Calculus of One Real Variable

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Lecture – 02
Countalbility and Uncountability

So, often when I am when ask to give some motivational lectures to much younger students may be high school students I often asked the students this following questions.

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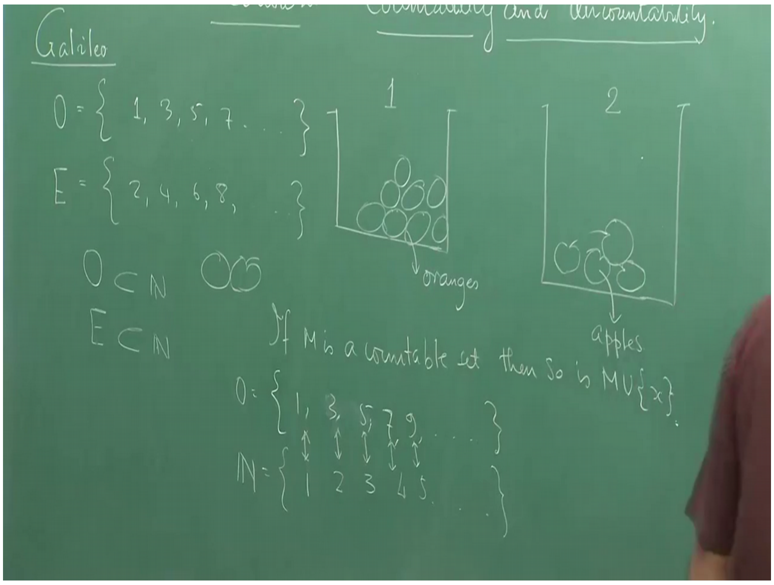

Suppose you have 2 crates of fruit, the 2 crates and there are fruits here. So, one consists of say apples or say these are so oranges. So, here your oranges and on other you have apples and so on do not take the drawings so seriously. So, here are apples now how do, so, this is crate 1 and this is the crate 2, which one has more fruits crate 1 or crate 2? Here; obviously, from the drawing it is clear, but suppose you do not know to count you have no idea about the numbers 1, 2, 3, 4 how do you know which has more fruits and which has lesser fruits, whether you have more oranges than apple or apple than oranges or the equal.

So, usually when I ask this question the usual reply is that we will weigh these 2 crates, but you know with very different amount of apples and oranges the crates can weigh same right and with equal number of apples and oranges the crates can weigh different this is basic physics. Because it you really do not have to the numbers really, do not have to be counted when you are looking for a weights. So, now, the question is what would you do sometimes some students call to the very the real answers and that is exactly the counting is, that you take a orange from here and take a apple from here put them club them together and keep them aside the same you keep continue doing it.

Suppose after when you see that I have finish in the crate 2 there are no more apples, but there are oranges still left. So, oranges are more, but if the reverse happens then the apples are more and or you see both of the crates are empty at the same time then you have the same number of apples and oranges.

So, this pros process of corresponding one object with the other is exactly the method of counting, in the earlier days when numbers were not there the shepherd used to these apply these methodology. So, when they would send this their flock of sheep they would take a for each sheep that is going out they will take a stone and put in a bag on when the sheep come back. So, what they will do that for each sheep coming in they will take a take the stone out from the bag and throw it, if all the stones finally, exhaust from the bag all the sheeps have come in if some stones re remained some sheeps have not come in or eaten by wolves or something has happen to them.

So, this method of 1 to 1 correspondence is what is exactly the counting is all about, what this numbers 1, 2, 3, 4 did is that they made a basic set through by which you can do all those correspondence. So, let us note all the countable set.

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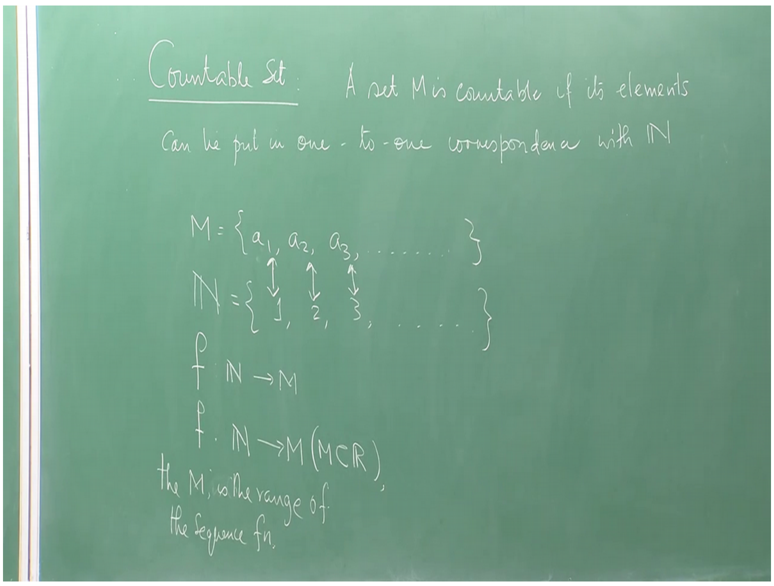

So, countable set is a set with elements can be put in 1 to 1 correspondence with the set of natural numbers. So, a set M is countable if its elements can be put in 1 to 1 correspondence with N which a set of natural numbers.

So, if M consists of elements and so on it could be a finite set, it could be a infinite set I am keeping that open, then the set of natural numbers is this then this is the correspondence, a 1 is the first element, a 2 is the second element, a 3 is the third element and so on. This whole process 1 to 1 correspondence if you learnt in high school is actually a way of creating a function. So, essentially what we have done we have created a function from the number N to the set M. So, this is the very abstract function, but if this a 1, a 2, a 3 are real numbers then we will call such a function as a sequence.

So, if I have a very specific scenario where M is the offset of say R. So, M where M is subset of R then the then m is the range of the sequence function. So, this function m is called a sequence function. So, I have not spoken about functions literally doing the definition, but you know I expect that since you have already are in the high school or above so you would have some idea what functions are. So, this whole procedure is a 1 to 1 function similarly, there can be inverse function it is a 1 to 1 function if inverse it is from M to N. So, a 1 f for a 1 would map to 1 a 2 would map to 2 and so and so forth.

The every strange things, you know if you have a subset say if you have 10 elements now if I make a subset of 5 elements you naturally know that the subset has less number of element and the whole set, but this may not be true when you have infinite of sets. So, infinite sets there is a very strange way of looking at things we will come to that very very soon. So, this is the method of counting, this 1 to 1 correspondence this method of counting says one way simple fact that if M is if m is a countable set then. So, is M union x how does gone through this fact that M is a countable set, then M union x is also a countable set that is if you add 1 element it does not affect the basic countability.

So, here I will give you a analogy of a hotel, it is called the Hilbert's hotel and this hotel business would actually solve all the problem that you will see coming, all your answers will come from the hotel business right. So, it is a nice stuff many of the good books speak about it not all, standard books do not I will not avoid.

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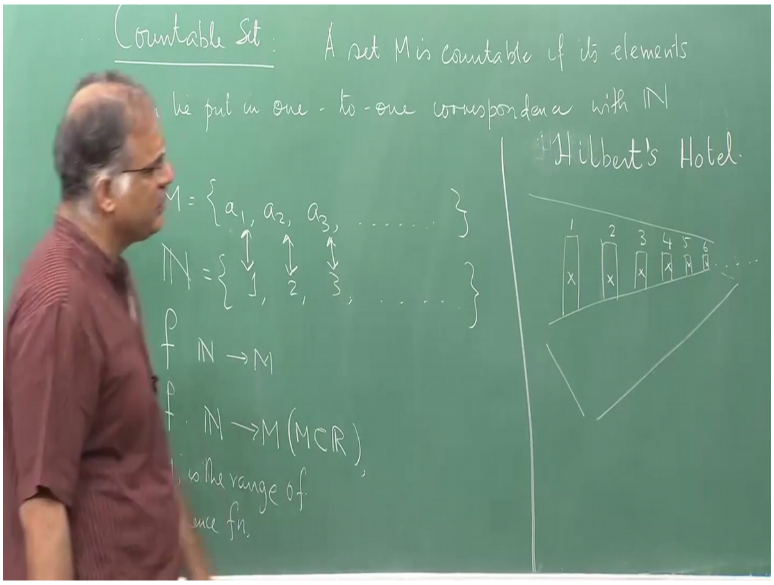

So, it is Hilbert's hotel, but those who do not know David Hilbert was one of the greatest mathematicians of the late nineteenth and early twentieth century, a very big German mathematician Hilbert's hotel. So, there is a strange hotel say beside nice hotel this is the balcony sort of thing and he here the rooms, it is a very strange hotel it is an infinite hotel and then so this is room number 1, this is room number 2, room number 3, 5, 4, 6 and so on.

Now, suppose all these rooms are now filled and then another guest arrives who is suppose to be a mathematics teacher and he says that he wants a room the manager says oh sorry we are completely booked, he says do not worry just shift the person in room 1 to room 2, room 2 to room 3, room 3 to room 4, room 5 to room 6 and so on so that room 1 will become empty for me, so he is that x right. So, if which means that these persons 1 the persons who are occupying these rooms where actually the elements of the set M and this new guy who come in is the set x, this is the simple criteria which solves many problems. So, Galileo observed for example, this very strange phenomena about the infinite world, which is this that if you take the set of odd numbers which is 2, 4, 6, 8, 10 so on and the set even numbers 1, 3, 5 oh sorry, I made a mistake the set of odd numbers 1, 3, 5 dot dot set of even numbers of 2, 4, 6, 8.

So, now these are subsets of M so suppose I take this subset O the odd and the even you can have the odd numbers as 1, 3, 5, 7 so forth and the set of even numbers are 2, 4, 6, 8 and so forth. You might think that these are subset of N, both O and E these are popper subsets of N. Now you might say there they would should count in less numbered elements which looks very fair whereas, standard judgment because our minds are very finite, but they are not, there are as many odd numbers there as many natural numbers the same the number of even numbers is essentially same as the number of natural numbers. There, there is no such thing like even number is less and odd number is more there is some sort of cardinality, that if you call it infinite cardinality that cardinality they are same.

So, there are as many odd numbers as there are natural numbers, as many even numbers as there are natural numbers. So, for example, so I take the odd 1, 1, 3, 5, 7, 9 so on 11, 13. So, the natural number set so, how do I do it, 1 is related to 1, 3 is the second odd number, 5 is the third odd number, 7 is the 4th odd number, 9 is the fifth odd number and so on. So, here I have been able to set up a correspondence. So, once I setup this correspondence means you have done.

So, there as many odd numbers are there, the number odd there is as many odds as there are numbers. So, this is a strange way of behaving of the infinite world, now what about the Z set of integers of course, this is countable. So, how do you prove its countability? Next apply the Hilbert's hotel principle. So, simply you once you have in which means if n is always a countable so n union 0 is countable. So, this is a countable set. So, n union minus 1 is countable and so on.

This Hilbert's hotel principle will actually prove the countability all those things. So, Z is countable also. So, so z is the, so z is the, so the set of natural numbers is countable set of natural numbers is also countable set of integers also countable, now come to the set of rational numbers so we have spoken about rational numbers irrational numbers. So, will discuss irrational numbers in detail in the next class, but here we are talking about the countability. You might ask why you are bothering about a countability and uncountability and what great things that they would do, but this method of proving countability and un countability for example, has huge implications in computer science.

So, you you will learn later when we talk about turing machines and all those things that these methods are very very this we will soon come to a method which is extremely powerful. So, now, we are going to speak about the countability of the rationales, that rational numbers also countable, remember rational numbers are dense in the sense that if you take any real numbers and take a open interval around it then you will always get a rational numbers there. Same being a rational numbers, but still rational numbers are countable in the sense that we have just told that we can put that mean 1 to 1 correspondence with that of set of natural numbers. So, here is one of the major theorems of mathematics the whole mathematics.

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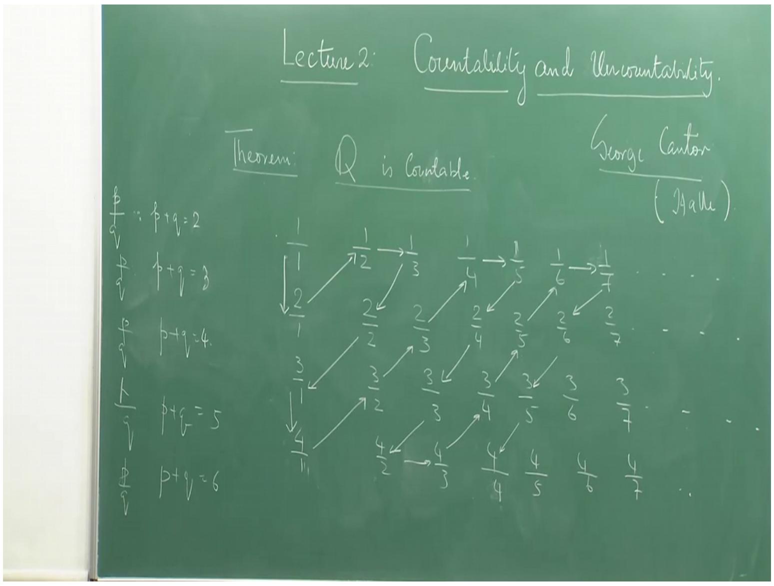

Actually Q is countable. So, how do you prove it, the proof is done by a simple diagram actually and this whole idea of proving was due to a man called Georg cantor, when we write this name Georg cantor or we just mention it, we hardly know the perils or the struggle this man had this great mathematician had.

So, when we tell that this thing was done by this person 1 might feel that he has a staring possibly coffin ccd or some cafe coffee day or while thought some place and he just wrote something with this sort of attitude we would like to take nowadays, but it is far from truth. Georg cantor a very brilliant mathematician whose supervisor was Leo prone Croneker, he could not get him a job in Berlin where is his place in Berlin.

So, because he could not get him a job in Berlin, he gave him a job in at that time a mo mid sight university in Halle and in there cantor for the first time in his life took infinity head on, he asked his question that I would assign some infinite cardinality to some symbol if the cardinality of these num, the set n in the set of natural numbers and he wrote this as the first letter of the Hebrew alphabet called aleph, aleph not this called aleph not. So, this is called aleph not. So, this and he said that any set which can be put in 1 to 1 correspondence with the set of rational numbers that is any countable set has the same cardinality. Now, this was a attempt to classify infinity itself a think which is Croneker never liked and so Croneker he and his whole gang had not allowed him to publish any of his ideas.

We will soon prove 1 of his greatest ideas that the set of all real numbers is not countable, you cannot put them in 1 to 1 correspondence with the real line with the set of natural numbers so; obviously, then infinity is categorized something is countable and something is not. This countability and un countability is a terms might look very straight simple English words, but in them lies a very deep factor what mathematics you are essentially classifying infinity. So, that is what Croneker never liked and as a result to which Croneker and his chelas I would say who would actually put Georg cantor into huge trouble. He was hugely mentally stressed to the extent that he could not publish anything in Germany, his friend from Sweden, Mittag Leffler actually arrowed publication in his journal called acta mathematic.

So, now to the I have been visited to Halleck many many time and seen Georg cantors handwriting he is writing to where sustrain to write to croneker to convince his teacher that no what is he thinking is correct only David Elbert later realized that what he did was great and then in the 1900 the second international commerce and mathematician Hilbert made this very famous statement, late no 1 evict us from the paradise which hair cantor has created for us.

So, the Georg cantor is to be treated so badly it also so mentally affected that that people would come when you take a class people would come and speak so badly with him that he would sometimes faint inside the class room. They would enter the class and speak in languages without come from berlins do this to him and go away and they were all fan followings of his teacher croneker, where ultimately cantors what remained to be the one which people use still it without these things mathematics would not be what mathematics is to be.

So, let us look at this counting business. So, I am, I am arranging the rationals, what I am doing is this so I am taking the number 1 then I am writing 1 by 1, 1 by 2, 1 by 3, 1 by 4, 1 by 5, 1 by 6, 1 by 7, 1 by whatever you want if you want, similarly I take the number 2, 2 by 1, 2 by 2, 2 by 3, you can say repeated 1 here 1 by 1 same as 2 by 2 that does not matter, but 2 by 3, 2 by 4, 2 by 5, 2 by 6, 2 by 7 and so on.

So, just let me write down 3 by 1, 3 by 2, 3 by 3 you see again 1 is repeated 3 by 4, 3 by 5, 3 by 6, 3 by 7, 4 by 1, 4 by 2, 4 by 3 so this is all dot dot dot dot that ultimately goes on 4 by 4, 1 is repeated, 2 is repeated here 4 by 5, 4 by 6, 4 by 7 and so on let me see how would I start during this job.

So, let me take the first number p by q. So, what you some this two, what is 1 plus 1 is 2 are there any p by q s where the sum of p plus q is 2 no. So, you will first start with this scenario. So, p by q which p plus q equal to 1 is this 1. So, start from here, now look for the cases p by q where p plus q is equal 2 sorry p by 2. So, this is 3 p by q is 3; obviously, we have taken 0 out, 0 is just like Hilbert hotel you just put use it no problem start with 1 and the next stoppage is 2 by 1. So, I come and take this as the next element and from there I start 1, 2 and then 1 by 2 also makes 3. So, I go here then I have to look for p by q this is the way the table is created p plus q equal to 4. So, what makes 4 1 plus 3 makes 4, I go there next count then I come down 2 plus 2 by 2 makes 4, 3 by 1 makes 4 and nothing else makes 4.

So, now I am again looking for p by q is equal to p plus q is equal to 5. So, what is the next number which is making 5, 4 by 1. So, 4 by I come to 4 by 1. So, after 4 by 1, 3 by 2 makes 5, 2 by 3 makes 5, 1 plus 4 makes 5. So, then again I do the job is 6, what number makes 6, 1 by 5 makes 6, 1 plus 5 is 6. So, you come here and what makes 6 2 by 4, 3 by 3 and 4 by 2. Then what number makes 7 4 plus 3 makes 7, 3 plus 4 makes 7, 2 by 5 makes 7, 6 plus 1 makes 7. Then is 8 what makes 8, 7 plus 1 makes 8, 2 by 6 makes 8, 3 by 5 makes 8, 3 4 plus 4 makes 8 and again 9 you just go up.

So, you see how simple this idea of counting is now nobody could come up with the pattern of course, does not matter if you count the a number twice basically, if you throw away a countable number of quantities on a countable set you have again a countable set right. So, if you throw away those you still have it. So, now, nobody could let come up even the early twentieth century away to evaluate the rationales. So, that I do not taking any rational which is not in the lowest form. So, every rational appears once so, this is the, that result is very resent this called the Calif Wilkm theorem which you will see in the notes, but I will not discuss it here it is a quite a recent result show how to do it using trees and all this thing.

Now, did I really prove q is countable, no by this diagram what I am proving is that the set of all positive rational numbers is countable. So, basically in some sense I have proved here. So, what about the negative rational numbers you have to put a negative in front of them. So, you take again the Hilbert's hotel principle, put 0 in and put minus 1 then put minus half keep on doing. So, each 1 of them would again be countable. So, the whole set of rational numbers q is countable once that is done I come to today's last and extremely important idea the idea is the following, that the set of real numbers unfortunately is not countable you cannot put the set of real numbers in 1 to 1 correspondence with the set of natural numbers.

So, this result was published by cantor in 1874.

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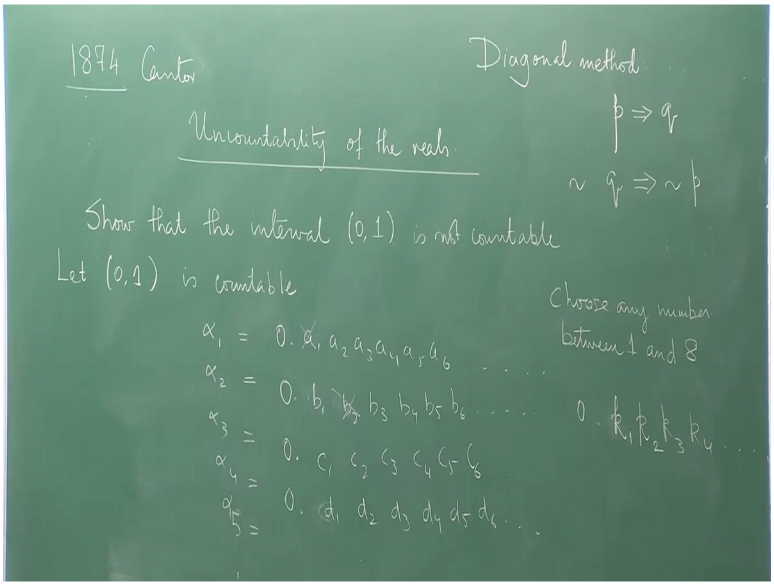

So, it is un countability of the rationales or of the irrationals un countability of the reals set of real numbers are un countable and in doing. So, he introduce the famous diagonal method of cantor, which will see in detail in the note. Now, if you take a countable set and take a subset of that countable set what do you expect that countable set to be what that, what do you expect from the subset to be, either it is finite or it is countable. So, if a subset of a given set is not countable the whole set cannot be countable, this is a simple fact of logic keep on using in a daily lives it is called very well actually a lot of mathematical proofs can be handled in this method. So, if I want to prove p implies q a statement p implies a statement q if I cannot prove it, then I should prove that it might be sometime easier to prove that the negation of q implies negation of p ,means not p implies not q.

So, if the thing is countable then its subset is also countable, a set is countable a subset of that is also countable say if the subset of set is not countable the set itself cannot be countable. So, what we are going to show which is a traditional thing in the literature is to show that there is countable show that the interval 0 1 is not countable. I think he first did not prove it with the diagonal method that we have going, but later he introduced it, but that is that is what it is it is one of the most beautiful proofs in mathematics, like many proofs in mathematics 1 of the most important tricks of the mathematician is to prove by contradiction that is you tell that your statement is wrong, whatever you want to prove is incorrect. So, you start with the incorrect hypothesis and then show that there is some contradiction. So, here we are assume that let 0 and 1 the set remember 0 1 means 0 and 1 are excluded is countable is sorry is no.

So, we are starting with this assumption we are assuming that 0 and 1 are countable see 0 1 and what this interval countable what does it mean. So, let us list the numbers. So, they are countable means I am able to match them with the set of natural numbers. So, I am matching them with say alpha 1, alpha 2, alpha 3, alpha 4, alpha 5 and so on. So, they all match with the set of natural numbers, they must be because they are countable. So, you can put a 1 to 1 correspondence between that set and the set of natural numbers and hence now these are all numbers between 0 and 1. So, that decimal expression would be 0 point something that we have learnt about the decimal extraction and so you understand and that also very basic school thing, which you were used to sometime you have to get used to things in mathematics not thinking too much about it.

So, here I just show I want to tell that you scale sometime you may classes that once a famous twentieth century great mathematician the star mathematician john von Neumann, who is also responsible for many thing responsible for many things for the development of modern computer and modern economies the game theory. So, he was once asked by a 1 of his 1 student could not solve a problem in US actually he could not he could not solve a problem in physics he was trying to solve a differential equation. So, his supervisor told him why do not you go and talk to von Neumann professor von

Neumann. So, he could finally, catch professor von Neumann in a party dancing with a beautiful woman and he goes and asks. So, what shall I do with this problem he said use a method of characteristics. So, he went and found out what is the method of characteristics used it and solved his problem.

But then next he calls up 1 Neumann and tells him that, but I could not really understand what is the method of this characteristics. So, he said that sometimes in mathematics you do not understand things you just get use to them. So, decimal expansion is something we do not many people might not really understand why it is, you just get him stood. So, so here I can start with the decimal expansion 0 point a 1, a 2, a 3, a 4, a 5, a 6 so these are numbers basically and you know that decimal expansion can be either infinite or if that are terminating then they are they could, either infinite or periodical or either what they are terminating basically.

So, sorry b 1, b 2, b 3 so what thought things we just terminated we can write, 0, 0, 0, 0, 0 as inspair in places where some terminated b 4 0 0 0 whatever so just I am doing d 2. Cantor data very novel thing he said come to a 1, choose any number between 0 and sorry choose any number between 1 and 8 choose any number between 1 and 8, 8 that is not equal to a 1 right. So, you put down that number whatever it is, say some some k 1 0 point some k 1. So, this goes now you come to b 2, those same thing choose any number between 1 and 8 and we which is different from b 2 and put it there and so you for the ex rest do for this do for that so and so forth. Now mike you might ask why are you asking to choose so 1 and 8 what is the possibility why 0 and 9, what suppose you take 0 0, 0, 0 suppose none of this are 0 and you put 0, 0, 0, 0 then you will get 0 and 0 and 1 0 is not in that set.

So, basically you cannot say anything 0 does not belongs to that set so, but if you say take 9. So, you have point 9, 9, 9, 9, 9 if you do that you might just get 1, you would say take 4 and then and this is 4 and 9, 9, 9, 9, 9, 9, 9 you will get point 5 which already there. So, that is sort of things. So, we avoid generating a number here which is already present there right. So, now, you might ask me what about 1 third, 1 third is what point 0.3333 or suppose I start taking 3, 3, 3 but the problem is you are just going down essentially counting down because you so if you take. So, when you are essentially counting down. So, if you have 0. 33333333 you cannot do that because what we do we have expressed everything there are an because now these are all countable. So, we have expressed everything in the form of decimal. So, 1 third here is also a expressed in the form of a decimal. So, when you come to 1 third you have 3, 3, 3, 3, 3, 3 immediately you will cancel 1 3 and put something else

So, that number cannot be repeated. So, what you have done. So, this number cannot be the first number alpha 1 because, because is not first number is not same as a 1 it cannot be the second number because the second number is not same as b 2 k 2 is not same as b 2 it cannot be the third 1 because this third number k 3 is not same as c 3, forth 1 because it is not a same as d d 4 and so and so forth. So, finally, we have created a number while looking at the decimal expansion it is a number between 0 and 1, but it is not in the list of numbers which are in 1 to 1 correspondence with the set of natural numbers.

So, which means that I have not being able to put all the real numbers lying between 0 and 1 in 1 to 1 correspondence with a set of natural numbers and thus proving 1 of the greatest conclusion in modern mathematics that the set of numbers real numbers between 0 and 1 is not countable and with this great result call called the cantors diagonal methods I would like to end the talk. There will be some more stuff in the notes which you can see when it will be uploaded.

So, in the next class we are going to talk more about irrational numbers. So, bit amount of fun there you can play a little bit of with polynomials suffers up proving certain irrationality of some numbers you have to use actually, your knowledge about polynomials that is what we will see in the next class. So, so the whole idea about numbers would end there and then we will go into doing the actual calculus functions limits and all these things.

Thank you very much.