Calculus of One Real Variable
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Lecture - 24
Fundamental Theorem of Calculus (Riemann Style)

So, we are gradually ending or may discussion of Riemann integral, and this is the last part where we are going to talk about Riemann integration.

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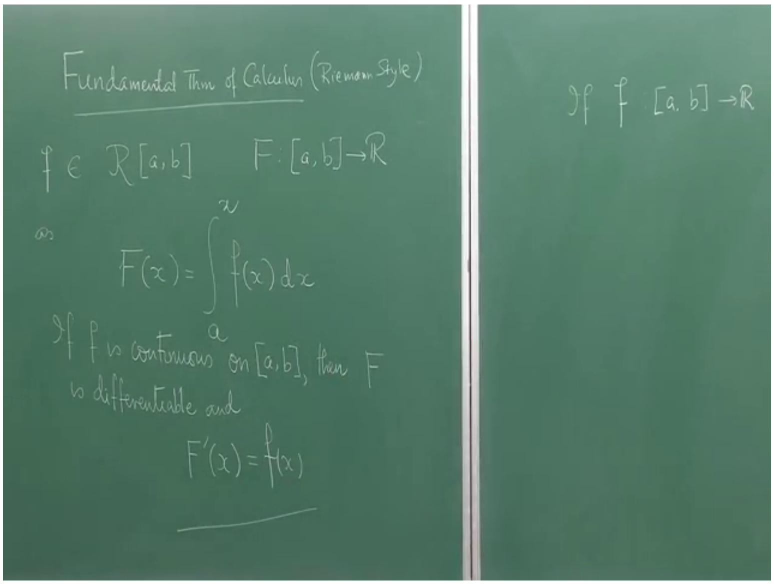

So, fundamental theorem of calculus in the Riemann approach; so, we are not going to prove anything here, but maybe one or two things will prove which are which you can do fast. Otherwise our aim would be to mention the theorem and our aim would be then to illustrate what the result tells us.

So, you have a function f which is in R a, b bounded function integral on R a, b. R a, b means this is a set of all bounded function which is Riemann integrable on ab. So, you have this function f on R a, b and you define a function capital F. So, you define a function F again from a, b to R as F x is equal to integral a to x, f of x dx.

Now, we have spoken about this fundamental theorem when we had spoken about the whole thing in our in Riemann in the newton's calculus, when newton's integration theory when we are speaking about integration we had spoken about this theorem. So, what we have proved actually there. So, that is why we are not repeating the proof here right.

So, what we prove what it says is the following if f is continuous on a, b, then if f is continuous on a, b, then small f is continuous and capital F is differentiable of course, right and left derivative is on the endpoint there are the these left and right derivatives. So, that is assumed.

So, f is differentiable and the one might start asking what do you mean by differentiability in the closed interval it is clear that f dash of a is f of a which is clearly defined and f dash of b is f of b which is clearly defined of course, f dash in that sense when f dash of a may be the right left derivative and f dash of b is right derivative that is all.

So, that is what this theorem says and we also give you some warning of how to use this theorem right; now if f is a continuous function and so, straight corollary is the following.

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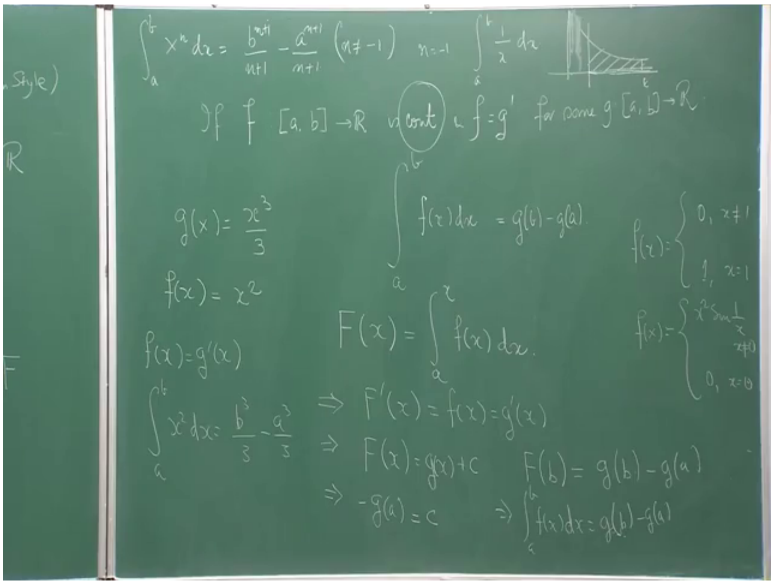

So, if f from a, b to R is continuous this is also we had done and when we introduced the idea of anti derivative basically, but you have you there was the crux I hope you would remember the example which had given the that was something very very important that you need to keep on keep in mind.

That f from a, b to this is continuous and f is equal to g dash means f of x equal to g dash of x for some g or some g from a, b to. So, if this is there then integral a to b, f of x dx is equal to g of b minus g of a. So, you might be wondering how to prove this. So, is g the function g is the function which is f no that is something one has to be very careful of g is not the f function.

There f and g capital F and G should differ by some constant. See let me construct for example, construct f x. So, never think that because of these that capital F which was given there must be this g. So, what does it say? By this previous theorem it implies that F dash x is equal to f of x is equal to g dash of x, but if two functions have the same derivative then they vary by a constant.

So, basically the derivative of these two functions the difference of f minus g the derivative of that is 0. So, f minus g must be a constant. So, it simply means f of x is equal to g x plus c. So, there is no uniqueness of this capital f that is something one has to be very careful of course, you know from here that if I put g equal x equal to a then f of a is nothing, but 0.

So, g minus of g a is equal to c because if I put a here f of a would become 0. So, it will be g of a plus c equal to 0. So, minus g a equal to c and when I put x equal to b, then you have g of b. So, F of b would be equal to g b minus g a or what is f of b? F of b is nothing, but the integral. So, that is your result.

One might simply think; oye; this is just simply think that what is this integral? Integral is nothing integral from a to b f x is nothing, but you have to find we have to find some function g whose derivative is f and then g b minus g a will give you the integral, but you have to be careful here the definition of continuity is very very important that f is continuous is a very important thing you cannot escape continuity here.

The reason is to fold because I am going to give you an example of a bounded function which is not differentiable as a is not continuous and hence not differentiable what has an integral. So, integral does not mean that it is just the difference of two quantities gb minus g a find some function which will you have g dash x and you can do that. So, it is very simple you take any take an example say f x is equal to 0, when x is not equal to 1 equal to one when x is equal to one we had solved this problem, we have shown that this is integrable.

So, once you have shown this to these to be integral, then it is clear that this function is not continuous and not differentiable everywhere, but still integration is there, but you cannot write that integral as g b minus g a. So, continuity is a very important thing here that is what I am trying to tell you there is no where you can write it as g b minus g a.

Of course, there are certain cases for example, if you have g x equal to x cube by 3, and what f x is equal to x square then f x is nothing, but g dash x and of course, that will simply tell you that a to b x squared dx is x cube. So, b cube by 3 minus a cube by 3.

So, now a induction you will have the standard formula which you know about integrals that is do not all any kid who has done some calculus would know that this is. So, you can prove this simply by induction, but you have to careful (Refer Time: 11:00) with be very careful here n equal to minus 1 is not something which you are going to bother about n not equal to minus 1, if n is equal to minus 1 this is undefined.

So, if n is equal to minus 1 then if n is equal to minus 1 we write this as integral a to b 1 by x dx now nobody knows how to do anything about it. Of course, you say oh I ever read integral 1 by x is log x. So, log by log a; so, of course, you can do that if a is strictly bigger than 0 and b strictly bigger than 0 if not. So, in general not knowing what a and b is it is not possible to say anything about this integral.

So, this is something you have to be very very careful about that just by applying that formula blindly, you may not always get something. So, because here there could be a scenario where 0 is inside it will be minus 1 to plus 1, 1 by x dx and then that is it. So, your function which is not continuous which is just blowing up. So, if you have. So, 0 to 1 say then also this function it is meaningless, but you want to some t then it is meaningful.

So, you can understand because this is nothing, but the hyperbola and you understand you want to some t. So, the area is clearly defined, but if I here I have the area the area keeps on increasing because the function blows up as I go towards 0. So, these are the certain things you have to understand that the arbitrary writing formula is not always fine. So, just if I know that f x is equal to g dash x and I write them as integral whatever be n we do not care.

I write it as g b minus g a is wrong lot of things have to be taken care before we actually apply the fundamental theorem of calculus. So, what we wrote is essentially the first fundamental theorem, now here we will go to something more what we have just written this for continuous functions. What can be said if we are just concentrated on integrable functions a bounded function, which is Riemann integrable whatever.

So, if you take any integrable function and if f is equal to g dash right if f is equal to g dash for some g then for even for such functions integral a to b, fx dx is gb minus ga. So, that is what I am going to write down. So, now, we are going off from Newtonian thing into the standard Riemann framework that it does not matter we do not have to bother much about continuity, only integrability of the function is important and only the fact.

Now, you can say that f is a function f is a; oh you might (Refer Time: 14:25) is not f continuous? No f is the derivative of a function; derivative of a function need not be continuous, every function did not have a continuous derivative you have a very pathological example this x sin 1 by x, when f x when x is not equal to 0 and 0 when x is equal to 0 say you take the function x square sine 1 by x.

When x. So, you take the function say f x, x square sin 1 by x, when x is not equal to 0 is equal 0 when x is equal to 0 is this is a differentiable function, but its derivative is not a continuous function. So, g is a differential function and its derivative is the function f. So, it need not be continuous, what we are expecting that it is a bounded integrable function.

So, even under that case even if you do not know anything about the continuity of f itself, you can still have the mean value theorem and that is why that is where we differ from what we have studied earlier, that is exactly the point where we have moved or we have made an advancement in the thinking from the Newtonian the sixteenth century think sixteenth seventeenth century thinking to the ninetieth century thing.

So, we push ourselves 200 years ahead by this just little change that, but you have to understand you want [FL] what is this just changed continuity are [FL]. No taking our continuity of function is a big deal in mathematics and its it makes a huge lot of difference, and that is why I am telling that this what we are now going to do and really want to prove is an important result.

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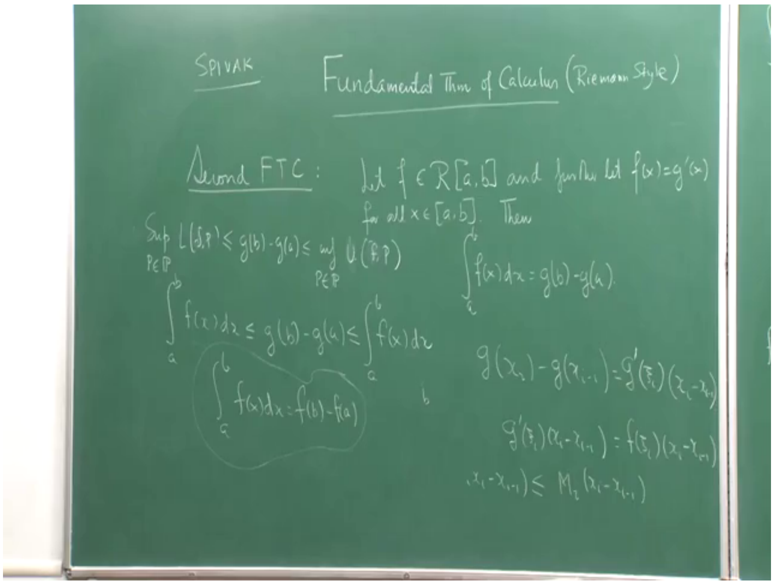

Second fundamental theorem of calculus second FTC. Let f the element of R a, b and further let f of x is equal to g dash x, for all x in a, b then integral a to b fx dx is equal to gb minus ga. So, this is actually a very important result. So, which we are now going to give a proof or proof would be from Spivak we are actually discussing this from the book by Spivak. So, this I write it down again.

So, usually found it other places also, but I prefer write it discussing it from Spivak where it is written in a proper way clean way. So, we will now start doing the proof for this, and with this once we finish the proof of this we can end this I am going to end this part we are not going to go and walk through the things, because will see if we have find time. So, we give some one or two examples it is ok.

So, we will start the proof of this cut (Refer Time: 18:34) though we proved this result, but I would like to tell you that this result is very pathological, it is not so easy to find functions f which are integrable not continuous, but is a derivative of some other function, it may not be so easy this could be one of the examples. So, generically continuity of the functions are important. So, you might see what we do now in a very simple way of looking at the proofs.

So, let me just try to do the whole thing very simple way. So, you have the partition P which is given same way as n points with this is the point a and this is the point b right.

So, now, once we have this, what we can now do is that we take g and compute gxi minus g xi minus 1.

So, by because g is differentiable, there is some xi I which is lying between strictly within xi and xi minus 1. This is nothing, but the mean value theorem of Lagrange that we have applied. So, what is g dash x. So, g dash xi i xi minus xi minus 1 is nothing, but f of xi, xi minus xi minus 1, but f of xi I xi minus xi minus 1 is less than equal to mi there is a supremum over this interval xi xi minus 1 and bigger than the infimum.

So, what we are showing here is that, if I do not even have even if I have continuity of f I i need not use the continuity of f I can only use the integrability of f that is a just the thing. So, once I have this; what I have? I have mi xi minus xi minus 1 is less than equal to now this is equal to this and this is equal to this.

So, gi. So, g of xi minus g of xi minus 1 [FL]. So, if you sum them up from one to n this will be a telescopic sum it will give a g b minus g a. So, you will get l pf is less than equal to g b minus g a minus upf. So, these two for any partition p. So, supremum over all partition p in p L p f is less than equal to gb minus ga less than equal to infimum over all partitions in p, L p f sorry; L f p I guess oh I am I think I will I am making a mistake this is the sign I am usually having it to L f p.

So, I was writing L f p throughout following Spivak's notations sorry u f p. So, this is,, but because the function is integral this is nothing, but the integral and this is nothing, but the integral by definition that simply means that a to b. So, you see these use not the continuity of the even if there is continuity we really do not require to use the continuity argument.

When a continuity of f there are other proofs. So, you can use the continuity of f otherwise you do not require, but you see its simply comes out here it simply means, but continuity is does a generic condition it is really effective when you have a continuous function it is not effective when a function is not continuous. So, we have already achieved something in the sense that we have the covered a huge ground in calculus, by that it is no mean feat that we have done it together and a lot has been done.

So, the course that you are learning it is not a even though it is a you can be think thought of as some basic course in one variable calculus, but is really not just a basic course because here I am not only tending to students who are just having wants for extremely rudimentary knowledge, but I am also tending to the ambitious. So, that is something I made very clear when I gave the two minute introductory talk that I am motivated by this title of Koerners book the calculus for the ambitious.

So, essentially that is what I want through also want to repeat taking the leaf out of Koerners idea that. This is a course for the ambitious that you really want to know calculus much better than many other people. So, we will stop this lecture here its slightly shorter because the next lecture will be slightly longer, because we are going to introduce something very different and you would see that things which are not integrable by the definition of Riemann integration would actually become integral when we use the our approach of the Henstock Kurzweil integral.

Thank you and we will be back in the next class.