Calculus of One Real Variable
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Lecture - 19
Integration-II : In the spirit of Newton and Leibnitz

In the last class we were telling that what is this issue of actually computing the interval and I told you that this computation is not simple even for a very simple situations of non-negative functions because, I told you about this bounce that we given in terms of the areas. The upper estimates of the lower estimate could we use that to do something. So, I am now asking would we evaluate the integral.

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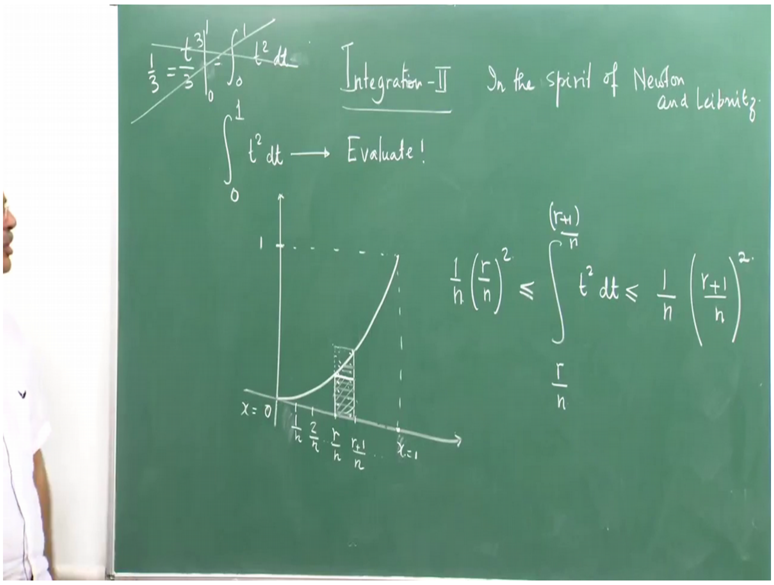

It could be x square d x. So, so do not get too much worried about this d t symbol. Come here, it is a symbol; actually we tend to give meaning to this. It is basically, telling a continuous sum of the function values. The length function values in to very small base and it is infinite decimal, very small base functional value does not change. So, that is the length of the rectangular, that you construct with the base, the d t. So, that is essentially the meaning as a sum in that sense.

So, do not get, do not put too much, you might thinking about it d t sorry what is d t and all sort of things. So, now, I want to evaluate this anybody who has studied for entrance exam would say; oh this guy is crazy I; obviously, know the answer it is simply t cube by

3 [FL] sorry to speak in Hindi, but in generally it would be how what is the big issue about it. Why make such a fuss.

Please understand that we have not yet introduced an ant derivative. We have not yet introduced many other things. We are in a privative state of affairs where we are simply telling. We are possibly in that very old classical set of simply telling that this is the number, this is an area, this is just a symbol of that number, how will you evaluate such things. I want to again assure you and I want to tell you that this was not done overnight, it needed a lot of work and proper estimation in the lower estimation business works and that is actually used.

Now, let me tell you that if this way of thinking that you, we are taking the historical steps because many big mathematicians as I told you (Refer Time: 03:18) and all these people they have actually given a lot of time to evaluate such things and then actually generalized it into an formula.

So, things have not come out overnight; please understand this very simple thing. So, once if you have table of formula, it is very easy to write the formulas, put it down, write solve problems in the exam, without understanding of the real meaning and real issues at hand. So, that is what this course is suppose to do. It tell you what calculus actually is; what are the real issues at hand. So, here I am. So, it is I am looking at the part of the graph y equal to x square. So, this is 1 say and this is length 1. So, this is 0 and this is 1. So, this is y equal to x square.

Now, we want to actually, now divide I have not divide, I will divide this 0, 1 interval each of length 1 by n here b is 1 and a is 0. So, it is length step length h is 1 by n. So, the first point here, where we cut is 1 by n then 2 by n and so, in the middle, I will have r by n and r plus 1 by n and go on. Now, just I will concentrate on this part and look in to this area what does this area mean for this function it means that we are essentially, if I write it in this form of symbolic the integral, it is r by n to r plus 1 by n t square d t, but let me now construct the upper estimate and the lower estimate of this area; the upper estimate means I construct a rectangle of this height.

So, which over estimate it. So, the rectangle of this height is so, it will be what it will be. In this 1 by n is the base and then this is what this is the length is nothing but the functional value. So, the maximum is achieved at r plus1. So, the maximum achieved is r plus 1 and the functional value is r plus 1 by n. So, it is r plus 1 by n whole square.

Similarly, here on the other side it could be 1 by n into r by n whole square, because here now the area we will be looking in to this rectangle this will be 1 is the upper 1 and 1 is the lower 1 lower 1 is that this 1. So, here the lowest value of the function on this range is r by n and this is the increasing function that you that you clearly observe. So, there was a little mistake I was just, I have speaking very fast, I just forgot to give the limits. So, this is how along to you (Refer Time: 07:28) most of the cases, but just forget about this do not think you were taught limit, having limits.

So, these are just in the situation where you are doing the real Math. You are actually seeing this estimation and trying to understand how to evaluate this interval. So, if I now sum it, if I sum this where r varies from you know 0 to n minus 1 in one case and. So, the r varies from 0 to n minus 1, then if I sum these up this part up this small parts they will give me the whole curve from 0 to 1.

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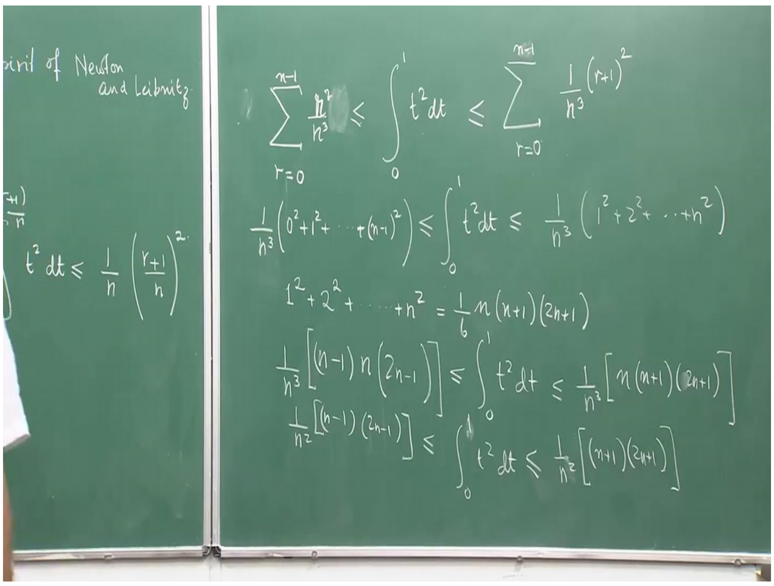

So, basically I will have summation r equal to 0 to n minus 1 1 by sorry 1 by n cube sorry r square by n cube r square by n cube and this is less than equal to, but if I sum these up this will be nothing but the whole area. So, it is a integral . So, it will be 1 by n cube. So, this is the summing.

So, n is not the issue it is n is fixed, you have varying. So, basically you have 1 by n cube to 0 square plus 1 square plus n minus 1 square and this is less than equal to 0 to 1 t square d t and that is and this is give you 1 by n cube 1 square plus 2 square plus n square. Now, here you have to use a very important formula which is taught at the high school level, but you see in those days it was not an easy thing to actually figure them out, this was research mathematics so many of you would possibly know. So, you see the amount of stuffs that is we used here. So, this is 1 by 6th of n n plus 1 2 n plus 1. So, here if I write it down, it will become 1 by n cube into I am writing for this part the left part. So, it is n minus 1 into n and 2 n minus 1, this is less than the integral 1 by n cube into n into n plus 1 by 2 n plus 1.

Now, what I will do is a simple trick I will divide this n by n cube by n square by n shall I, 1 by n square in the denominator and then I will split 1 by n square in to 1 by n into 1 by n and then divide 1 by this and 1 by n with this and 1 by n with this and 1 by n with this. So, I will have 1 by n square n minus 1 into 2 n minus 1, this is less than equal to 0 to t sorry 0 to 1 t square d t 1 by n 1 by n square into n plus 1 into 2 n plus 1. Now, I will as I told you, I will split it up and hence let us come to this side of the expression. So, now, here we are coming again from here; so from here, if I just have a hash tag.

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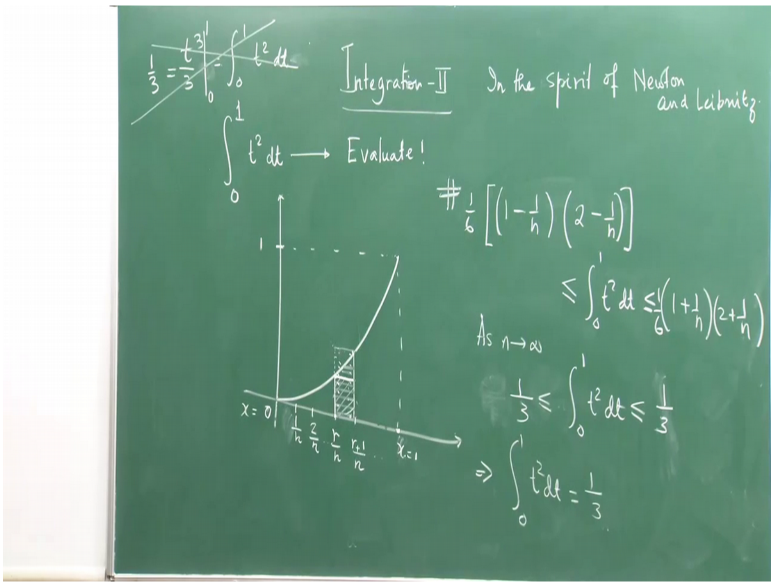

This language is understood by everyone hash tag. So, here I will have we divide by n 1 minus 1 by n if I divide by 1 n this part then this part and two minus 1 by n this whole thing is less than 0 to 1 t square d t and this itself again is less than 1 by 1 plus n and 2 by 1 plus n.

So, 1 by 1 plus n and 2 by 1 plus n. now, you take the limits as n tends to infinity. So, what we have missed is some 1 by 6th here. So, 1 by 6th has been missing. So, I have to put 1 by sixth here 1 by 6; so just applying the formula 1 by sixth. So, here also it will 1 by 6 here also I will have 1 by 6th. So, now, if I take the limit as n tends to infinity this will go to 0 1 by n and this will go to 0 1 by n things. So, it will be 2 by 6. So, 2 by 6 which is 1 third is less than equal to 0 to 1 this is independent of that n is less than equal to sorry less than equal to here also x 2 by 6 which is 1 third.

So, which means what this means 0 to 1 t square d t is equal to 1 third now, I will tell you that I will give you another integral I will show you another example of geometrical help and without any thing. I will do something interesting without much help of any of these sort of working and you will see the number pi again appears in front of you I immediately write a more complicated expression. I ask you to evaluate the integral what does this give me what does this 1 third tell me does 1 third tell me anything is there some information in there nothing it just tells me that this is the area number.

So, here we are just figuring out the numbers. So, integration is as the symbol which is actually a number right. So, I ask you to directly a much more complicated not simple function

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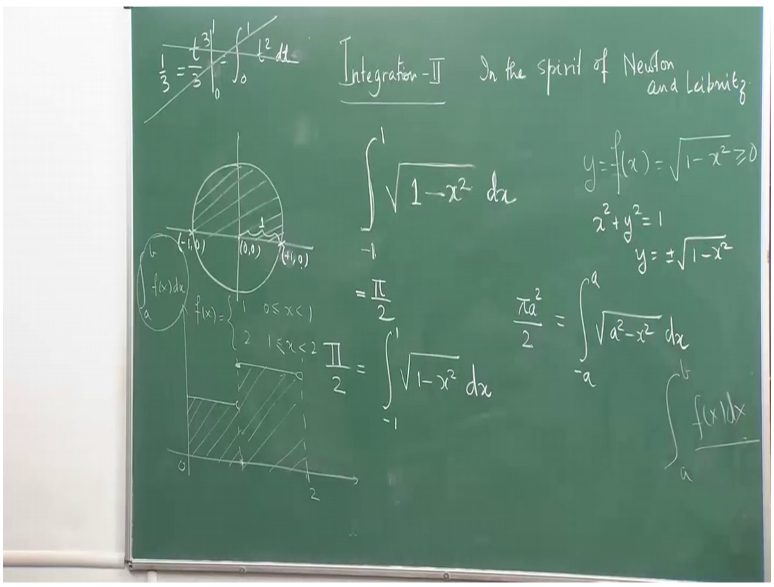

You say oh my god; what is this certainly you have, that is a change you have to observe that this function f x if I keeping it if I am keeping x between minus 1 and plus 1 this function is always non negative and so whatever we have learnt can be applied here, but do I need all these strategies to evaluate it the answers simply is that you do not need any of these strategies that two we have just learnt to you do not need any of these strategies to do it here you see. I always want to emphasize it might be my view or that is the way I can understand things that mathematics cannot be learnt well without visualization mathematics is all about visualization geometry is that the core of things. So, here what does this symbols if I have this functions y equal to root over 1 minus x square this simply means x square plus y square is equal to 1.

So, it is telling me to find the area. So, this is the circle, but I am only looking x between minus 1 and plus 1. So, between minus 1 and plus 1. So, y can at can have plus minus this and I am only looking at the positive part instead of if i. So, here I will have y from here y plus minus root over 1 minus x square I i am only having the positive part because I am not put the minus sign. So, we just put the positive part of 1 minus x square; obviously, has to be nonnegative otherwise you cannot evaluate the integral sorry cannot be evaluate the square root.

So, what does it mean what is this integral telling me it is telling me that this is nothing but half of the area of the circle and we have just learnt today that any of the circle pi is a constant we know is. So, this is nothing but if so this is plus 1. So, this is x is. So, this is the point plus 1 0 and this is the point minus 1 0. So, this is with the center 0 0. So, this is the circle of radius 1 that is from this equation circle of radius 1. So, it is area is pi r square it is pi, but I am only looking at the half part it is symmetrical because I am just evaluating the positive part. So, this is nothing but pi by 2. So, pi by two is minus 1 to 1 root over 1 minus x square d x see such a complicated integral looking integral and we immediately figure out.

Similarly, if I for example, if I take minus a to a root over a square minus x square d x it is the same thing now the radius has become a. So, it becomes pi a square by 2. So, here is a complicated functions are immediately integral is evaluated once you know some geometry. Now, I can use this thing to evaluate the area under an ellipse. So, you see we are getting more into more complicated stuff without much effort once the geometrical view is clear see all this things that you have written they are all geometrical views and I later on this is a geometrical view coupled with a limiting process and that is a allowing me to evaluate the areas .

For example if you have an ellipse.

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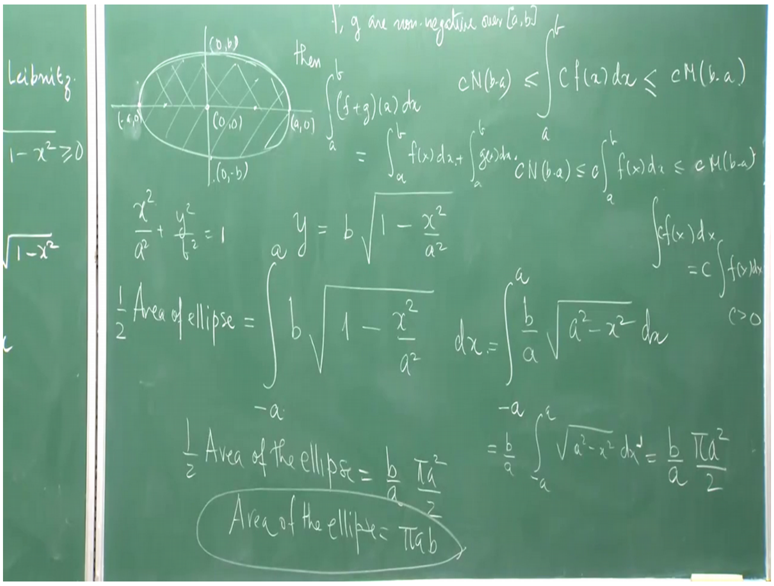

Sorry. So, these are the two foci right focus of the ellipse and let me write the answer. So, this is a 0 and minus a 0 and this is b 0 sorry 0 b and 0 minus b. So, these are the major axis and minor axis and the question is x square by a square plus y square by b square is equal to 1, but I want to evaluate this area, but what we are experting not only we are experting this, fact that we know the structure of the equation I know what sort of curve it is giving basically now I want to find that this area. So, what sort of curve I may what sort of things I am experting the key feature that I am experting here is symmetry of geometrical objects and because of that I can simply write this is half of that it is the symmetry of the geometrical objects is playing a very important role and symmetry is a really very important concept in the natural sciences.

So, here what I do. So, I will just take this part an upper part of ellipse I wouldnt bother about the next bottom part I i just know a facts about nonnegative functions the whole notion of integral is valid for me when the function is nonnegative. So, I will look at y is equal to b into root over 1 minus x square by a square. So, basically now I am integrating from minus a to plus a b in to root over 1 minus x square by a square d x. So, this so, area of ellipse this is my area of the ellipse and what is this; this is nothing but integral minus a to plus a b by a into root over a square minus x square d x of course, and again you know now we have we do not know what would happen if there is a constant in front of the function that is what would happen; how do I evaluate this. See, the best intuitive to look at it is in this way how will you see there of course, we will do more details about all these, but way to figure this out is the following.

So, you know if you move the function you scale the function by c then your maximum value and your minimum value is also scaled up by c. So, it will have say c times m into b minus a and c times m into b minus a. now, you already know that n times b minus a is less than equal to integral a to be f x d x c times it is capital M times b minus a. So, if I multiply by a positive number c. So, I am multiplying by a positive number here b and a are positive numbers. So, if I multiply by a positive number what is happening you are getting this equation. So, once you have this you know that these two equations look same. So, possibly and intuitively and it is correct that you can immediately conclude that f c of f of x d x is equal to c time integral f x d x since c is positive.

So, this is simply an intuitive idea it is not like a I have not done any rigorous prove the rigorous proof comes later on this is just an intuitive idea. So, once this intuitive idea is known was this intuitive idea is known what am I suppose to do. So, I know that that is what I have. So, I can simply write his to be b by a minus a to a root over sorry area of ellipse is not this. So, area of the ellipse is this half of the area of ellipse. We are just considering this part. So, we are just finding first area of this part then you make it half of the area of the ellipse is what we are getting. So, please note this sorry. So, this is nothing but minus a to a a square minus x square d x and that is what we have already evaluated pi a square by 2. So, the answer here is b by a in to pi a square by 2. So, half of the area of the ellipse is equal to b by a into pi a square by 2.

So, a cancels because these are positive numbers area of the ellipse is simply pi a b see we have done everything without any effort and this is what where a beauty of mathematics lies at lot of things if you viewed in the right way can be done without effort thus, 3 minutes to this lecture and. So, I will not start doing anything new, but I will ask you to go back home look at the definitions that I have given and prove something which is interesting as Koerner mentions that there is something about area what is what is that about areas which the Greeks did not know, but that thing we know about integrals and that is the following that if f and g are nonnegative functions over a b nonnegative over a b then integral a to b f plus g of x dx that is f x plus g x basically that is same as separation separate evaluating that two integrals and adding that.

So, I would like you to prove this is something is immediately interesting with the Greeks did not knew about the areas if you add these two areas under two curves, the new area that is found is actually area under the cu some of the curves or some of the functions the curve that is generated by summing the functions area under that is area this sum of this two component 1 and that is the key thing that the which the Greeks had no idea about and so it tells me much more about areas that this this is a big statement about areas just not just some [FL] just that I have written a sum and forget about everything it is. So, I would like you to go back home look at the limiting definition and can you use the limiting definition which I had given earlier a plus n each that form and show that this is possible with those will come in the next lecture.

So, tomorrows lecture we will talk about how would I define an integral when f need not be positive need not be nonnegative when f could be anything what would be the answer how would I do it we shall do that and many more and we shall get into this whole idea antiderivative through the notion of what is called a the fundamental theorem. So, this there will be 3rd version of this talk. So, here we are doing things in a slow manner, but you see how things are getting evolved that once you know what it means that what area means a lot of things can be achieved that if the integral actually represent the area lot of complicated looking integrals can actually be resolved.

So, we end this here today and after tomorrows class in the next week lecture, next I think the 5th week, we will start talking about Riemann integral, Riemann for example, 1 of the greatest mathematician of the 19th century famous for introducing Riemann geometry, I will tell you story of a bit because he also talks about the Riemann hypothesis, which might not be understood by most of you, but some stories can be told he thought about this particular case. Suppose, I have define a function say from 0 to 2 and some 0 to 1 it is 1. So, between at 2 at 1, it takes it does not take any value, here at 1 it takes the value 2 and then at 2 it does not really take any value. So, here the f x is equal to 1 when x is this and is equal to 2, when x is within 1 and 2.

Now, does this have an area under this is the discontinuous function. So, you have already seen some discontinuous functions does this have an area you can say this is this actually this having area this if we say that this is has area we are actually making a very big statement and those who actually if somebody is watching this who has actually advanced math training would really know that we are talking about that if there are some. So, called setup measure 0 or few points at which the function is discontinuous you can just disregard it this function is the bounded function it is not an arbitrary 1. So, values are bounded now you can say does not matter why we bother about this 1 point; what is the problem, take this area, this rectangle and take this rectangle and sum them up that define that 2 area.

So, if this is the function how do I then evaluate what you, what meaning then I give to this because till now we have been talking about nice continuous functions we are not talking about anything else, but now we are talking about something else we are talking about functions which could be discontinuous and that will start talking in the 5th week and this will give this story would be the story of Riemann integration basically, it is a oh Riemann m a, I am forgetting the spelling it must be like this Riemann integration and Riemann by the way the this Riemann integration came in his thesis.

So, he has he thought that he concurred Newton's game by showing that most of the functions which Newton could not do anything about or Newton (Refer Time: 32:53) I have aware to integrate them, because give he was only talking about bounded functions and every continuous functions are bounded and so but (Refer Time: 33:02) bounded function need not be continuous. Here is straight an example in front of you and these are. So, I have won up Newton then his PhD supervisor Gustav Diri Dirichlet gave him a small function when f x is equal to 1 when x is rational and f x is equal to 0 when x is irrational it is before his PhD examination and told him just see whether this is integral you can integrate i was not integrated though it was a bounded function it was not integrable in the sense of Riemann.

So, the story of integration does not stop with Riemann it goes on goes on to we finally, in 1902 , who a man French man called envy Lebesgue Henri Lebesgue. I would say for the French would call Ery Lebesgue by they think h and r are silent I are silent. So, this man lebesgue (Refer Time: 34:19) in 1902 write a thesis where he ques where he looks into the difficulties of Riemann integral and devises a new way of looking at computing integrals.

He wants to integrate functions or give meaning to the area under the curve of functions which are not even having area under the Riemanns sense then. So, you see the whole notion of area is getting changed when a whole what do we mean by that. So, how for various types of functions now what he did Lebesgue. So, this story has to also Riemann though I would not be teaching you that he did a completely different thing in his thesis he said these guys Riemann newton all these fellows are actually partitioning the x axis I will partition the y axis can then I do something improvement he just changed it and that is exactly what he does.

So, that is story I will tell at the end of this story. So, what they are few integraval intervals which (Refer Time: 35:07) integral, but most of the integrals are all are leveled integravaal and he is till now the official integral of the mathematics, but there is some unknown not. So, very famous mathematician called Kurzweil and Henstock, who did a little bit of thinking that could. I improve the Riemann integral in such a way that those functions which were not Riemann integral in the sense of Riemann and now become integral in the sense of Riemann that is called the generalized integral or the Henstock integral or Henstock Kurzweil integral.

So, where I will tell this fact a bit when we talk about Riemann integration where those functions like the 1 which Dirichlet give it was not Riemann integral becomes Riemann integral in this generalize sense, but unfortunately for some reason or the other because these all these are question of tradition in mathematics that Lebesgue has entered into the Plastivan mathematician in such a way that is very difficult to get that him out and accept the generalized Riemann integral as 1 of the top 1s.

So, it is good to know some story about things. So, with this whole idea of integration which you might or just finding area which might look very triviar to many people outside mathematics is actually very deep and very interesting thing.

So, I hope you enjoyed the lecture and next time we will start talking about other things, but the Newtonian spirit could be maintained and till the next week when we will come to Riemann’s famous integration idea.

Thank you very much.