**Calculus of One Real Variable**
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**Lecture – 18**
**Integration-I: In the style of Newton and Leibnitz**

So, today we are going to start one of the other important aspect of calculus integration many of you know possibly; it is a reverse of differentiation and blah, blah, blah stuff, but integration as the way we know; it had started not exactly in the way we think taking the anti-derivative and this and that it let us go back and think in the old classical style in the time of when as if we are in the time of Newton and Leibnitz. So, Newton and Leibnitz are I do not have to; obviously, tell you who Isaac Newton is and Leibnitz is also German mathematician and philosopher who actually had developed this science d y d x; this use of this signs of calculus; I think he started calling them derivatives rather than flections which were called by Newton and there was a lot of fight between who is the true inventor of the calculus and Newton make made sure that the royal society guys gives him the credit.

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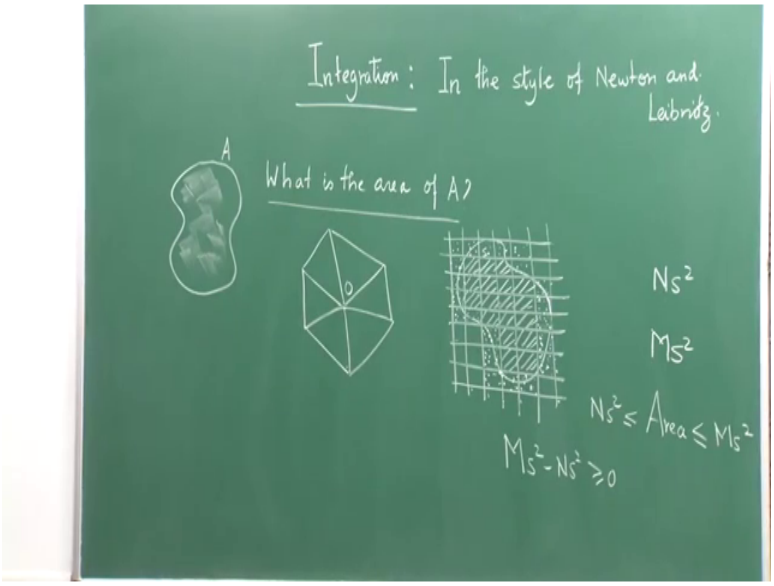


So, whoever it is. So, we are going to talk about what integration is integration actually comes from very old geometrical idea which was known to the ancient Greeks. So, if I take a closed body like this and I ask what is the area. So, if I call this body as A. So, I ask what is the area of A I ask; now what is the area of A. So, now, with if I ask this question to many of you; you would say or something I; this is you get try to get something; something inside something like that, but truly speaking area is a very intuitive notion and it is it is not so easily definable. So, what we want to say by the term area is that we are trying to give a measure to the space enclosed by this curve.

And we had the Greeks had defined areas of certain objects and with those elementary objects; we can talk about areas of much more complicated objects. For example, if I look at this area; if you take a polygon. So, how do I know the area of a polygon take a point here join all the sides with this point is the center of the polygon and then you know the area of triangle you compute the area of the triangles and you know the add those areas and say this is the area of the whole polygon.

Now what would happen if a not such a regular body is given to you; what are you supposed to do if I get a body like this; the best thing is that if you take a transparent body on a paper actually put a graph sheet over it a grid. So, essentially; for example, you put a grid around it, well each grid has a same length and breadth. So, it is square just drawing it; it may not be very clear.

So, what I would like to do? I would first like to look at these this squares which are all inside they are completely inside the body and we know what is this suppose say; they are each of side S then this is S square. So, if see if each of them is of side S; these squares, then if there are N such things here then you have N S square as the area that I have just covered.

But then I also want to take some rectangles in which the body is contained, but of course, I will not take a huge one maybe the nearby ones. So, let me take over take those rectangles which are just slightly outside the body that the body is contained in it. So, maybe I will mark these rectangles these are the dots. So, here to they are actually; this is how mathematics is developed and this is the way it should be learnt. So, this is called as this or for example, this one the slightly outside the body. So, we are not taking this body which is a slightly outside the body. So, we had not taken it inside this is completely up. So, we are basically covering these rectangles which overlap with the body will not cover any rectangles which are away from the body which do not overlap with the body. So, only those which are have an overlapped with the body if for example, here it there is only one say one point which is passing or little bit. So, we will take this one, but we do not take this one because it does not overlap with the body.

Now, or neither we take this one because it does not overlap with the body now the dotted region suppose that M such rectangle. So, the dotted region; so, the dotted region has those rectangles which are overlapping. So, now, let M be the total number of blocks that is both the dotted region and the region which is already inside. So, I consider those which is already inside and those which is inside plus those which are overlapping. So, M N is the number of total blocks which are intersecting with the grids when it square blocks which are intersecting with the body. So, N is the number which is those blocks which are completely inside the body. So, this has A area M S square S square is the area of one square.

So, the area of the body is bounded by these 2 quantities and then I can make this area I can choose my S in such a way you know that this would be; obviously, bigger I can make them A I can choose this S in such a way that of both sides these this would becomes smaller and this would becomes larger the more more rectangles would come inside the body and hence finally, they would converge to in the limit to the area that is essentially is a concept.

By the way this concept is what Archimedes used to find the area of the circle and hence also the area of yeah also the number pi, right, we are not going to maybe we can discuss a bit about that if you wish. So, let me just this is a good thing because it is not always; there is not always chance that you speak about how you talk about the area of a circle. So, this is how Archimedes did it. So, this is one method which I take from the book by Korner Bulk helpless for the ambitious Korner’s approach here.

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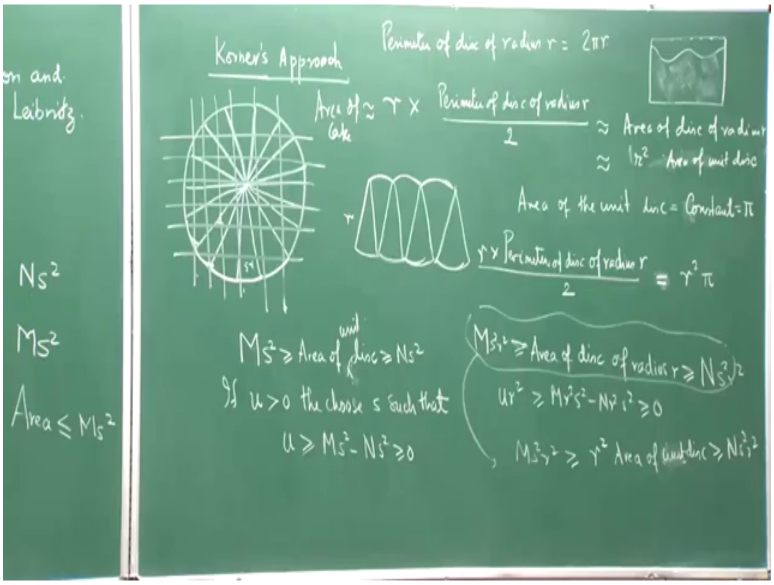


So, this is an approach which I have not seen in any text; I will speak about the Archimedes approach later on. So, you might ask me what why are you do intending to do all these things because that will give you a fair idea of how I am developing the idea of the calculus how developing the idea of integration where integration is a term where we are trying to find a area under a curve and by and bounded on 3 sides by a rectangle it is like you know something like this; it is as if you know the water is flowing and in this section you know water is flowing the river and you are looking through some mirror and the water is flowing by u and then how much water is there that is passing by you. So, these are the questions while integration helps you in answering such questions now what is Korner's approach; Korner's approach for computing areas of the circle is that you take a circle is that you take a circle my drawings may not be all correct due to full looking, but what I do.

So, this is say of some radius R does not matter radius 1 also does not matter even radius 1 does not if any take it to be radius 1. So, what do we do we cut into slices which are basically gives a cake cutting thing. So, you cut it off into slices I am not going to waste my time completely cutting them off. So, if I cut them off into slices; then I can actually write down the slices like this I can arrange the slices like this basically arranging the cakes on the plate and so on.

So, there the then all these slices here now how do I use an approximate idea see if this slices are very small that is so, let me. So, what I can do actually what I can do duplicate the circle as an body which is closed by a curve I can simply do the same thing I can write the grids and do the same story. So, if I put grids over it. So, first I cut it into this and arrange it like this and then I put grids in the same way and I do the same thing with the size S.

I do the same thing. So, I put the grid and then you see what happens is that I can simply say that again in the same way M S square is bigger than the area of the disc area of the discs or circular disc and this is greater than M N S square, but I can choose S in such a way that if you give me any number u, right, any positive number u if you take any u greater than 0 then choose S such that then choose S such that what would happen that u is controlling the growth of this difference u is controlling the growth of this difference u could be greater than or equal to this. So, this u is now controlling the growth for this difference cannot be if I keep on choosing the S; I can make u smaller and smaller and smaller, but that is I can push u towards 0. So, this difference should you know smaller and smaller you can make u smaller and smaller. So, that these difference the difference between these 2 becomes smaller and smaller and finally, converts to the area that is that is A area.

So, now if I put a magnifying glass which magnifies make this bigger. So, instead of radius one I have radius r. So, I have a sphere of radius R. So, what is; so, the area of the sides or the squares now gets magnified by an amount R. So, everything is dilated by an amount R. So, now, area of the squares the sides would now become R S if it is dilated if it is dilated; I will dilate by an amount R the squares would become R S.

So, what would happen area of disc of radius R; now is bigger than the same you could have the same M and N only the dilution has dilation has taken place this N now it is S R is the size. So, S R into S R S square R square. So, this inequality continues to hold right and of course, from here you will find that u R square. So, what you have done. So, area of the R radius; what is this basically multiplied by R everything. So, I can write this statement this particular statement this one this line can be now written as M S square R square is bigger than R square times area of disc of radius 1. So, these are disc of radius 1. So, that the disc of radius 1 is also called a unit disc. So, area R of unit disc which is greater than N S square R square. So, this is what I will have.

So, what we will finally, conclude; form here, we will finally conclude form here is that the area of disc of radius are is equal to R square; the area of unit disc that is what we have concluded from our discussions here now how does it help.

Now if you look at this thing this some sort of this cake cakes being put on the table if you look it from the top that cake should look like that and it can be viewed as some rectangle very rough it can be viewed as some approximation of a rectangle not really, but let us just view it as some sort of an approximation of the rectangle and then what do I have. So, this is the length R and what is this; what is this part. So, this part comes from this part of the circle it is the bulge that you see comes from this part of the circle; this sides of the cake. So, if I look at this thing. So, what do I have these are nothing, but parts of the circumference of the circle.

So, this length if I had laid down all these cakes side by side. So, this side is nothing, but roughly it is perimeter half of the perimeter of the circle. So, this grid this whole thing R into perimeter of disc of radius R is by 2. So, this is this thing or what is this; this is nothing, but the same as the area of this circle are of the disc because we have cut out the cakes and laid them down. So, they must be the same give me the same area roughly. So, I am just writing rough, but this is just an approximation please do not. So, it is really not our rectangle. So, if there is a roughness. So, it is approximately this it should be like this. So, area of this area of this part; I can call this as area of cakes say area of cake is roughly this, but this is nothing, but the area of is the product of the sides of R. So, it is this is roughly the area of disc of radius R.

Now, this is nothing, but R square into radius into sorry area of unit disc; now area of unit disc is a fixed quantity; it is a constant quantity and let us call that constant quantity as pi now. So, area of the unit disc; now R square into area of the unit disc is roughly this, right. So, perimeter now let us. So, area of the unit disc is constant and let us call it constant pi of course, you can ask me how this pi will tell u a bit area of the unit disc is pi. So, some constant right; it is just a constant. So, let us call it constant; just let call it constant if you want to and let I am just writing this as pi because you already know something about it that it is a constant number right so; obviously, it cannot be changed.

So, what does it mean it shows that perimeter of disc of radius R radius R into R by 2 is equal to. So, as you will do better and better approximation in the limit this gets better and better because as you make these things smaller and smaller the approximations become better and better. So, this approximation finally, let us see becomes equal these are all just approximation argument by the way and this is nothing, but that R square into pi constant; now what does it mean from here; it simply says that the perimeter of this curve radius R; R is equal to or the circumference of the disk of radius R perimeter is same as the circumference is 2 pi R.

So, which means the perimeter of the disc divided by the diameter of the perimeter of the disc of radius are divided by the diameter is always constant. So, whatever be your radius R; it does not matter the ratio of the circumference by diameter is always constant and this is done because you have some idea if you are using the idea of area in just this particular way; it is the area idea that the way you. So, you just you are adding you know about the area square and once you have defined that you can do a huge amount of thing and that is exactly what we are going to do with functions and so, the question came when calculus was developed that you know sometimes curves part of it for example, can be defined as some function a contiguous function, then how do I go about finding that area under a curve of a function which takes non-negative values between 2 points. So, that is exactly what we are going to now discuss.

It is the same story that we want to write a several ways to look at it and let me tell you what are the ways; how are you see it is the same idea that will go through you can put the squares, but I will not put squares; we will just deviate a little bit more comfortable.

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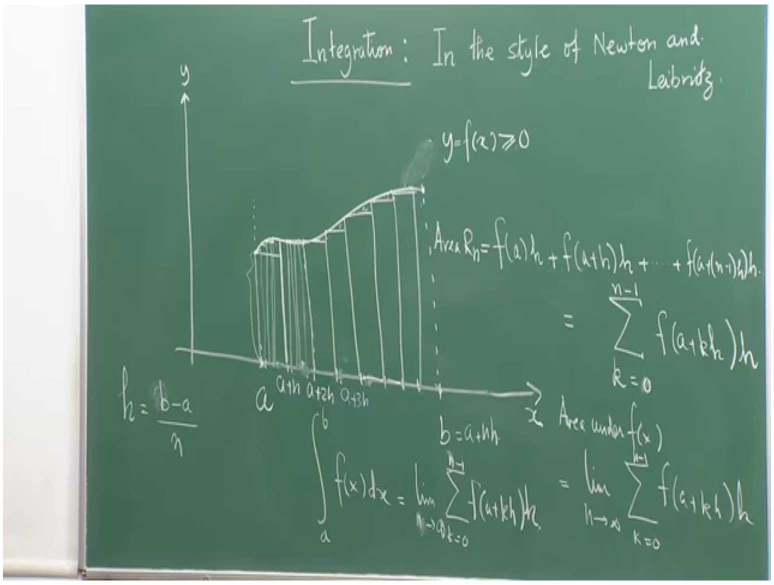


So, here is my x axis and y axis and here is a here is b. So, here is a line x equal to a and line x equal to b. So, let this be a curve a nice continuous curve in those days these sort of curves were what people were using; they were not talking about this continuities and all these things by the way a huge amount of mathematicians actually devoted their lives to this. So, we cannot just in one session really finish what the integral is. So, we actually split it into 2 sessions the Newtonian style in 2 lectures maybe one more does not matter and then we will go to the next step which was given by (Refer Time: 25:05) in the 1800s. So, this is what was going on in the 1600s; 400 years ago that is amazing well.

Not 400; 350 years ago (Refer Time: 25:17); the beauty of mathematics is that it endeavors, it is permanent, theories in physics change, theories in chemistry change, but in mathematics things endeavor and that is the beauty actually. Now how do I (Refer Time 25:34) draw rectangles it might not be always good idea to have rectangles, but then we are not been able to use the function value. So, we have to make use of this function value that is known to me. So, this is the graph of the function y equal to f x what I will do; I will partition this into say equal parts a plus 1 by n. So, I will take a nice division which is called b minus a by n. So, I will make equal divisions of this. So, a plus h a plus 2 h a plus 0 h is this a plus 3 h and so and so forth and this is b is a plus n h.

So, what I do is I draw rectangles I measure being this should be equal [FL] about this and this whatever; please forgive me for unable to write 2 things equally, but does not matter the idea the idea is a main thing. So, I draw a rectangle here and then I take this side a and draw perpendicular on this line. So, I construct one rectangle; similarly, I take this one and construct another rectangle, another way I can construct another rectangle and go on you see and so and so forth; I am constructing the rectangles and going on all the rectangles has the same .

So, now what is the height of this rectangle this f of a; what is the height of this rectangle; f of a plus h; what is the height of this rectangle f of a plus h by a plus 2 h? So, what I am going to ask you. So, what when I club the areas; what do I have I have this. So, f of a into h plus f of a plus h into h because these are all h plus f of a plus n minus q h into h. So, this is my area this rectangle are of say rectangle total rectangular region which I call R n area R n is this. Now I can make my N larger and larger. So, this h becomes smaller and smaller. So, these are the leftover areas you see over overestimation and some underestimation if you use modern language see; it is underestimation here there it is a little overestimation and then all are underestimation. So, some area is not there. So, if you have more small h you have more and more areas covered in the same way.

So, what is this? So, you can write this as f of a plus k h, right where k is n equal to 1 to n minus 1; sorry is equal to 0 to and that into h. So, that is exactly what I have now; what, but we have said that area finally, comes out in a limiting sense in what we are doing. Now you say where is your overestimation underestimation in that form what did you do does not matter we can see that if we just keep on doing this and increasing make n increasing the large underestimation overestimation gets smaller and smaller and smaller and smaller.

So, ultimately the area under the curve; so, area under f x is limit n times to infinity summation a is equal to as I told you everything; name is given to special kind of limits and this limit is what this limit is what we called the integral from a to b of f x. So, what is this? It is nothing, but an extension of this sum you are elongating the sum you are just making the sum loop this it is so because it is now and it is so large that there is no discreetness it is almost continuous; it is a continuous sum.

So, a continuous sum of integrals will give you the area. So, this means; so, here if the this is we are only concentrating functions which are non negative plus functions which are non negative; my area is very clearly defined functions which are either fully negative also, then it has clearly defined, but function which are both negative and positive, then there is some issue we really need to take care. So, this is f of a plus k h into h; this is what we are doing? This is what we have this is the definition of the integral. Now I can say; I can get back the same ideas in a slightly different way some people would say; let us do something else give me those estimations when you have to understand; you have to use some property of a continuous function; if it is a nice function use its properties one of the main properties of the continuous function it is that it is an upper bound and a lower bound once it is defined over a closed interval, right.

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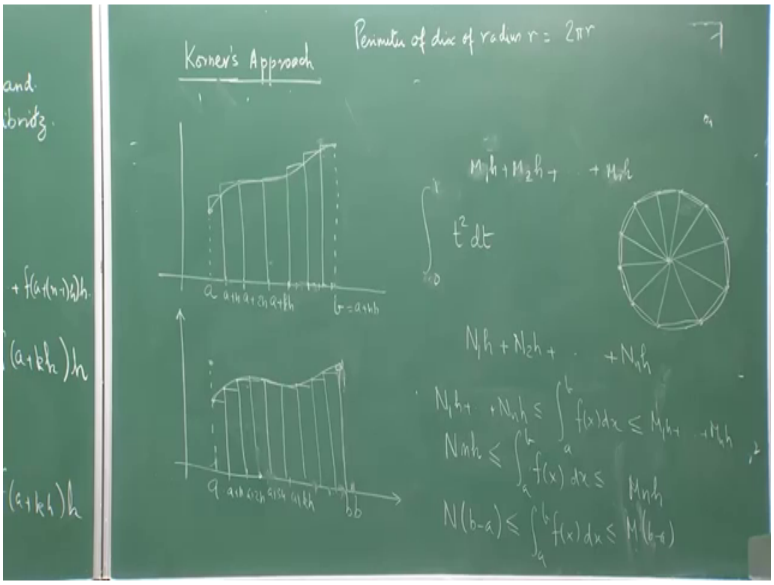


So, then we have the same. So, now, what we will do this is the idea which we will later on use when we will talk about Re bounce in approach to integration is that let us make this same divisions a plus h would generalize this approach also this approach of dividing a plus sum a h N dot, dot, dot, dot, dot, a plus N h with this divisions. So, with every division you see; which is the maximum value of the function; which is the minimum value of the function.

So, first concept the maximum value of the function this and create a with that; the rectangle here you see the x m value create with this; the rectangle you see this; these rectangles are overestimating maximum value here also maximum value is here and here the maximum value it is here. So, you see the rectangle is overestimating see the maximum value is this; this would be the rectangle. So, if the rectangles finally, are overestimating.

Similarly, you can draw. So, what you have taken the maximum value and the minimum value. So, this line; this is x equal to a; this is x equal to b; maybe I will draw the same sort of curve. Let me look at the curve and draw same sort of curve. So, now, we will do the same division a plus h a plus 2 h a plus 3 h a plus k h and so on. So, take the lower value. So, you see this rectangular area; now is underestimating some area is already left inside. Now you take the lower functional value on the whole interval because there will be a bound and do it this function is a plus 3 h is here and then it is again the lower the inner value is slightly here, it is like this and here again; the lower value is here and it is like this and it just goes on like this; see this will be all underestimation now.

Now, what is this? What is this integral is that is if you want to (Refer Time: 36:16) what are those (Refer Time: 36:18); what are those final integrals basically maybe this line should be b. So, what are; what is these. So, how do I calculate the area of these? So, the function overall has capital M as a upper bound and small n as a lower bound and now how do I calculate the area of the overestimating rectangles and overestimating rectangles. So, overestimating rectangles we took M i.

So, is the upper bound for; so, M 1 is the upper bound for this and so and so forth. So, what we did M 1 into h plus M 2 into h plus M N into h and what is this ultimately this rectangle these rectangles; these rectangular area and this these rectangular area is suppose; I take or let me capital M 1 capital M 2, capital M. So, these are the maximum values over these intervals and these intervals are close on the intervals over which the function is continuous and hence they will have maximum value which is obvious.

So, smaller 1 N is the lower bound. So, it is N 1 h plus N 2 h plus N h. So, that is what you have and the integral of the actuality is lying between this value and this value. So, it is N 1 h plus N h is less than equal to integral a b because the area under the curve; we are not looking at only non negative functions is please keep this in mind and then this would become. So, this is the class; sorry, this is the classical way of doing the stuff M 1 h plus M N h. Now this all these N 1 into N; N is finally, bigger than the actual bound one of them is a actually bound the over the whole. So, I would have this whole thing is actually bigger than N of h, right. So, I put N, but if I add these h S up. So, what we will have happen I will have N of N h is less than equal to integral a to b where N is the lower bound over the of the function; over the whole a b and M is the upper bound over the of the function; over the whole a interval a b and this would be less than sorry M N h.

But what is N h for my definition this be minus a. So, N times b minus a is less than integral a to b f x; d x is less than equal to integral, sorry, is less than equal to M; M of b minus a. So, because you will sum those see here. So, what you have done here all these M 1; M 2; M n are actually less than capital M because these are all over the interval and name is over the maximum over the whole and N is the minimum over the whole. So, the lower bound over the whole. So, that is; so, you see we had used this limiting idea to know that finally, we will get to the area; what is the whatever is the case and this is exactly the way what we had done with those squares on the closed body sounded by a closed of a plain body surrounded by a closed curve the way we tried to measured the area of that the same idea is been put here.

You see that bodies area is lying between the 2 things actually if you look back and look at to into Archimedes style of finding the area of a circle what he does is takes certain points divides the perimeter equally means each of these arcs has equal length and then it joins them it slices them like the cake, but does something else also. So, this is. So, historically integration came earlier than much earlier than the notion of the derivative now we joins these 2 points over the curves; adjacent points with the curves. So, what you get in is a polygon, but some areas are left out some area of the circle. So, in the next step you will increase the number of points and again do have a polygon well which will cover further more area and keep on going and doing. So, you will cover the whole area.

So, what happens? How do you find pi that you have already known that is the circumference and the radius of whatever circle you take will give you a constant number. So, you take any circle take you can do the experiment home take some 2-3 circles and find the diameter and their circumference put a put a thread; take a thread and put it across the diameter. Now just slight allow along the circumference; take it out gradually make it straight and see what is the length on a scale and when you take that ratios you will see more or less that you are getting the same number, but because; obviously, there will be errors in your measurement because once you measure something you have to admit; there have been errors in measurement. So, you cannot get exactly pi because pi S value cannot be exactly calculated because it is the irrational number, but you get somewhere an approximation of pi.

So, here is the fun. So, here you see how this. So, idea of integration came historically much earlier than the idea of differentiation and that is why this is very important that this is the way first it has to be introduced. Now once this is known how do we go from here to arbitrary functions that is what we will discuss in the next class and we will put some formulas in the next class and then we will use actually this idea; this simpler idea to actually evaluate in integrals for example, I will evaluate for you an integral of the form; sum of may be 0 to on t square d t; a simple function (Refer Time: 43:29), but mind you as Korner mentions that this to evaluate an integral of this form big minds like Koshi (Refer Time: 43:39) Euler had given their time.

So, it is now a joke; it is not so easy to actually evaluate and you just know these are definition and it is not easy to calculate limits like this, but how are we; so, this is the only way you really have to evaluate an integral. So, every integral of this form had to be evaluated in this fashion and equal also not easy and that is what we will do in the next class. We will see; we will end with calculation of this one and; but first telling how to get on with things which are you know how to go from non negative to the standard general one.

Thank you.