Calculus of One Real Variable
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Lecture - 21
Integration Theory of Riemann – I

So, we had discussed integration of continuous functions and what the ideas of Newton and Leibnitz. We have done pretty well actually we did a lot of stuff. We will do more about those things later on when we will talk about indefinite integration. But now how to move beyond continuous function? Continuous functions are bounded, but every bounded function is not continuous as we have shown an example of a step function in the last class, whether we can define anything called an area under that. So, this work was first move from continuous or bounded functions were achieved by a very famous mathematician of the 19th century.

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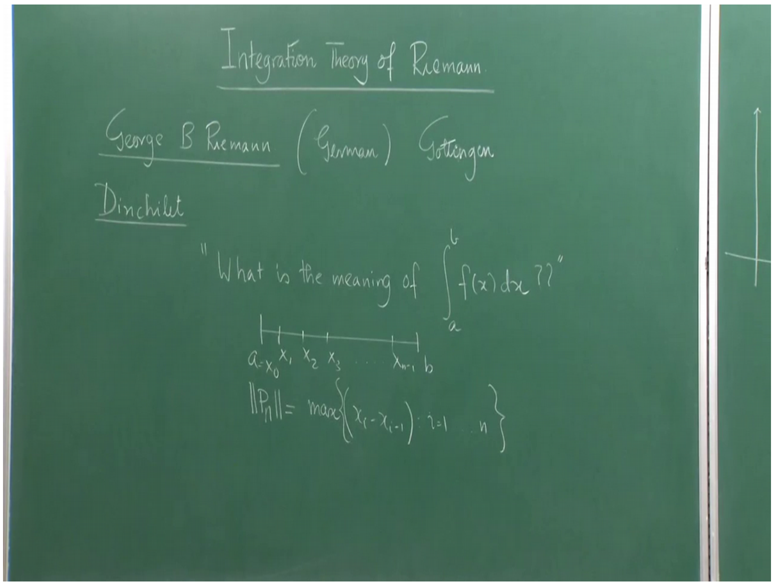

George Borner Riemann in fact, his birthday falls on September, he was born on the 17 September 1826. I would say 1826 or 24 something. So, George Borner Riemann is a German mathematician, and his was brilliant enough to study in one of the most famous places of mathematics of old times, and specially during those times Gottingen. Gottingen was famous; it was not only it hosted some of the greatest minds of 19th century mathematical minds of the 19th century, but it also hosted Carl Friedrich Gauss who was called the prince of mathematicians.

Please understand that Riemann was a very homesick child, he couldn't stay too much away from home he used to go back to his home pretty often. And during his PhD days he was his supervisor was Dirichlet. So, his PhD thesis was title the following translated to English it should simply mean, what is the meaning of, So he was thinking beyond Newton and Leibnitz.

He was not bothering about whether the fu a function is nonnegative or not he was just asking this following question. So, what did Riemann try to do here? What was his idea of the integral? He obviously, knew what Newton did, and the limiting sums that we had written. But in the la in the last class we wrote the; took a positive function and built up the rectangles and wrote the sum of the area of the rectangle and took a limit has the number of points kept on increasing.

So, the maximum size of the intervals decrease and hence the limit if it exists is the integral is the area actually. So, integral becomes a symbolism of area. Can we take out the integral from that symbolism of the area? These integrals have a standalone meaning other than the area, that is the question.

So, for that he considered f to be a bounded function f, f he considered a function from a b to R.

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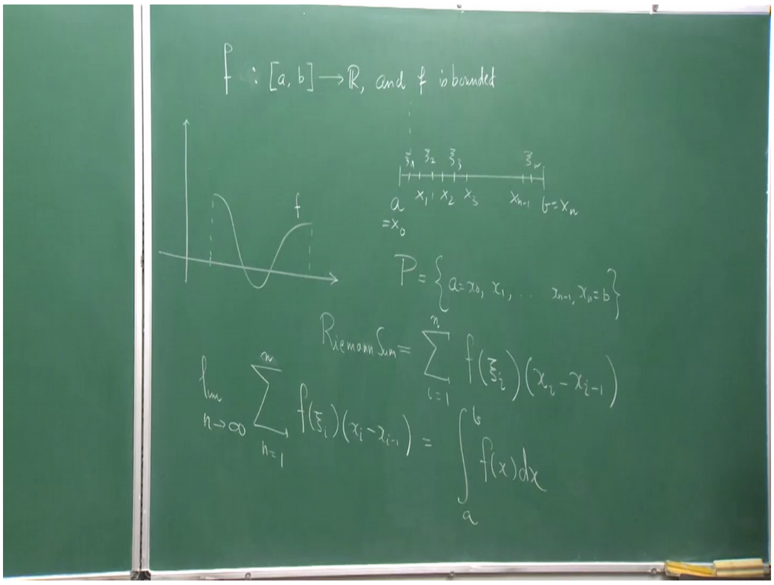

And f is bounded remember that every continuous function is bounded on a closed and bounded interval, but every bounded function need not be continuous. Now of course, he didn't bother much about what was the function was assigned function could be like this does not matter. So, what he did? So, what he did was to look at this interval a b and divide it he put some partitions inside right. What he did was he divided the whole thing into some partitions. So, I will put a equal to x 0 and x 1, x 2, x 3 and x n minus 1 and b equal to x n, this is my partition.

So, what is the partition p? It is a collection of points a equal to x 0 x 1, x n minus 1 x n equal to b. That is simply the meaning of a partition. So, that is how mathematics progresses from one concrete idea of an area you are moving in a direction to free the idea of the integral from the area. And you want to say that if the function is nonnegative then the integral the integral equals the area, right.

Now, what we; Riemann did was to choose a point in this interval say first interval x 0 x 1 did not matter, you could choose it here or say choose i 1; here I choose i 2, you need not be an interior; it could be any of the endpoints also I am just writing in the interior is i n and so on so forth.

That is how we choose, right. So, this partition when you have chosen some point is called a tagged partition. So, what he chose was at tagged partition and he constructed the sum which we call the Riemann sum, which later on become famous as the Riemann sum. So, this is known as the Riemann sum. Riemann sum, Riemann sum as per the partition Riemann is not only Riemann is not only famous for his integral he is famous for his geometry he is famous for his Riemann hypothesis which still books the best mathematicians of the world. So, so this is a Riemann sum and this is nothing but the area of the rectangles which you construct with the height bring f of xi i, right.

So, this is rectangles you construct here. So, whatever be the function you construct a rectangle with it could be this direction it could be in this direction does not matter. So, some there could be some cancellations of plus minus also done correct. So, it is just a Riemann's sum. So, what Riemann defines? So, if you take the limit. Now there is a concept call see if I keep on putting more and more points that is us n tends to infinity. Then when I take the limit of this, and if the limit exists the limit exists sorry, not the infinity it is n. So, n goes to infinity. So, number of points of the partition increases and this if the limit exists we will call this the integral or f x this is the definition of the integral as per Riemann.

Now, you can make it look much slightly more different. For example, if you take a partition p, we take a partition P then call the norm of the partition P, right. Call the norm of the partition P is the maximum of the length. So, maximum of x i minus x pi minus 1 i varies from one to m. So, it is a maximum of the length of a partition. So, you can keep on choosing. So, if you have n points you can call the partition P n if you have n plus 1 points you have con call the partition P n plus 1 and so and so forth. So, this can also be defined as limit norm P n tends to 0, that is as you put more and more points in that is you increase the number of n this length would keep on decreasing this norm P n would keep on decreasing.

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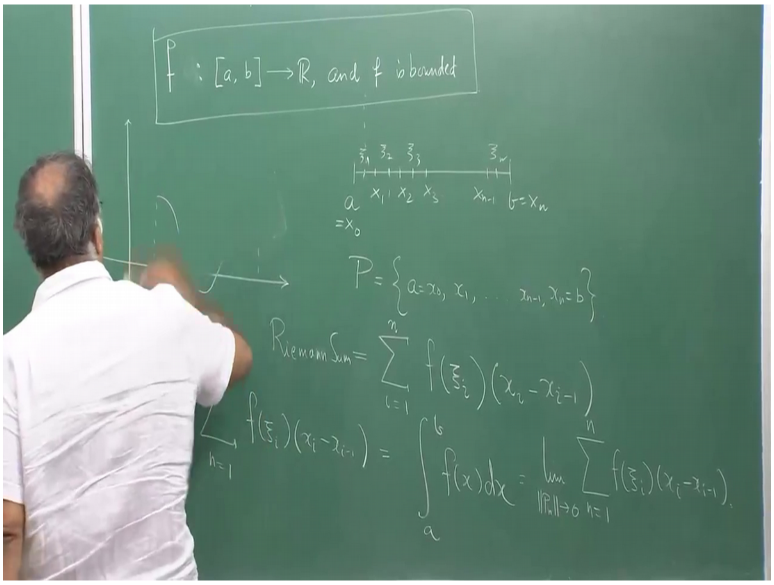

If you take; this is the same as what we have written here. So, some people can also write some authors also want to write in this fashion.

So, Riemann sum, but is this a way we really study in our books? The answer is no. Go to any book, you will see something else they talk over upper sum lower sum etcetera, etcetera, but Riemann didn’t use any notion of upper sum and lower sum. What Riemann did?

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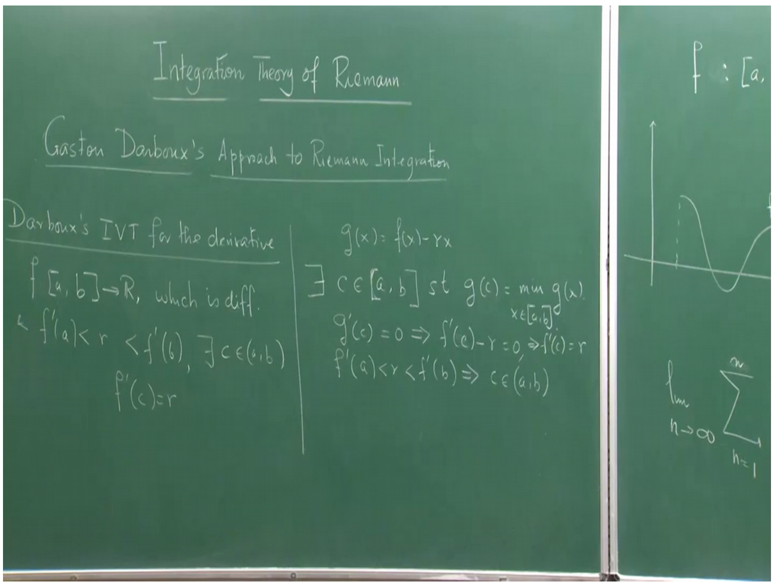

What the things that we learn in the books is the approach to Riemann integral due to Gaston Darboux. So, Gaston Darbouxs approach to Riemann integration. So, this is exactly what you learn in books What Goston Darboux did.

I do not remember a or find the life and death dates of Goston Darboux. Riemann died quite young when near at the age of 36 or 37 Goston Darboux. On the other hand was of quite a famous French analyst he did other work too.

So, there is some Goston Darboux is not just known because he did some modifications or did provided an easy approach to the Riemann integration theory, but he was known for something else. So, study some I have just we are want to celebrate Gaston Darboux famous result I am sure you would enjoy this. Now we have learnt the intermediate value theorem we have learnt that let f be a continuous function, and if continuous function from a b to R in f of a is strictly less than R and f of b is strictly bigger than R and then there exists a c between a and b, because line in between a and b not equal to a or b. So, that f of c must equal to R that whatever be the values that line between f a and f b there is a point on the interval a b which will take. So, whose function value exactly called that that is called the intermediate values theorem that we have already discussed earlier.

Now, suppose I have a derivative I have the, I know the function is differentiable and I know that say f dash a is strictly less than R and f dash b is strictly bigger than R and is there some c such that lying between a and b such that f of c is equal to that R. So, this thing was not known because I have not assumed f 2 because that derivative to be continuous if I do not assume the derivative to be continuous, how can I apply the natural question is how can I apply the intermediate value theorem. But Gaston Darboux used an elegant approach and he proved the even if you do not have continuity of the derivative you still have the intermediate value theorem.

So, it is Darboux’s’ they just for a little bit of just a side tracking of basically we are taking a little d 2 sometimes when you take our journey you get into some by lines and try to discover what is in there. So, Darboux IVT for the derivative; so, it is very simple things take 2 lines to prove it. So, you have a function f from a b to R which is differentiable. Assume that the function is from R to R, but we are resetting it from a b to R and assumed that is differentiable that the derivatives are there. And f dash a is strictly less than R and f dash b is strictly greater than R then they are exists, a c element of a b such that f dash c is exactly equal to R.

We have not assumed the continuity of the derivative. Of course, f would be continuous because it is differentiable. So, how would you prove this little fact? Now consider a function g x, f x minus R x. Now this function g x is differentiable because the function f is an R x is; obviously, a differentiable function. Now what we can immediately understand is that f, right. So, this f is a continuous function. So, g is also a continuous function. So, g the function g be a continuous function on the interval a b right. Would have a maxima or minima in that the it will have an upper bound and a lower bound and the bounds would be at end. So, they are exists c in a b in a b, such that g c is equal to the minimum of g x or x element of a b.

Now, what is happening is that once you So, you know this. So, you know that g dash c because it is differentiable it is 0. So, that would immediately imply f dash x f dash c minus R is equal to 0, which will immediately imply that f dash c is equal to R. And you know that f dash of a is strictly less than R and f dash or b is strictly less than R. So, on that would immediately imply that c must be in a b and that is the proof of the Darboux intermediate value theorem. So, again I want to repeat that though what we are going to learn about Riemann integration is not really Riemann's story about in the books it tell you the story that was written by Gaston Darboux. And this is the one of the most famous theorem that would Darboux proved which even eluded Cauchy that If you have a differentiable function and the derivative behaves like this then there would be intermediate point for which f dash c would be equal to R.

So, this is called the intermediate value function for our derivative, but you do not need the continuity of the derivative this is very, very important result. So, we are just taken as it will d 2. So, this is the additional information. So now, we will go back. So, this is what you have seen this is what is written this is what Riemann did, right.

Now, we are going to write the story of how what Darboux has done about Riemann integration, but how would Darboux view Riemann integration. And In fact, he proved that taking the limit do we approaching the limit of this is same the value of the integral approve that is obtained by this and the value the integral that he obtains by his approach are the same. So, that we will not prove that theorem would write it down. So, let us write it down here I hope that you do not mind me rubbing of the intermediate value theorem because this was just a d 2 on our journey.

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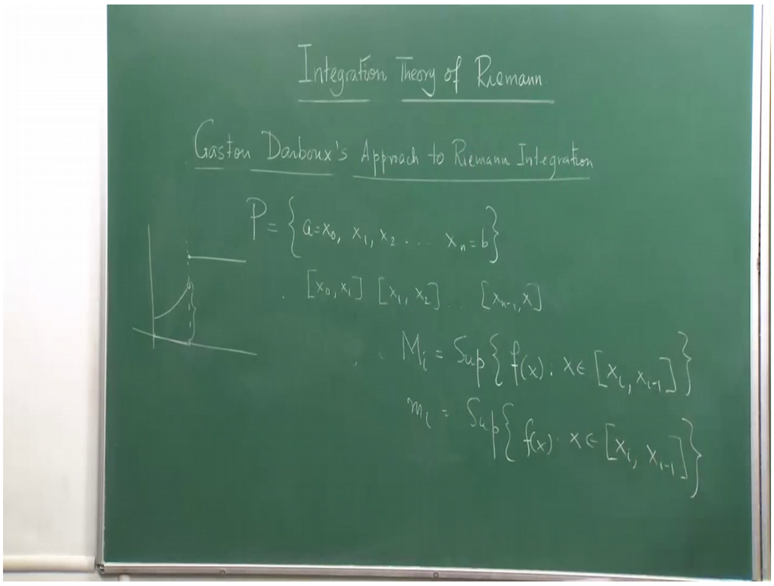

So, again take any partition P between a and b. So, a partitioning a and b. Now what is happening is that for every between every partition x 0 x 1 x 0 x 1 is the interval. So, basically the partitions are of this form. Sorry, in each of this interval the function is continuous and hence would have a maximum value and a minimum value. And that is what we are going to exactly use. In fact, we will not call a maximum value or a minimum value we will call a supremum value; an infimum value because the function is not continuous is bounded. So, though it could be that there is a discontinuity, but the bound is there, right.

So, for example, it could be like this. So, it is here. So, on this interval on this particular interval the function is bounded by this value, but the function never achieves this value. So, that that is what I am trying to say. So, we will just use the letters M i is equal to supremum of f x, such that x element of x i x i minus 1. And small m i is equal to supremum of f x, x element of x i x i minus 1. So, all these results that we are going to write down a very standard in the literature; is there any enumerable textbooks whatever textbooks you are open in to be the same thing or we will.

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Sorry this would be infimum, sorry, I made a mistake just; so obviously, because the upper bound and the lower bound. There will be a bound because the function is bounded please understand our function this is a standard of function of when we do Riemann integration I am not talking about any other transfer functions, right, but keep that in mind we are not. So, bounds are always there. So, this is the bounds. So, once I have the bounds this allows me to create what is called the upper Riemann sum and the lower Riemann sum.

In the upper Riemann sum So, for a given partition P the upper Riemann's are new P f or U P f usually I would have the many, many ways to do I would rather follow here the notations from Spivak and the results from this famous book by Spivak which I was saying. So, he does it right he said U P f.

So, I will just follow that in case some of you have just see Spivak. I wish this book has an Indian edition and people could all if students in our country should really buy. And read this book, but any way this all this description that I will give is very, very standard in found in most of books all books still in my calculus U f P.

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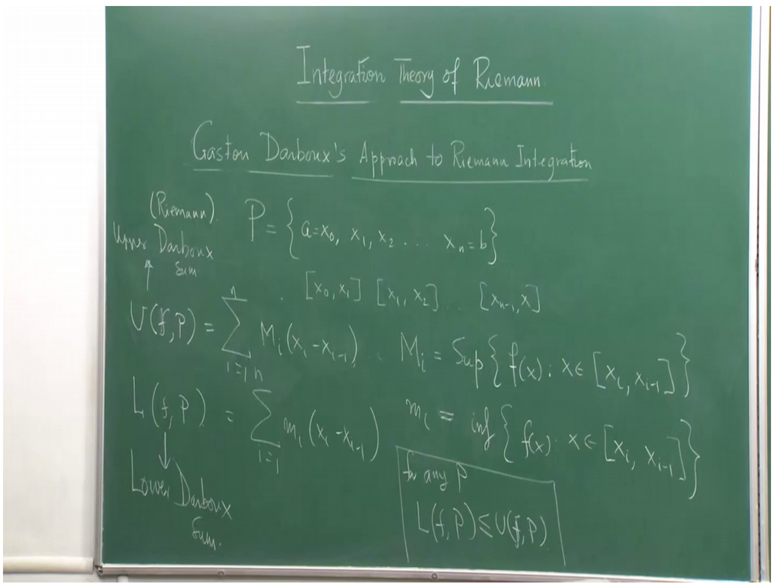

So, given the function and given the partition the upper sum, sum this is actually in the literature some people call it the upper Riemann sum, but actually it should be called the upper Darboux sum. Because when I did it; when I was a graduate student I was told that this is called the upper Darboux sum. It does not matter if you replace the word Darboux by Riemann it is also accepted in the literature.

We are ultimately you are talking about the Riemann interior. Now this is nothing but summation i is equal to 1 to n M i x i minus x i minus 1. And the lower Riemann sum or the lower Darboux sum right it has the lower Darboux sum. So, you can immediately understand that M i should always be less than capital M i small m i should be always less than capital M i. So, one result about the nature about the Riemann sums or the Darboux sums is clear that for any p, the lower Riemann sum is always written lesser than the upper Riemann sum. Now how do you define the integral?

The integral is defined as follows now let me tell you we are not talking about areas anymore we are called we are just giving mathematical definition of the integral. And so, I am removing what Riemann had actually done I will follow Darboux procedure. The very end of the lecture on when we have studied enough on Riemann integration we would in the integrate Darboux method will show or at least write down the result which is also duo to Garston Darboux which says that his approach and Riemann's approach are the same. So, they integrate the same class of functions.

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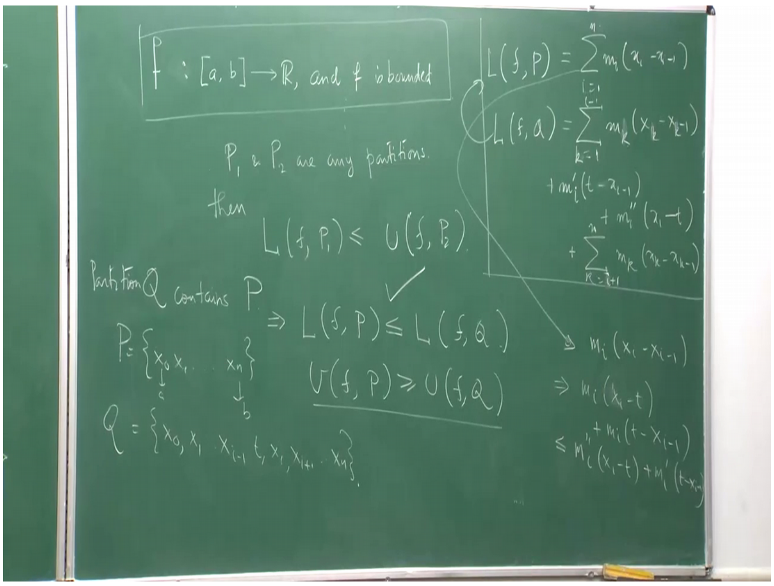

So, what does this mean? What can we do about it? There are certain issues are that has to be settled before handle that what we want to prove that?

So, P 1 and P 2 or any partitions; then lower Riemann sum is always big lesser than the upper Riemann's, I mean it does not worry if you can change P 1 or P 2. It does not matter you can change them and an interchange P 1 and P 2, it has got nothing to do with this. So, we have not even defined the integral we are trying to understand the nature of the upper Riemann sum and the lower Riemann sum. Of course, this is the same type of area when we have studied the ideas due to Newton and Leibnitz, and we are trying to press the area between the upper rectangular or some and the lower rectangular area. But that is beside the point that is how I would ask you not to think not to think about in them in areas just think that these are some sums.

These are just sums which are defined like that, and it does not matter whether M i could be negative this M i could be negative also it does not matter. So, first result that we wanted to show is that if Q contains P. So, partition Q partition Q contains P, that is pr at least one more point than Q at least one more point it has more points. So, all the part of points of P contains all the point all the points of P are contained in Q, then you have then it would imply that the lower Riemann sum is less than.

So, I just want to mention that I am using the book by Spivak calculus by Spivak. So, I am using a symbol because I you I usually read U P f. So, that does not matter. So, what am I supposed to do? How do how a how am I going to prove this fact? So, first step is just an induction procedure. So, the first step is to take our take P that x 0 which is a x 1 x n right a this is b.

So, take Q to be x 0 x 1 x i minus 1 P x i x i plus 1 x n. Just one point more just one point. So, in between team x i minus 1 and x i x i I just added one more point t. So, all the points of P are called is also in a partition of t, now what would happen? So, what is your L P f of Q? L f P; right. L f P you know is of course, summation M i x i minus x i minus 1; i is equal to 1 to n. And what is L f Q? That has to be now thought about.

Now here I have, I have 2 partitions to be now added x i minus 1 to t and t to x i. Now the minimum over. So, x i minus 1 to t has a minimum also, but that minimum is bigger than the minimum over the whole interval, similarly for this interval that the minimum here is minimum over the big interval or equal.

So, L f P has something like this. L, i is equal to 1 2, you come to this point one to I minus 1 m i x i x i minus 1 plus m. So, I can say m dash phi you got put k also in this place maybe where I should put k it is much more easier to understand k, k one dash one and M i dash t minus t minus x i minus 1, right. Plus m i double dash x i minus t. So, what is happening? What is happening? So, if I write down this whole thing and then this plus the remaining terms plus summation m k, x k minus x k minus 1 k would vary from.

What is that point? Now next I will have t plus 1 t plus 1 to n. So, this is the Riemann sum. Now this particular term here in the ith case it will become M i into x i minus x i minus 1. So, if you just take, take a fig flee from here and you just look at the ith interval came here. Then it will be x i minus x i minus 1. Now for that case I can replace this as M i into x i minus t sorry, sorry I can replace this as to sorry, x i minus t plus M i into t minus x i minus 1. But you know M i by the very definition M i is less than M i double dash x i minus t plus M i dash t minus x i minus 1. Here sorry this is this is I plus 1 to n, right.

So, what is happening? So, this term this term here is bigger than the ith term here, and the remaining everything remains the same. So, immediately we will see that this term that L f. So, this term is bigger than this term. So, this immediately holds. I would leave the rest of the part for you to prove this part. And then what you do? So, suppose I have for P to Q I need to add say 20 points. So, just keep on adding one point and thus keep on writing down the results and finally, we will reach P. So, I am not doing it in detail there is time constraints also there in this course.

So, once you get that, now f is a function which is bounded on a b with that is a standard thing. Now how do I prove what we had started to prove that any if I get any 2 partitions? What I am I supposed to do what I am going to get. Once I, so I want to prove now this, once this is established how do I prove this let me see, how do I play the game the game is simple actually.

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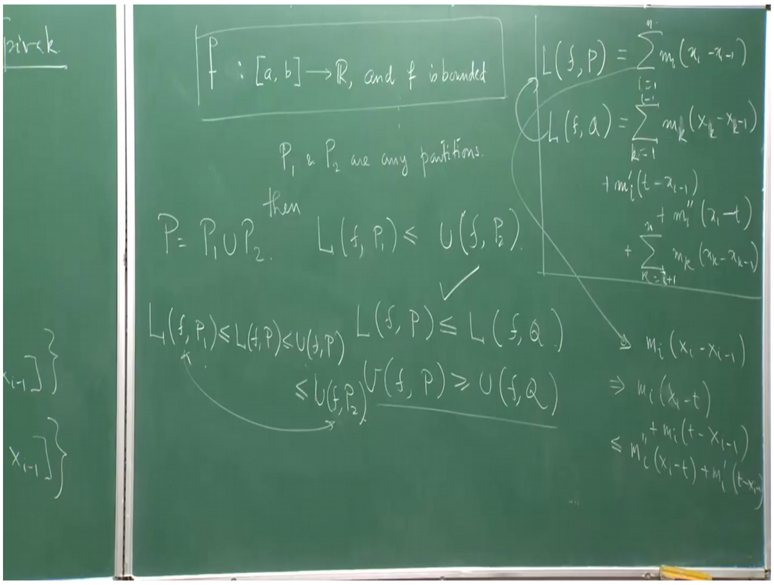

Now, once you have a partition P 1 and P 2 construct a partition P 1 union P 2. So, this P 1 union P 2 partition which we are now calling P contains all the points of the partition P 1 and all the points of the partition P 2. So, L f of P 1 is less than L f of P and which is less than U f of P and U f of p; obviously, is perform this is less than U f of P 2 and just connect these 2 inequalities and you are done. So, it is just a application of these 2. So, once you know this the remaining thing is very simple. What does this result allows me to do? So, here is a powerful idea. The; what does is this? This is a Darboux insight, that this is this is a true mathematical insight.

So, what does this result allows me to do? This allows me to define thing you want to a integral by the way we will take around 4 lectures to complete, 3 my integral and one lecture would be on is generalization. So, that is the plan. And then we will get into these applications around this stuff in proper, integral and all the other issues.

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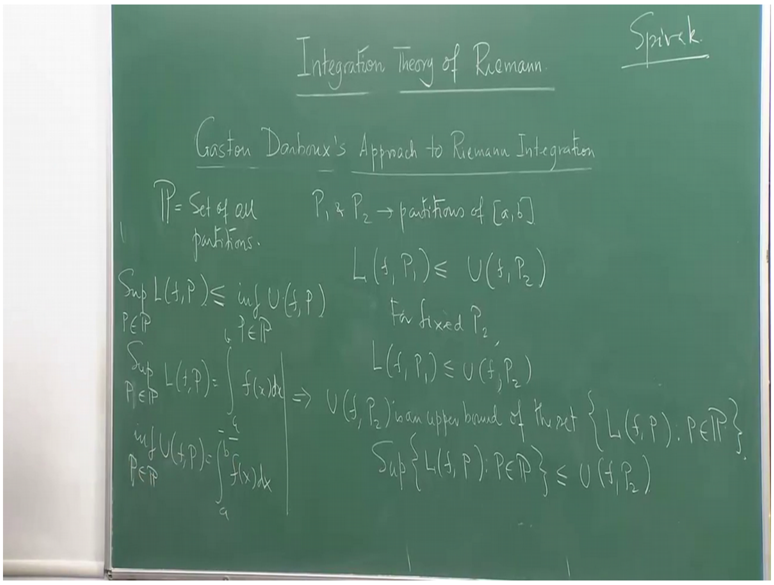

How to calculate indefinite integral? Some important things applications, all these things would come out after that.

So now what does it say? So, for any P 1 and P 2 just put your partitions or a b; so, what I have? L f of P 1 is less than U f of P 2. Now if I fixed my P 2 I can vary my P 1, it does not matter this relation is always holding. And suppose I fix my P 2 and then I very my P 1.

So, P is the set of all partitions is the set consisting on all possible partitions of a b of course, this is a countable set or we can have 1 point, 2 point, 3 point, 4 point, 5 point whatever. So, for fixed P 2 let me write it down once in c for fixed P 2, L f P 1 is less than U f P 2. What does it imply? U f P 2 is an upper bound is an upper bound of is an upper bound of the set of real numbers the set L f P, p element of this set P right. So, if we had an upper bound there must be a supremum. So, where suprimum is a least upper bound it should be less. So, supremum of L f P the P belongs to P is; obviously, less than U f P 2.

Now, this P 2 must chosen arbitrarily, it does not matter. Whatever P 2 do I choose I can come to the same conclusion? So, supremum of this thing acts as a lower bound to U f P 2 for all P 2 belonging to P. So, which implies that So, from here I will prove that supremum of L f P, P belonging to P is less than infimum of U f P, P belonging to P [FL]. So, this is the nature of the Riemann or the Darboux sum. So, Darboux says that if equality holds between the 2 then there a common value is the integral. So, supremum P element of P L f P it is called the lower Riemann integral. Sometimes written like this. This is just fancy think we just do not really I do not really reply for writing such things, but P element of P U f P is called the upper Riemann integral. And some people had the lower Riemann integral is always lesser than the upper Riemann integral.

If they are equal we call it the Riemann integral. So, it is nothing Darboux is defined it in this way if these 2 values are equal then the common value is defined as the integrals.

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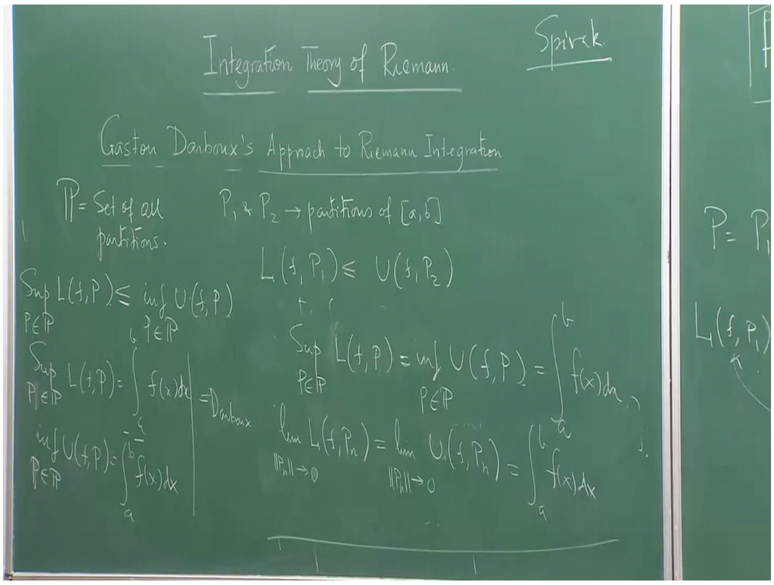

So, if sup P element of P is L f P is equal to in f of U f P, p element of P this happens is there equal then their common value is the integral. This is the integral according to Darboux. And he says that these are Riemann integral, but to us at this one this is an integral for a boundary function defined by Darboux.

At the very end what Remann's for Riemann did something else in his thesis and I wrote down in the very beginning that name of the thesis was what is the meaning of integral a to b f x d x is a just sees it in 1854. Now then we will at the end we will write down half page and end of the next class has the Darboux been theorem which shows that Riemann's approach and his approach are same right. So, what Darboux also showed that if you take the limit we have defined what is called the norm of the partition right. If you take L f c n limit with norm P n when here is the partition with the n points right. And the largest integral length of the integral of the partition that goes to 0. So, it is not infinity.

His aim is just intends to infinity nothing else, it is a fancy over the way of writing intends to infinity (Refer Time: 39:58) sometimes mathematician wanted to be more geometrically visible this is what Darboux proved. Darboux proved and not only this is true this is actually true, but if the Riemann integral exists. If this integral exists is this is true. And if this holds then this also holds. So, with this we end our talk here. So, we have just introduced the Riemann integral. So, we will not get into too much of what Darboux says and what with these things will come at the end. And so, this is this integral is actually this integral Darboux says if this happens.

Then this must also happen and this is exactly equal to the Riemann sum that is what Darboux would prove that is Darbouxs one most one of his; another famous theorems. Now what we are going to do in the next class is to introduce examples, showing how to calculate one integrals. What is the necessary and sufficient condition for an integral to exist, which is nothing but which is a very clever restating of the definition. So, that is also Darboux theorem. And that actually helps in really cal really finding a Riemann integral really showing that a function as Riemann integral able. So, we will do that in the next class.

Thank you.