these are dummy variables. So, do not bother with dummy variables you can just write it as f of minus x dx plus 0 to a f of x dx now you see how the reasoning goes, but the because this is odd f of minus x is f of x is minus x.

Student: fx dx.

So, this will become now will become a 2 0 sorry will become 0 to a because of the minus here. So, this will become f of minus x dx plus 0 to a fx dx what you see f of minus x the function be not it is minus of fx. So, this is nothing, but 0 to a minus of fx dx plus 0 to a fx dx which is equal to 0, that is it. That is, that is solves the problem actually.

So, you see we had some discussion of how to find areas what are the relations a little bit of proof at the end. So, that you we would have some time. So, with this will stop and in the next class, we will try to discuss more about these applications for example, finding volumes, how will you find volumes. So, this will come in the next lecture, that how would you go ahead and find the volume of the volume of a object right. So, let us that would be very important. So, how just you can use this integration to find the volume of an object.

So, we will learn some stuff for finding the volume of sphere our cone our cylinder all these things. And let us see what would how we go about it. So, so we will take time and the in the next classes also we will keep on doing this integration volume of cylindrical shells and the these applications of how, in a definite integral just how just of single variable can we use to find a lot of good things. So, maybe some of the applications they have to physics these sort of things for 2 3 classes 4 classes will discuss about applications, how things can be applied.

So, will come back to the next class and we will we have discussed volume will discuss surface area. So, this this will continue for 3 4 more classes. And then we will go to sequence series Taylor's theorem and all the stuff. So, some idea what power series and we will come to all those things in possibly in the last section. And so we have 5 5 10 lectures more of which I guess if we do not take this lecture last lecture into in of which I guess couple of lectures would be 3 to 3 lectures, or 4 lectures would be an integration and the remaining lectures would be really on power series sequence and all those stuff. So, will come back to this very soon. So, will start looking at volumes in the next lecture.