Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

Lecture – 17
Optimization Problems

Optimization as a subject is very close to my heart, optimization is a generic name given to the process of maximization and minimization. Long back more than 24 years ago or not more than 23 years ago, I landed up in the Indian Institute of Technology, Kharagpur not knowing what to do as a PhD student of mathematics, well remember my teacher taking me down to my doctor also taking me down and handing over a paper which actually changed my life and I had no idea about this subject of optimization what does it mean and what does it do and why it is important as a. So, optimization is not a just an application of calculus it is also subject by itself.

Today through this dialogue with and I am going to have with you I am going to bring about the nature of optimization problems and how we deal with them the very very subtle things that we need to know about optimization problems and that is exactly what I am going to talk about.

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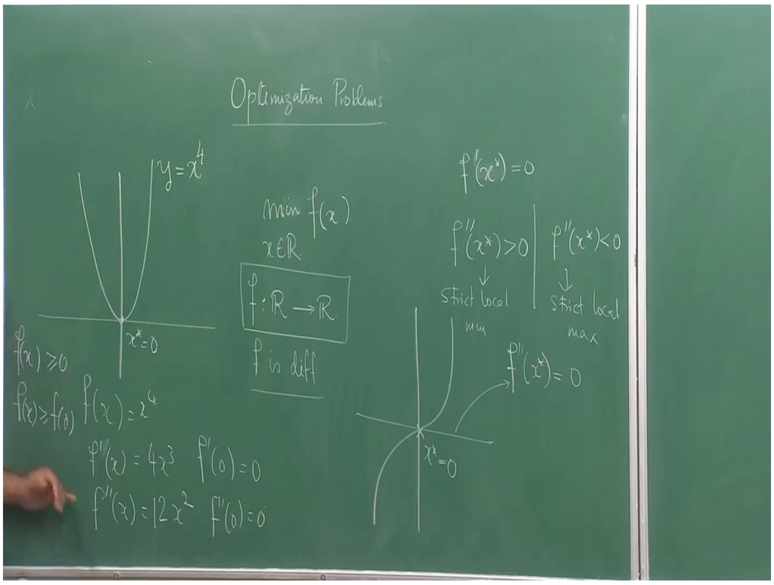

So, what we have been discussing till now, while finding a maxima and minima is a following that I want to find; I have a function from or 2 R n f is differentiable and if f is differentiable, now trying to find its maxima and minima, we have already written down if you just recall that our first step is to find out a point x dash such that f dash of x is equal to 0 and the second point is to make a check f dash of f double dash; f double dash exist at x star either it can be this or if it is this, then this gives you a strict local mean and this gives you a strict local max; this tells you that x star is take local min and x star is take local maximum.

But what about the situation by having second order derivative may not always be of help you might. So, this is the situation of x star is possibly not a maximum minimum because most of the cases in such situations in your mind you will keep the function f x is equal to x cube. So, at f x equal to x cube this point x star equal to 0 is; obviously, neither maxima nor minimum of the function decreases it becomes negative after 0, it becomes positive on the right hand side of 0. So, and you know in this particular case, f dash f double dash f dash f double dash x is equal to 0 because f dash x is 3 x square and f 6 x is f double dash x at 0 it is 0, but that usually not the issue, I will give an example of function where the first derivative is 0 at a point at 0. In fact, the second derivative is 0, but 0 is a minimize.

If this function is slightly steeper, then y equal to x square and this function is this is the graph of the function y equal to x to the power 4 or f x equal to x to the power 4. So, if f x is equal to x to the power 4, my f double dash x; sorry, f dash x is 4 x cube and f double dash x is 12 x square. So, f dash 0 is 0 f double dash 0 is 0, but you know that x 4 f f x is x 4. So, it is always greater than equal to 0. So, f x and f 0 is 0. So, f x is greater than or equal to f 0. So, 0 is a minimizing point. So, x star equal to 0 is a minimizing point, but f double dash x is equal to 0, it does not give me any gaining information.

So, what I going to do try with third derivative fourth derivative; what here once if when is once the second derivative also fails to give me some information, we should look rather that you draw the graph by current softwares or we should rather try to look into some more other properties of the function which might just tell us that if f dash x star equal to 0; if the function has this additional property, then you get your minimizer and what is that property; what is that property that will tell me that thing that I do not your second order information has failed, but maybe there is some additional property which I can attach with f dash x star equal to 0 which will tell me immediately that x star is the solution such property of a function is call convexity.

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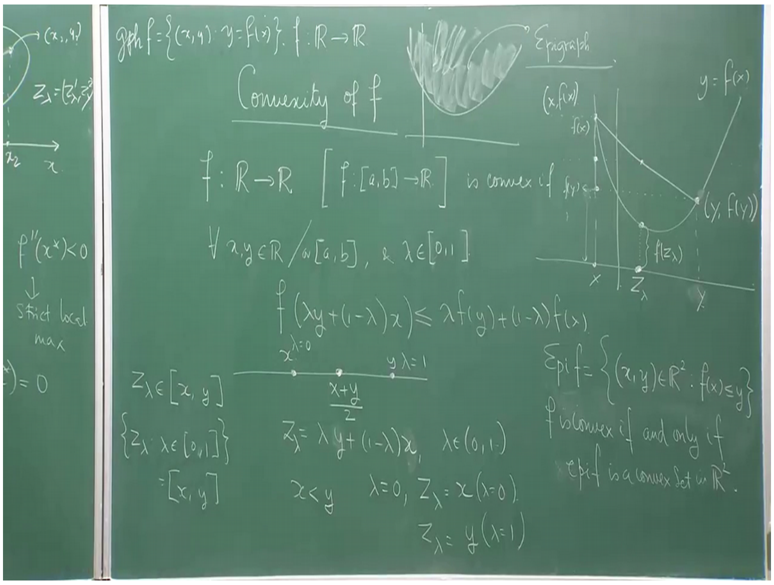

So, by discussing our optimization on the most, I bring you to one of the most important aspects of optimization that is studying of the optimization of convex functions.

So, what is a convex function? A function f say from R to R or if you want a function f from a, b to R is convex if for all x, y element of R or a, b does not matter what you want and any lambda given in 0 1 f of lambda y plus 1 minus lambda x is less than equal to lambda f y plus 1 minus lambda f of x this is true for a any lambda between not 0 and 1. So, if you take 2 real numbers you must have studied in coordinate geometry, we take 2 real numbers x and y and if I make this combination Z equal to lambda x plus 1 minus lambda y where lambda is some number between 0 and 1 may be actually call it, Z lambda and what sort of points Z lambda satisfies where the Z lambda lie is Z lambda less than x here I suppose I know that x is bigger than x is lesser than y.

So, then I know this then is Z lambda less than x or Z lambda in between this 2 or Z lambda is outside this is something that is figure out. So, what happens? So, suppose I choose lambda equal to half and what is happen is it tells me; it is nothing, but half of x plus half of y which is x plus y by 2 which simply means is the midpoint, it is a midpoint. So, which means now if I have put lambda equal to 0 I get y sorry or maybe I may be; I should write it like this lambda y plus 1 minus lambda x that is what I have put there sorry I just make this change. So, if I put lambda equal to 0 Z lambda is equal to x and if I put lambda equal to 1; this lambda equal to 0. So, if lambda is equal to 1; then Z lambda equal to y and between 0 and 1 half lies.

So, you can immediately see from intuition that if I move. So, this x actually corresponds to the point lambda equal to 0 and this corresponds to the point lambda equal to 1 c, if I have moves it from 0 to 1 lambda from 0 to 1 the Z lambda points are all the line; all the points which are lying between x and y. So, basically then Z lambda is z; Z lambda belongs to this or rather the collection of Z lambda where lambda belongs to 0 1. This set is nothing, but the interval closed interval x y. So, what does this definition signifies, it tells you I want you to figure it out yourself that if you tale 2 points on the graph of the function and join them by a chord.

So, this is my x and this is my or this is my y and this is my x. So, this is my x f x and this is my y f y. So, let me take some Z lambda. So, this is Z lambda. So, that is lambda x plus 1 minus lambda. So, what is the value of f of Z lambda? This is the value of f of Z lambda and what is lambda f x and lambda f y. So, this is my f y value this is this is my f y and this is my f x. So, lambda f y plus one minus lambda f x is some point here.

And if I draw line parallel through it to the x axis it will cut it here. So, the functional value of this; this 2 is here is bigger than this that is exactly the meaning and you see what it says it essentially says that this is only possible, this fact is only possible if this is only possible if I draw the graph; if I draw this chord; this chord always lies above the graph of the function in the sense not above the graph the part of the graph of the function is always below the chord that is the point Z lambda where you have the Z lambda that would always lie below that part of the graph would always lie below this straight line or more beautiful and simple explanation comes if we just go slightly into higher dimension for a while. So, we were been in r, but you already conversant with the x, y plane. So, let me go into higher dimension define for example, what is call a convex set.

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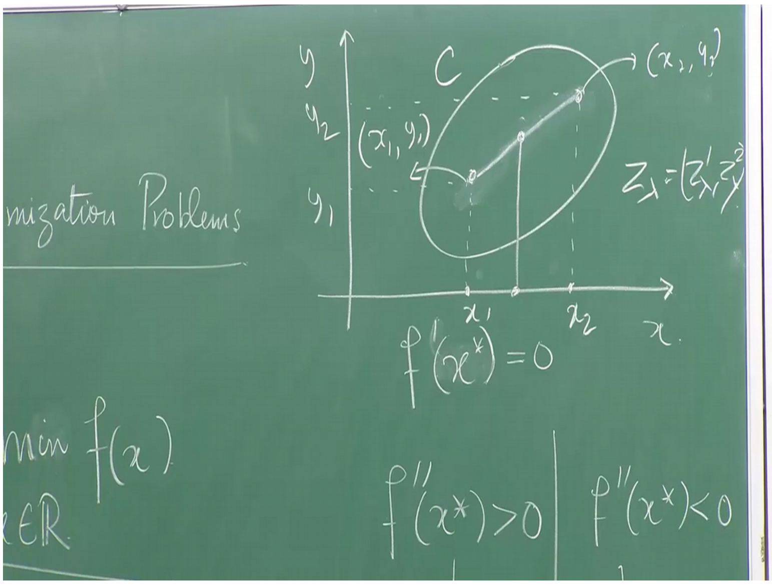

So, let me go in 2 dimensions, right. So, let me go in x, y plane, let me look at a set like this. So, now, this is a point and this is a point and this point could be x, this is my set c which has all this points inside and I take any 2 points and join them by a straight line. So, this is the line segment joining the 2 points. My drawing is so bad. This is the line segment joining this 2 point; how do I express this point and you know this very well, it is a; you know this very well from you coordinate geometry that if the say if I want to take any point here. So, how do I describe it? So, suppose this point is given by x 1, y 1 and this point is given by x 2 y 2. So, this is my x 2, this is my x 1, this is my y 1, this is my y 2.

You know that suppose this is my some point Z lambda, right Z lambda what is a 2 dimensional point with Z lambda 1 and Z lambda 2 as the 2 coordinates; now what is happening you just drop a perpendicular here. Now this divides the line into 2 ratios m by n and then you know how to find the coordinate of this point it is m of x 1 plus n of x 2. So, if you have 2 points x 1 y 1 and x 2 y 2 then any point.

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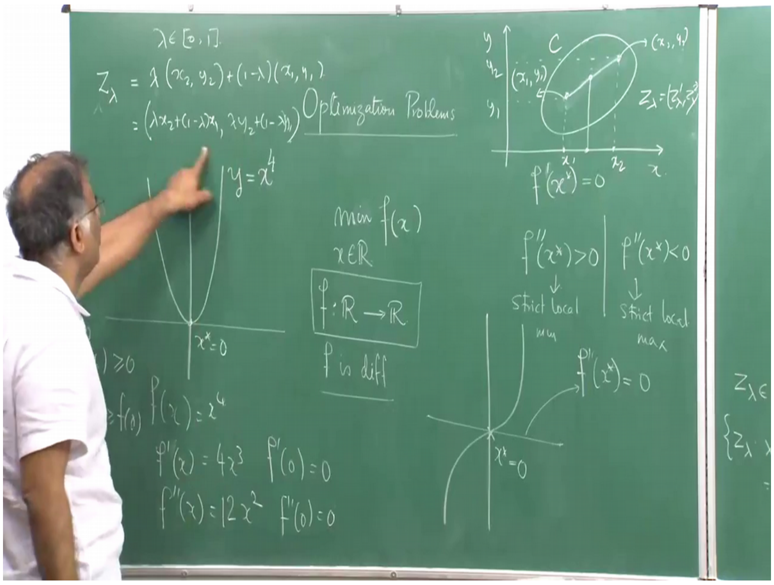

Z lambda this point is whose the coordinates would be of this form lambda x 2 y 2 plus 1 minus lambda x 1 y 1 which is same as where lambda is; obviously, between 0 and 1. So, I am joining the line segments, right and here if I do some additions of a vectors are added say if you have x 1, y 1 and x 2, y 2, they represent the 2 vectors, then the new vector, then you add the vectors, then the standard way of adding to geometrical vectors and the new vector the tip of the new vector has the coordinate x 1 plus x 2 and y 1 plus y 2. So, I can anyway multiply this lambda inside; basically you are increasing the length of the factor. So, it will become lambda x 2 plus one minus lambda x 1 would be the nu coordinate of the vector this is nothing, but scalar multiplication of the vector and lambda y, it 2 plus 1 minus lambda y 1. So, this is the coordinates of the point Z lambda where lambda is between 0 and 1.

So, convex set is a set a convex; set is a set where if you take any 2 points; any 2 points and join them by a line segment, then the whole line segment lies in the set the human body is not a convex set because if I take a point here on my head; here a point in my shoulder you join them, then you see the whole line segment except this 2 points outside my body. So, body is not a non convex thing, but convex sets are very common in mathematics.

So, a set is convex; if I take any 2 points inside it; a set in R 2 is called convex; if I take any 2 points inside it and then any this points must also be inside the set c for any lambda between 0 and 1 and where the x 1, y 1 and x 2, y 2 are 2 arbitrary points, Now what is so important about caring to this definition; see another way of recognizing a convex function is this you take a function like this and you shade the part above the graph of the function. So, now, once you shade the graph of a function if you take a function from R to R, then what is the graph of a function?

So, the graph of a function fl it is the set of all coordinates x, y such that y is f of x and. So, this is now in 2 dimensions is in our cross R is in the Cartesian planes if you shade this part and just check whether this is the convex set, then you can immediately guarantees that the function is convex function and this definition exactly tells you this and this anything which is above the graph is call the epigraph. So, function is convex if and only if it is epigraph is convex.

So, this is the very very important statement which I have want to write it down here. So, what is my epigraph; epigraph; epigraph of this function f is a set of points x, y in R 2 such that f of x is less than equal to y and f is convex, if I am just considering functions of R to R; R to R. So, we can later on built for a, b to R; you can do those extra technicalities; if I only if x is if and only if a p f is a convex set in R 2.

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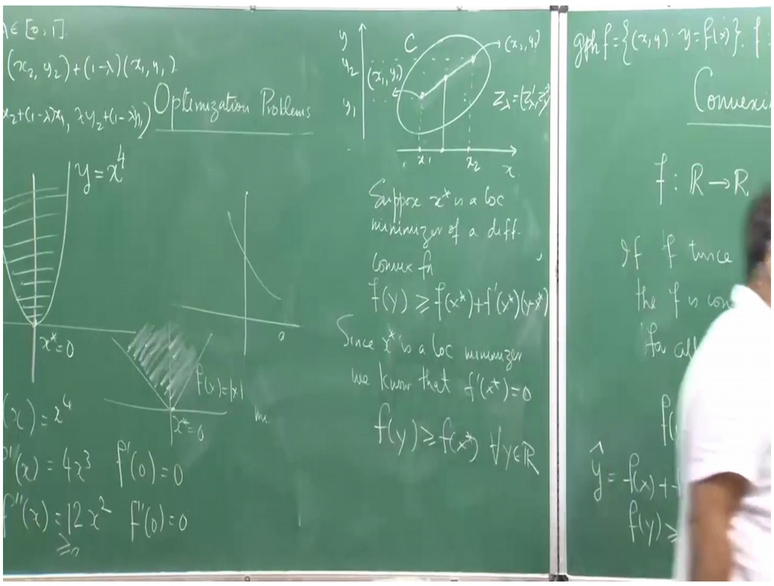

So, this is a very important learning and I will tell you why we are so much bothered about convex functions. Convex functions brings us into a world where there is no local minimizer where is there does not exist any local minimizer that is if you want to minimize a convex function over a convex set for example, an interval is a convex set. So, in real line convex set is either the empty set by conventions of single point or closed interval.

So, if you want to minimize a convex function over an interval a, b, then there is no local minimize, every local minimizer is global if f from R to R is convex, then so, it does not matter. So, even if I do not take this a, b; forget this a, b; let forgot this interval, then every local minimum is global; this is a very important point. So, f is a global minimum point. So, how do I really prove this fact when you consider convex function which are differentiable of course, there are convex functions which are differentiable I will imply give you the most standard example f x is equal to mod x at x star equal to 0 the absolute value of x f x is the absolute value of x.

I am give you a have x star equal to 0 at its minimum, but you know that x star equal to 0 there is no derivative, but if you look at the epigraph immediately know that the epigraph is convex and this is the function is convex set. So, convexity also takes us into a different real completely new real in optimization in mathematics, it takes us into the real non differentiable optimization that we have to phase the hard reality that there could be non differentiable problems which might have an optimum point or minimum point and they may just talk our at the point where the function is not differentiable and this happens pretty often with convex function, but we are not going to get into such details.

So, now how do I prove this fact? So, my function is differentiable and convex is if you want to say is differentiable and convex then if the local minimum is global. So, how do I prove what; on what basis I am making such a statement this is the very very amazing statement. So, if your second derivative fails; for example, you look at x 4. So, look at the epigraph; the epigraph is a convex set. So, immediately you know if it is; there is a convex function derivative second derivative has failed and if the convexity check comes true, then I am done, but how do you check the function is convex of course, you can draw the graph and try to do this; what I just told you if I have a twice differentiable function if f is twice continuously differentiable then f is convex if and only if f double star x is greater than equal to 0 for all x in R.

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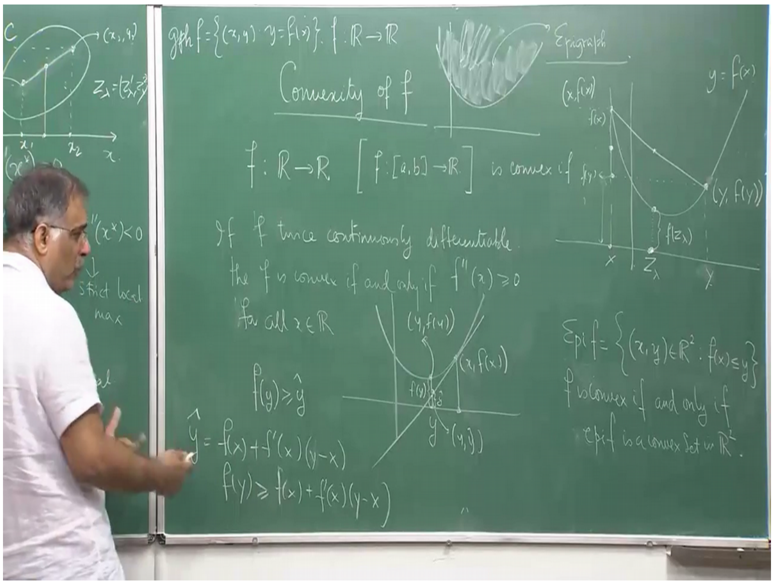

So, this is a very very important concept; this is one of the important application of Taylor theorem which we will show you and we will discuss Taylor’s theorem. This is a very important application of Taylor’s theorem. So, I am just building up the subject you see the certain things which will come later and we will show that these results are actually applications of a lot of mathematics lot of analysis lot of optimization essentially itself on something called Taylor’s expansion which will come towards the end of the course and so that is the one way, for example, if I take x 4; what is x f double dash x x; this 12 x square.

So, no matter what is your x; you are always getting a non negative; this is always greater than equal to 0. So, immediately you know it is convex. So, once it is known to be a convex then any point which satisfies f dash x equal to 0, we there is a local minima that will satisfy this and that immediately will give us a global minima, why; I will tell you tell you; the reason here I can we will use or geometrical ideas now suppose I take a convex function and now, let me take a point x f x this is x and this is a point x f x; let me draw a tangent through x f x.

Once I have drawn a tangent through x f x observe what is this; observe this; observe what is happening; the tangent line is lying below the graph of the function or lying below the epigraph you want to say like that lying below the graph of the function. So, how do you express this fact take any other point y and this is the y value on the tangent and this is the y value of the function say let me call this y hat and this distance is f y. So, what is there; f y is greater than or equal to y hat or what is y hat you know that y hat; we have already founded the equation of the tangent line.

In the previous lecture also is nothing, but f of x plus f dash x which is the slope of this tangent line into x minus sorry into y minus x. This is exactly y hat, right at the point y the at this point y, this is the value of the; so, corresponding to this; y the draw perpendicular line; it cuts the tangent at the point y y hat. So, this is the point this is the point which is y y hat and this is the point which is y f y. So, what does it say that f y is bigger than this y hat? So, what does it say? It says that f of y is bigger than equal to f of x plus f dash x into y minus x.

But this relation say f is convex this is true; if it is a convex and differential function this is true and this happens for any y x because we have just chosen 2 arbitrary points, this is a geometrical proof, this is not a rigorous proof, we will give you a later on such rigorous proofs when once we talk about Taylor's theorem, but what I want to tell you is this very fact is that you see that once I have done this; this is true for all x, y in R; it is not some for a particular x, y because I have just show the arbitrary x, y you can take x here, y here or y here, x here; you do the same thing; you do the same thing; you get the same answer.

Now suppose there is a local minimizer x star is a local minimizer of a differentiable convex function of a differentiable convex function. So, then how do I say that it is a global minimize; you see I will simply say let me take any point y in R, then if I take the pair for y; if I take the pair y and x star. So, for that particular pair, this relation will whole true because f is convex and differentiable. So, f dash of x star into y minus x star, but now since it is already a local minimize; we know that since f is a local minimizer, we already know that f dash of x star is 0 since f is a local minimize; sorry, sorry, sorry, since x star is a local minimize; we know that f dash of x star is equal to 0 and lets plot that 0 in here.

So, we will implicate f y is bigger than equal to f x; sorry, for f of x star. So, this does not matter whatever I have chosen some arbitrary y in R. So, this is essentially true not essentially true, this is just for language the way I speak this is true for every y in R because here y was chosen arbitrarily does not matter whatever why you choose; you make the pair y x star on it, you apply this fact and put f dash x star equal to 0 because this is local minimizer this will always happen and hence you will immediately get the fact that x star is a global minimizer.

So, what does it say? It not only proves that a local minimizer is global, it says that whenever you get for a convex function of point f dash x star equal to 0 that point is also global minimizer. So, you have 2 learnings that there is no local minimizer for a convex function every function has a global minimizer and whenever f dash x star is equal to 0 that point is a global minimize. For example, if you take a say cube the point 0 is nothing it is not a global or local minimize, it is called point of inflection where the curve changes shape, we will not bother that amount with such things, but this is very important 2 learnings that any point f dash x star of the convex function if it is 0, then x star is a global minimizer and hence every local minimizer is global.

Thank you and hope that you have enjoyed this new thing; usually not old if it such detail in calculus courses.

Thank you very much.