# Calculus of One Real Variable

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## Lecture – 26
## Calculating Indefinite Integrals

We shall make use of the second fundamental theorem of calculus, which we learned in the last 2 weeks lectures to actually compute integrals. You cannot be just telling you Do this do that and this is it will happen with abstract quantities or functions.

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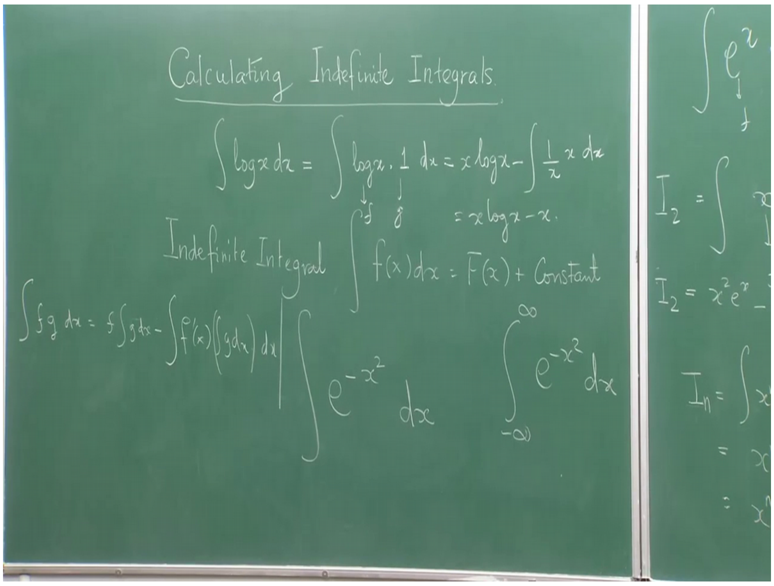


We need to also tell you how to go ahead and actually make exact calculation, because that is what you really need. So, if capital F is a function such that it is it is means differentiable and it is derivative is equal to f of x, and we say that the indefinite integral, indefinite integral integral f of x dx is nothing but f of x plus a constant.

So, what we are now doing it is called integration in elementary terms. Actually it means that for simple situations like this you can actually do some integration. But for example, if I look at a function like this e to the power of minus x squared dx this integ this cannot be integrated in this form, because you cannot find a capital F which is differentiable for which f dash x would be exactly equal to e to the power minus x square. So, for those cases where it is possible we say that indefinite integration is possible, and then we can actually write down the real stuff.

So, of course, if the function f is continuous you would you already know that you we will you will have if you can you know you can always have set these sort of primitives. It is it is can be always done, but for continuous function if you can actually find a function a 2 x the way we did the mean value theorem then of course, you know that that thing actually work that capital F works basically if you actually can figure out this. So, that is a continuous function, and then f is differentiate then if this is this; let us say it is f t dt.

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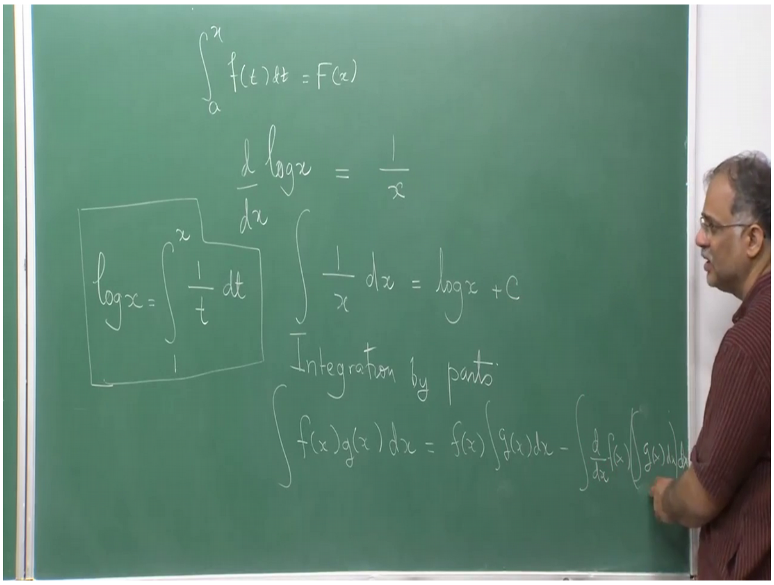


So, then see some in are figure might ask why I am not writing t here because you have taken x here. So, these are all dummy variables even if you write x does not matter and once you understand the situation. So, then of course, you can do all this elementary stuff, but it is not So often that it is not that for every thing that you have any dam function continuous function you are only you are able to write anything that you are able to write you will just always get elementary function, I may with for which you will always get a capital F for which f dash x would be that function.

So, here is one example, but by the way when we will learn improper integral this integral with certain limits. So, minus into plus infinity becomes an important integral. So, for example, if I have ha thing like this makes huge sense this answer would be root pi if I have not forgotten it. So, this makes huge sense has a applications in statistics the normal distribution, and the interesting part is that one is just when you just take of this 2 things this makes no sense. And these are the things that you have to keep in mind while doing your math and So, let us start with 3 simple things for example, log x.

So, for example, if you differentiate a log of x, you take the ddx of log x. So, what do you get you get 1 by x of course, you know that you have your function x cannot be chosen to be 0 of course, you know that x cannot be chosen to be negative either even the 1 by x makes sense if x is negative what log of x would never make sense. So, then you write the integral of 1 by x dx is log of x plus c. In fact, this whole this thing leads to what is called the logarithmic function sometimes log in a modern way you can think that log can be calculated in this format.

So, if you are give a given x and a log x and you simply numerically evaluate this integral there is some techniques by which you can evaluate integrals we will show them as one of the applications on how to evaluate integrals. So, they will come as one of the weeks lectures here that how do we evaluate integrals and then we apply the apply those stuff. So, what I want to tell you is that once you do that you would realize that you can actually compute log x by computing this integral. So, though logarithms were known much before calculus, but you can actually use this very simple idea of the fundamental theorem of calculus to actually compute log x.

So, this is this sometimes called logarithmic function. One of the greatest thing that happens is the use of the integration by parts. So, I have already thought you what is integration by parts when we are doing definite integrals, but if I just write down what is integration by parts in this case in the standard case it will like this.

So, you just do not have to put the limits, there will be limits that is it and what the same story would go on. Now here you would have. So, I am integrating f of x and gx. So, dx So, what you do? You keeps the first function fixed say integrate the second. So, I am sorry integrate the second function minus integrate first you take the d dx of the first function fx and then sorry integrate G x dx. So, this is whole thing with another dx. So, there are 2 dx is here basically you integrate this then integrate that. But when you integrate this thing inside you do not write the constant because the constant we will finally, get a absorbed in the constant at the end.

So, that is what you do it is it is here you do not put the limits you see do not you do not have the limits anymore. So, that is that is what it is for example, if I take an arbitrary function I remember my own younger undergrad days or So. So it I think I learned this in high school, but So say e to the power x, x squared dx called a c it is not.

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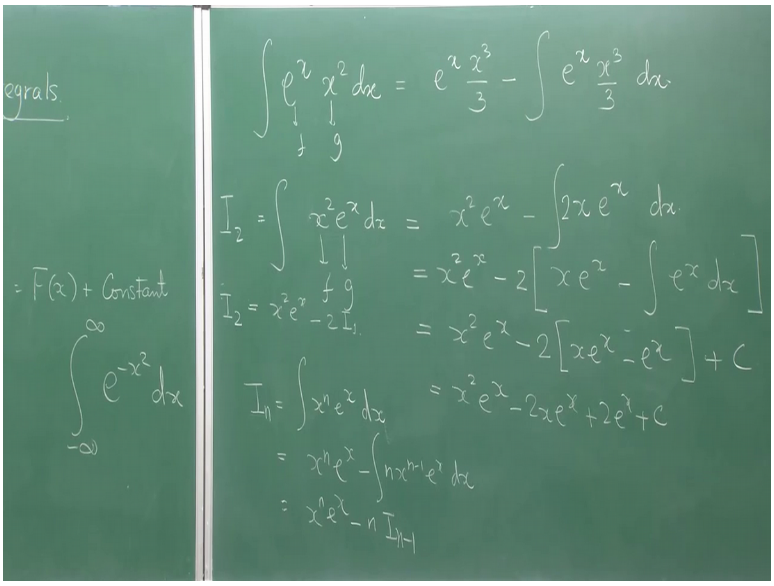

So, easy to immediately say 1 1 can. So, I will put the x square is t x is and you will have 2 x. So, e to the power t, but you do again be the same situation. The substitution is the substitution method of integration that you learn at high school, we will just not work here So easily.

So, what you really have to do is to think about it in terms of integration by parts, because you can integrate terms. So, keep the first function there if you integrate this it will become x cube by 3 do not put the constant in because the constant finally, is a part of the whole thing that you can just add at the end. So, minus integration the d dx is e and e to the power x and x cube by 3 dx.

So, this is So, this is what is happening. So, you can say I am stuck with a similar situation. So, you keep on applying. So, this will keep on increasing. So what? So, this sort of integral. So, what is happening is that you can have some relations between them. And then you are collate to get some better want for example, maybe I will try something else well take a more simple example, maybe I will take instead of x square, because So, you have come to x cube.

So, you might think oh I made a mistake it is increasing. So, if I again try it integration by parts I will another x 4. So, how will I go? So, here there is this. So, you have chosen this has the first function. So, this was your f and this was your g. So, when you face such situations where powers are increasing and you know things for just getting out of control, you can try to switch the f and g mode. So, you switch this mode and you rewrite this integral as the integration my parts is one of the most important tools, you rewrite this integral as 1 by as your x square into e to the power x dx, right.

So, this now becomes f and this now becomes g, and that you would give you let me call this as the integral I. So, this will give you x square e to the power x minus, now you have take the derivative of my power is goes down, x e to the power x sorry, 2 x. So, so what is happening is my power has now on down. So, that is a question for jubilation of course, I am not putting the c you can put the c at the end. Minus 2 or now a I now go back to these considered this x as f and this as g again I reapply the integration by parts which we give me just as this x e to the power x minus integral. Now if I take the derivative of x will become 1. So, e to the power x dx.

So now, if I rewrite this. So, what I am having? X square e to the power x minus 2 of x e x minus ex plus c. So, ex plus c is whatever the c will go out plus 1 c. So, basically now you have the integration by parts formula. So, if you want to write it down somewhere let me write it down. Integral f g dx and just writing in code not writing fgx you can now understand. So, f into integral gdx minus f dash x into integral gdx. So, this is the formula.

Student: (Refer Time: 12:01)

Yeah gdx into dx. So, that is a formula which we just wrote down and what it here. So, we are doing the same thing here basically. So, here you will have x square e to the power x minus 2 x ex plus 2 e to the power x plus c. So, how do you check that this is true? You take the derivative of this and check that will give you give you by x square e x.

So now, you can actually develop what are called recursive relations here. Basically if I define I n as integral x to the power n e to the power x dx, then how you can have a relation between I n and it is next one. I n minus 1 possible here because you can again switch this whole thing to have x n e to the power x minus integral n x n minus 1 e to the power x dx. So, I can write this as xn if I call this as I n and the power is n then this is nothing but integral I n minus 1.

So, here is a recursive relation. So, basically when I is 2, here it is I 2 basically this for I 2, essentially if I go by this notation. So, I 2 is nothing but x square ex minus 2 into I 1. So, that is the relation. So, these integration by parts makes certain relations between certain typical type of integrals. So, you could have for example,. So, we suppose now I give you ask you to do this integral very typical on a famous example. So, integral of log x dx, log x essentially we are meaning that log to base e in calculus is that is what is the meaning base 10 does not matter, but base 10 we do not really want to work with because that has you know you can do a lot of things when you have that x exponential as the base.

So, log x dx. So, how will you do this? Log x dx business how are you who how do you know which function is your derivative stuff. So, here you are going to have to invent something. Here you are going to write log x as and how do you this it is not. So, apparent to write of function whose derivative is log x. What you are going to do is you are going to write log x into 1. So, this is your f and g is a constant function which is G x equal to 1, for all the x and then you see you apply the integration by parts. And you see the power of this whole technique. Then what you do is just like here you keep the log x fixed. So, log x and if you integrate this is x. So, x log x minus integral ddx of log x which is 1 by x into x dx. So, x is gets cancelled. So, it is x log x minus x.

So, so surprisingly not So simple looking integrals can have a very simple solution once you have these kind of stuff. So, I do not want to get into more of these issues, I would rather concentrate on doing showing some other examples. So, I have just halted the camera for a while and just wanted to write down these integrals.

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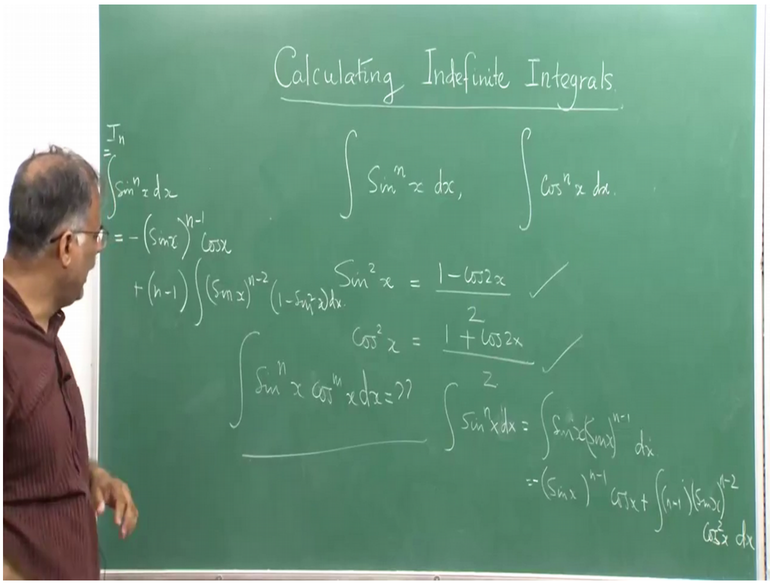


These it integrals of trigonometric functions because it is asking sin x to the power n cos x to the power n integrating you want to find the indefinite integral or you want to find the capital F. These formulas become very handy to evolved. I would like you to any practice some integration to do some of this trigonometric integrals.

So, I will not get into too much detail. So, what I will do I will take a problem from spivak an example and show you that how this integral can be handled with these formulas. So, first we will take an even power of sin, and then we can take an odd power of sin and then we can do the then we can apply sort of other techniques. So, here So, for example So, integral that is the example which spivak uses is this one.

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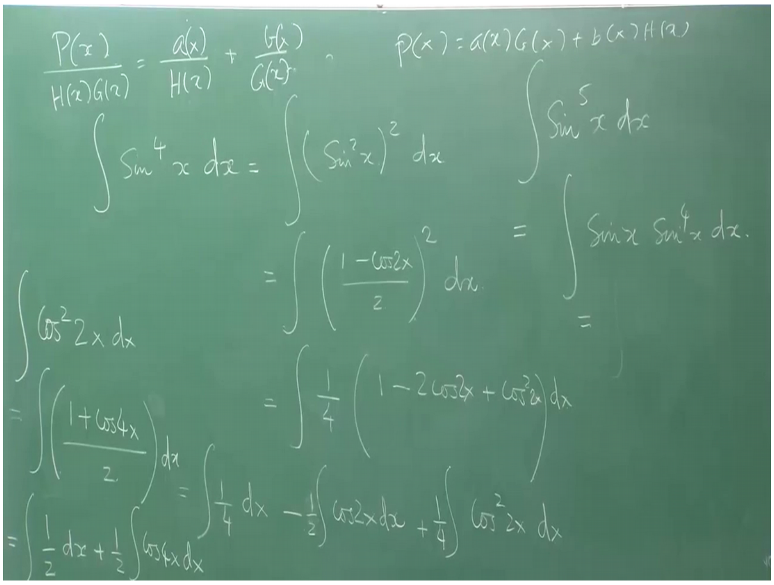

So, what we will do? Is write it as integral sin square x whole square dx. So, here I know what is sin square x it is 1 minus cos 2 x by 2. So, it is 1 minus cos 2 x by 2 whole squared dx.

Now you just have to open the bracket expand the sum. So, it is one 4th into 1 minus 2 cos x plus cos square x dx. So, then of course, it becomes very simple one 4th d. Sorry, sorry thank you cos 2 x cos squared 2 x. So, this is just one 4th of dx minus integral 2 of cos 2 x dx you know what cos 2 x dx is. You see when now you have this additional the 2 bar 2 by 4. So, so it will become half here and one 4th of this part, cos square 2 x now it is cos square 2 x? How will you how will you handle this situation?

So, here you put 2 x here you put 2 x it will become 4 x. So, basically if you take the integral out. So, you can just you just really have to compute this integral this is what you will know what is a integral is this is minus sin of 2 x by 2 1 over half of sin minus sin 2 x. So, you know these integral this is you know this is the elementary thing here. Again you have to use this formula you see in the same formula we are using making use of both of these formulas. So, cos square 2 x dx can be again written as integral 1 plus cos 4 x by 2 whole square. So, I have reduced taken the integration now the power has reduced. On the power square of square it has now come to just the square. And so, you can write down this again in.

Student: (Refer Time: 19:26)

Sorry, sorry, sorry, sorry, thank you. So, from power of power. So, square of square it has come down to just the square here. So, here I is a just need this not square thank you. So, I will just use, simple thing now it will become half of dx plus half of integral cos 4 x dx and I am not going to write this formulas for you these are standard high school stuff which you can write. So, I would really. So, you can ask this is even, but if it is odd. So, if you have sin say what 5 x dx. Now I suppose to do? What I am So, the only thing I am suppose to do is to write it as sin x sin 4 x dx. And then integrate now then, So for this part you know the integration right.

So, this part you already know the integrations sin 4 x. So now, now you go by the integration by parts. So, you can look into formulas like sin n to the power x dx sorry n to the power x cos m x dx you should try this at home these are some good exercises, but I am sure it is not a good exercise for old men like me at the board at this hour. So, I would rather leave these things to you. Now you can actually say is for sin nx instead of taking these sort of roots, here we have use the formula and taken some roots, take a taken one approach. We when used directly integration by parts if I am faced with this problem on always use integration by parts. So, I am trying to tell you how powerful integration by parts it would you which is I think the central jewel of these elementary calculation approaches; sorry.

So, here you can actually do sin n into sin n minus 1. It does not matter, sin n x into sin x into sin x to the power n minus 1 dx, and then just do one. Keep this for your derive in a second way keep like this as a first function and then take the derivative of this and just go ahead. So, basically you will simply have here. So, take this as f and this is g. So, you will have sin x of n minus 1 into cos x minus integral. I take the derivative of this it will become n minus 1. So, it will become n minus 1 into sin x sin x n minus 1 minus 1 n minus 2 into cos x, into derivative of sin is cos x and you already have cos x from here this integration minus.

So, here I will have cos x into minus cos x which is coming from here. So, it will become cos x square and outside that that will come and become plus. So, cos sin x cos square x I mean cos x whole square into dx. So, that is where what the formula looks like. So, what would happen? I can just now write integral sin nx dx is equal to minus sin x. So, this is my integral I n. So, I n is equal to this one this integral. So, it is sin x n minus 1 cos x plus n minus 1, sin x integral sin x n minus 2 and cos square x can be written as 1 minus sin square x. And then that will give you the recursive relation.

So, these recursive relations becomes very important in elementary calculation into the days when people were calculating thing by hands. These were actually the feedback systems. So, xn is f of xn plus 1 these are algorithms for in those days, while people did integration. They had a actually invented the way we now do algorithms.

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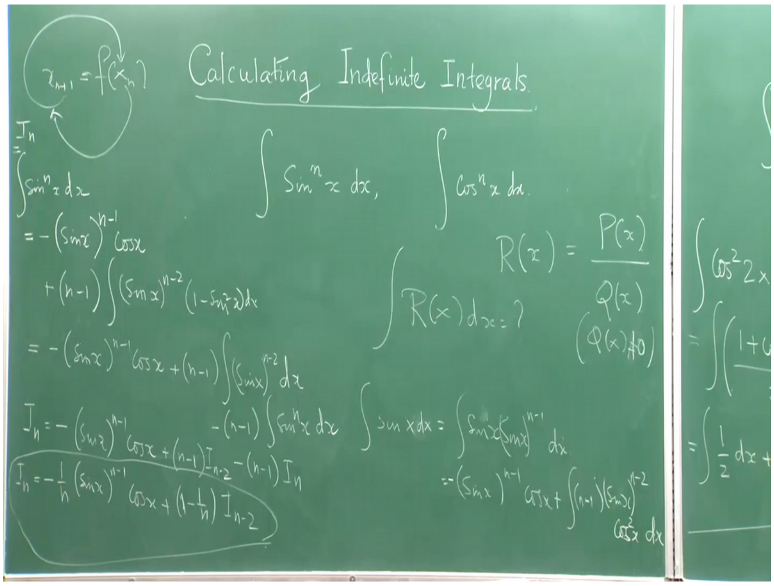


So, you have a function f where you put xn you get xn plus 1, and you have an re put xn plus 1 here you will get xn plus 2 that sort of feedback system. So, that is exactly the feedback mechanism that you are getting here. So, that is finally, minus sin of x n minus 1 cos x plus n minus 1 integral sin of x n minus 2 dx right. Plus sorry, minus n minus 1 integral if I multiply this will become sin n of x dx right this is.

So, what I am getting? So, this is I n is equal to minus sin x n minus 1 cosin x plus n minus 1 into I n minus 2 minus n minus 1 into I n. So, you take it to the other side. So, what will have? I n So, I n into I n plus n minus 1 into I n. So, you will have n I n right. So, then you divide by n. So, it will become minus 1 by n sin x n minus 1 cosin x plus n 1 minus 1 by n this is n minus 1 by n right. Into I n minus 2. So, that is the recursive formula. So now, you can find. So, if you want to try I 4 if you what we are thinking. So, I just put 4 here that is it.

So, you do not have to do all these things. So now, the last part is about partial fractions which is a very, very important class of. So, stuff I would again do it from spivak here some one or 2 good examples, there are some results. So, mathematical results where I know now nowadays people might not be too much interested in doing to details of things. But So, they are looking at functions in this way. So, you are going to integrate what are called rational function. So, a rational function is a function R x which is represented as a the ratio of 2 polynomial functions.

So, and then we are looking at what is we are asking the question how to integrate this. Where Q x is of course, not equal to 0 and both are polynomials So, certain degree. So, Q x cannot be 0. So, both are So, these are polynomials of the of some degree. So, that is standard example that is given in many many books, one this is this is one of the most important class of functions. So, what people do is given class there is a technique of writing it. So, if we go into the technique of writing it will become terrifically complex. The idea is that you have the polynomial Q x and then you factorize the polynomial. So, suppose I have I have been able to write P x in Q x into H x into G x right.

So, you now want to do a very simple calculation now you want to write P x, H x, G x as a of H x plus b of G x. And you want to figure out what is this a and b essentially right. Depend not these no sorry these all functions of x sorry I made a by mistake it has to be function. So, so you want to find out what is this ax and bx. So, basically you are writing you are writing the last part is function. So, basically what is happening? G x into ax plus H x into bx is actually your P x. So, P x is essentially this polynomial is ax into G x plus bx into H x.

So, if we have to figure out this. How does how can one do that? So, this is an example. So, these examples we do not have really time to really compute out the examples here, but this is an important class. So, some examples will be given will be handed out in the notes. I know there is a demand of notes in the forum, I will start sending one or 2 papers already one or 2 chapters, but these sort of things where we are not in the class. I would request my ts to take some examples of these rational functions, and actually write them up scan them and put them on the notice board.

So, these examples everything cannot be done in the class because we actually if you want to break certain things it will take a huge long time which is not feasible, because we are already exceeding our class timings why quite frequently we are doing so.

So, we have some idea now I specifically I wanted to show you the power of doing this whole calculating indefinite integrals the key idea is to show you how powerful the integration by parts method it is. In advanced mathematics many many things which require integration integration by parts come into the for a. So, this is very, very manipulative thing here really we are not going to look into the much of mathematical understanding. Of what there is a particular way you can write this whole thing into and then you can write, write down this integration in certain way. But I will ask the my ts to put in a some examples of these just they will scan to 3 examples. So, you can see those examples.

Thank you very much.