**Calculus of One Real Variable**
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**Lecture – 16**
**Newton's Method**

So, we had been looking at the application of derivatives. The key idea which, we have been emphasizing is that derivatives are the derivative at a given point actually linearises the function; that is you can approximate the function values near say x f x or near at the near x the point x, where you computing the derivative by the value of the tangent lines drawn at x f x on the graph. And in fact, the tangent line has to be expressed to derivatives.

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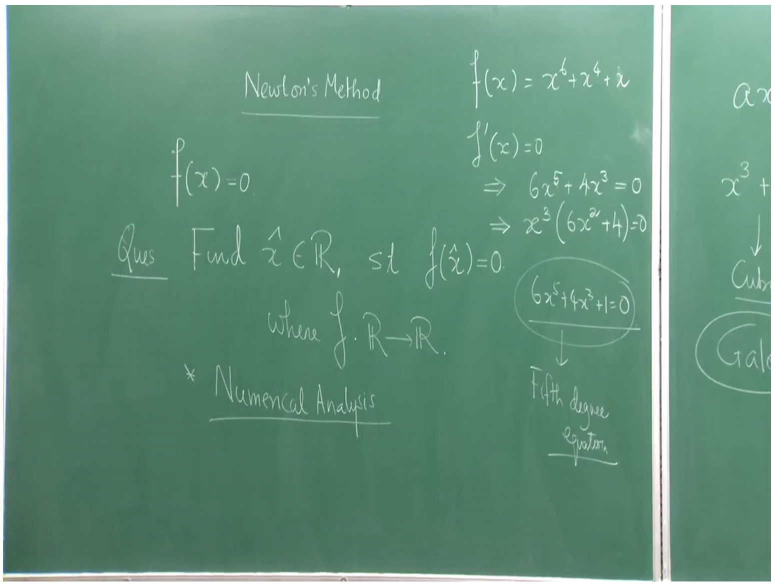


Today we are going to talk about how to solve f x is equal to 0? So, the question is find x hat in R; such that f of x hat equal to 0, where f is a function from R to R. Now, why such a study is required?

So, what we are going to do today; we are taking the firs-t step into a subject call Numerical Analysis, a very important branch of mathematics; required everywhere. Engineering, Business even in Sciences; you might ask what is there in solving a equation? I have been solve many questions for such a long time in school I have been solving equations; what was I doing in school? I was solving equations.

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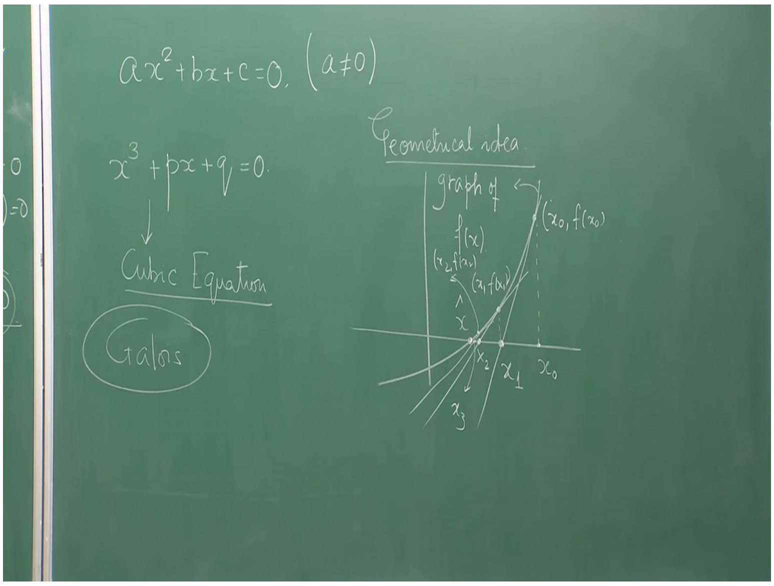


Now, I would turn around and say that; I will just give you an equation would be very difficult to solve. For example, you are aware that you can solve quadratic equations of the form a x square plus b x plus c; you can use the Brahmagupta's formula or the standard quadratic equation formulas, but there are several methods; one of them is by a famous Indian mathematician called Brahmagupta was a court mathematician in Ujjain.

So, here of course, we should have a not equal to 0. So, this is a polynomial equation an algebraic equation which you know how to solve. There are other equations called cubic equations means called a cubic equation in an standard form, but it is not easy to solve it like this one; the quadratic equation and the idea of complex numbers, which you have already some information about; in high school the numbers of the from a plus ib; they also arise while trying to solve this equation; not by, usually say that there is a there was a desire to solve an equation of the form x square equal to minus 1; of course, everybody knows if I put a real number x and square it; it cannot be a negative number.

So, it came while trying to solve this; there is a famous solution called Cardano's solution or Tartaglia solution. Italians are pretty good in all these things by the way of course, you can talk about a algebraic equation or polynomial equation; whose highest power of x is 4; this is called the quadratic equation, this is cubic equation; fourth degree equation and that is it.

Once a fifth degree equation comes at above; it becomes impossible to solve it; that is the famous theory of Galois; a famous French mathematician who died only at the age of 21 in a very stupid duel and he did a large part of his work just before he was dead. So, so Galois said that; here what is happening to express a solution, I am using square root here I will be using cube root; in the fourth degree polynomial I will use the fourth root what about Galois said that once he have a polynomial equation or degree five and above, there is no way you can solve it through radicals.

Radicals means taking the root; root of that particular power that is you cannot solve it by rooting fifth root or a six degree equation; you cannot solve it by taking the sixth root. So, what am I supposed to do? That is the big question remained. So, does that mean that I don't need to solve equations of the form f x equal to 0; where x could have higher power of power higher than 4.

For example if I simply asking to minimize f x equal to x 6 plus; x 4 plus 1; if I ask you to do that you immediately know what to do this is the differentiable function and say first step would be to check out what is this? And this immediately will give you 6 x 5 plus 4; x cube equal to 0. So, here you would immediately have x cube; 6 x square plus 4 equal to 0.

So, either your x cube equal to 0 or this equal to 0; so, you will be able to solve it. Now, suppose I had much more higher degrees; suppose I put in more other degrees. For example, x 9 or x 9 or something like that and your first degree; if I put for example, x here additionally instead of just plus; I put x here. So, then here what will have this equation will be on 6 x 5 plus; 4 x cube plus 1 equal to 0 and that is it; you are done, you have finished; you have no way to solve it just by factoring or getting things like this.

So, once you have one of constant here; you are done; this is the fifth degree equation; fifth degree polynomial equation and you cannot solve it by radicals. How would rather you would solve this equation? Then you have to get resort to what is called Newton's method. Let us understand that Newton did his method earlier; much earlier he was just trying to apply the notion of the derivative and they were doing these for trigonometric functions and all those things. The Newton's method become very important when they realize that if you want to have degrees above 5, where if we want to solve polynomial equations of this form fifth degree and above, then you really have to resort to this Newton's technique.

So, what is Newton's technique; so, let us look at the geometrical idea behind Newton's method. This idea I want to emphasize again; rests on the fact that if I take points very nearer given point x naught, then I can actually replace the function value by the tangent at x naught f x naught; essentially it says it builds on that particular idea; that very near x naught, I can actually approximate the function values in terms of the values on the tangent line. Though we talk about exact sciences, our natural sciences are called exact sciences, but I would let me tell you that lot of important things in exact sciences are just done by approximation methods.

So, what is the idea behind? Of course, you can ask whether there is a solution or not of this equation; that is the first question one should mathematically ask. But let us start with this very naive assumption that it has a solution. So, may be this is the solution and this is the line. So, this is my f x; this is a graph of f x; now what is the key idea. So, here is the solution; I really do not know it, here I can see because I just drawn the graph purposefully I do not know what the solution is.

Now, what I want to now do is take any point as my starting point and then progressively move towards the solution. So, let me take starting point as x 0 on the graph x 0; fx 0. So, this is my starting point; so, this is my solution x hat this is my starting point x naught.

So, what I do? At x naught f x naught I draw the tangent and see at what point the tangent cuts the X axis; that point I rename as x 1. So, now, what is the corresponding point on the graph of the line? It is this one. So, this is now x 1; f x 1, so now, I draw a tangent through x 1; f x 1 and call this point as x 2; you see I have come quite near the solution. And basically take the tangent at x 1; f x 1 and then see where it cuts the X axis and that is where does this tangent line becomes 0 and that is the point x 2.

And similarly you can do it for the point which is x 2; f x 2; this point which is x 2; f x 2; I do not want to draw it; it will look very clumsy and if you draw tangent line; you are very very near to the solution this point is x 3. So, what we have generated is a sequence of points which are moving towards x; it looks very simple, but let me tell you that this has its; these are some more ideas involved in it. So, we have never spoken about what is the meaning of a sequence of real numbers, but we are telling that here we have generated a sequence of real numbers; if we apply this methodology. So, we will give you a very brief idea what is the sequence and what is the meaning that the sequence is moving towards the given point and that would then help us build the Newton's method.

So, sequence you understand are so in general the way we have started with x 0; x 1, x 2, x n; may be the starting point could be x 1 does not matter.

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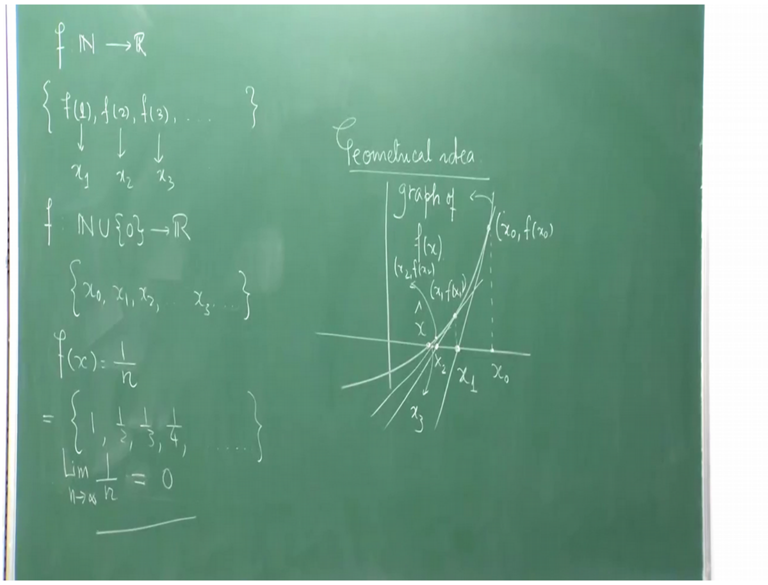


Sequence; if you take a function from the natural number n to R; this is called a sequence that is you take every number 1; that is f 1; you list all the numbers f 2, f 3 and so, on. So, this one we call x 1; this one we call x 2, this one we call x 3. In our case, we have defined what is called a sequence; it is infinite, it is countable, but infinite; we have already spoken about countability and uncountability in a detailed way. So, you understand that we are producing a countable set.

So, sequence is a countable set and here what we have done; we have for ask we have devised a mapping like this. This is also a sequence; does not matter whether you take this or this essentially. So, in this case we would have x 0; f of 0 and x 1 then x 2; then x 3. For example, the sequence are have generated here zeroth point, first point, second point and third point and of course, for example, I will give you a example of a simple sequence; f x is equal to 1 by n. So, maybe I cannot start with 0; obviously, it does not make sense; in this case, I will start with n equal to 1 and you have one half, one third one forth so, on. But observe; keep an observe vashon on this sequence that this values are constantly going down and going down towards 0. In such a case, we say that 1 by n converges to 0; this is the descritised version of the fact that 1 by x converges to 0; if x goes to infinity and each x is positive.

So, it is nothing, but some sort of discritization of the graph or of the function 1 by x on the positive part of the X axis. So, this is a limit of 1 by n, we would say as n becomes larger and larger which we indicate by the sign n tends to infinity is equal to 0. Now that is what is meaning that the sequence is approaching a point; that is the meaning that sequence is approaching here; that does not mean the sequence will be only here. Sequence for example, if I take it from x 3; I could have a point which may just cut across; it may just come this side and again it will go back to this side. So, it can come in both sides of x hat; it is not just that it has to be on one side of the x hat.

So, this is a very brief idea brief idea what is the meaning of convergence. So, now I will start explaining to you; what is essentially a Newton's method?

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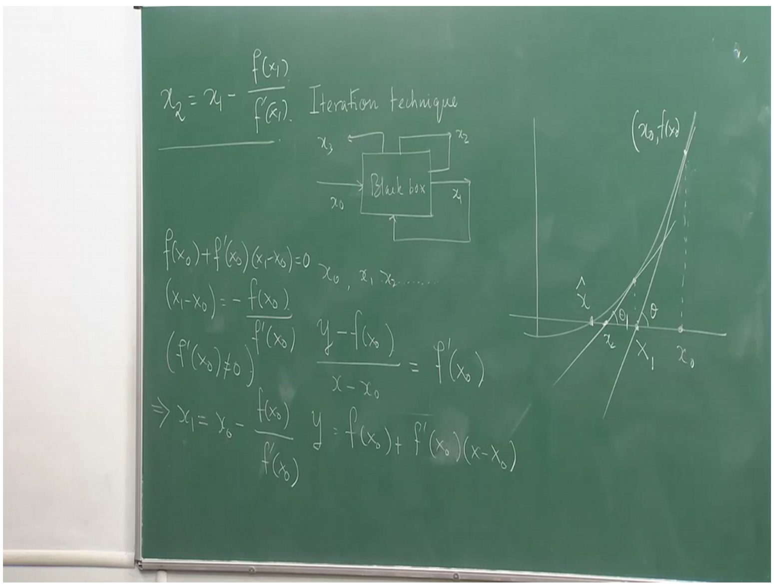


So, Newton's method is essentially an iterative technique. So, what is iteration technique? So, there is some sort of a black box; which is an operator where we do some operation on a given point. So, you put in some x 0 and outcomes x 1 and I feed back that x 1 into the same mechanisms and outcome is x 2, which you when again feedback a x 2 into the black box outcomes x 3 and so, on. So, this is called iteration technique and by doing so, you generate the sequence x naught, x 1, x 2 and so, on. How do we do in the Newton's case?

So, let us look at this first starting point. So, here is my graph and here is my starting point; if here is my x hat and here is my starting point here is x naught. So, as I told you; I will take the point x naught f x naught and draw the graph; not that draw the tangent. So, what is the equation on the tangent? So, take any x y on the x; f x on the tangent, so, x y. So, tangent line is given as y is equal to. So, very simple we have to find out the slope. So, the slope is y minus f x naught by x minus x naught and that is nothing, but what is the slope of the tangent? Slope of the tangent passing through x naught f x naught is nothing, but the derivative at x naught. So, y is equal to f x naught plus f dash x naught into x minus x naught.

So, our first attempt to get x 1; so, x 1 is my solution to the fact that this must be equal to 0; that is I want to ask the question what is that x for which the tangent line cuts the X axis? So, if x 1 is that point that; so, then in that case f of x naught plus f dash x naught; x 1 minus x naught is 0. Now, suppose I am you can see from this. So, for example, a theta; this tan of theta is not 0. So, f dash x naught is not 0; so, I can write x 1 minus x naught is equal to minus f of x naught bringing the f x naught to the other side dividing it by f dash x naught because f dash x naught is not equal to 0 from the diagram and this simply means that x 1 is equal to x naught minus f of x naught by f dash x naught.

So, my mechanism is like this; so, my general mechanism is like this. See if I now want to find I found x 1. So, I know what is x 1 f x 1 on the string and then I draw tangent through x 1 f x 1 and if I want to find what is x 2; you know the you have to just now to replace x 1 with x 2, x 0 with x 1. So, x 2 would be equal to x 1; assuming again that look at this theta 1, where you do not have the angle is not horizontal that it will become 0 to derivative.

This derivative cannot be 0; the tan of theta cannot be 0. So, it is x 1 minus; so, you have I am just replacing x 1 with x 0; with x 1; x 1 with x 2; f of x 1 minus f dash x 1. So, this is the general, this is the scheme; so, this is the feedback mechanism. So, I am just replacing. So, if I want x 3; I will put x 3 is equal to x 2, f x 1 by f dash x 2.

But this allows me to write a very general scheme for getting the Newton's meth for computing something through the Newton's method.

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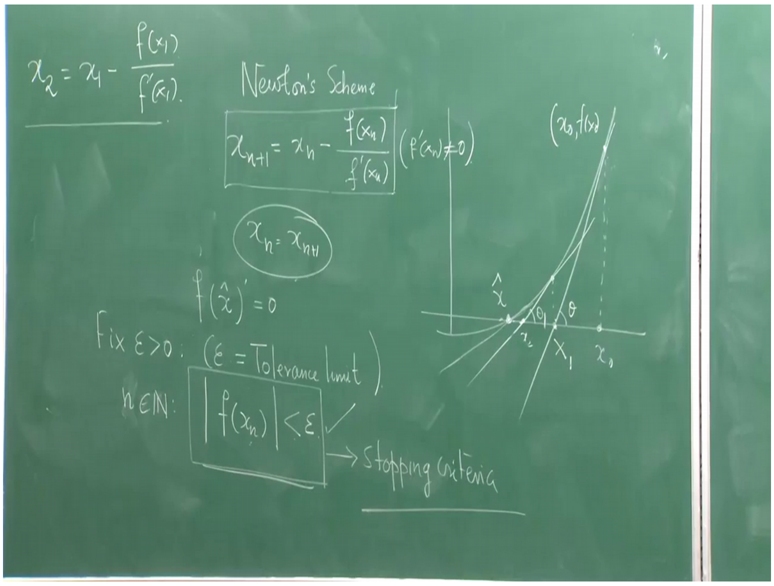


So, in generally if I am at x n; I want to know what is my x n plus 1 and the general scheme is that x n plus 1 is x n minus f of x n divided by f dash x n and of course, we know that f dash of x n must be non zero; now this is my Newton's scheme and you might ask me you will just keep on doing it; when will you stop?

The real stopping thing would be when x n would be near x hat. So, what you have to do? When will you stop? So, you stop when x n would become equal to x n plus one. So, you have actually reached this point because when you reach this point; if you draw a tangent, it will not cut the X axis again; you reach x hat and that is the reason you stop.

So, this is the time when you stop, but who knows will you ever reach this. So, you might take infinite; countable infinites sets have to reach this; that is the meaning of something going towards a limit or when n goes to infinity. So, what will you do there? That is the question; how will you stop? When I was in the doing under graduation studies; we were told that you just keep on looking at the values of x you generate. And once you see the stabilizing the decimal part is stabilizing stop it; that is not really a proper answer not a proper mathematical answer; you really have to find f of x hat equal to 0.

So, the main the way things would be done in an algorithm is to fix; epsilon greater than 0 and this is called that an epsilon is called that tolerance limit. Now, in practice I might or most cases I would never reach at x hat. So, what I am suppose to do? I am suppose to be happy with some x k; k greater than 0 or let us say n greater than 0 oh sorry; n element or n rather such that f of x n is strictly less than epsilon.

If my functional value is less than epsilon; the absolute value of the functional value is less than epsilon. So, the function value could be the xn could be here and here very near that point basically; then it is all right for me and that is where I stop.

So, when you do a Newton's step. So, every time you have to calculate x n and you calculate f x n and you see whether mode of f x n is less than your tolerance, when you reach the tolerance you stop. So, this is what in the modernity language would be called the stopping criteria and this is what you need to follow; x n see you understand very well for example, if I want to find out the square root of 2; I cannot really go on have finding I would never reach a irrational number.

Because I will try to generate sequences which are rational; because if these are integer numbers and they could be just rational numbers and might be that I will never reach that root two points; it is impossible. So, what I would do? I would stop somewhere and take an rational approximation as my happy solution. Because what I would ultimately get here is a rational approximation of; at every time I am getting x n plus 1, which I represent through a decimal which may not always be rational of course, but I will be happy to just to have a rational approximation of actual solution.

So, that would surprise for me. So, an approximate this is essentially called if you take an x n like this coming out and having this solution; then this called an approximate solution. If f of x n is 0 anyway you satisfy this; whatever your epsilon. So, let me now; once it is there it may now give an example how your things can work, but there is a little catch before I will give the example let me tell; you cannot, the Newton method was very fast and Newton's methods work extremely fast.

Now, means you start with an x naught and you just reach immediately to the solution very fast in few alterations. But there is an issue, you have to be clever enough to choose your solution x naught. Sometimes if you are very far from the original solution; you might not converge to the solution and might go in some other direction.

So, from point of view of more advanced mathematics in the study of theory of equation. So, the solution of equation it has been told very clearly; that means, shown if there is a small radius or small delta around x hat. So, if you take x naught to be a point within that delta neighborhood; then you will converge very fast; if not we do not know. Now, you might say; what is the use of how do I know that delta around x hat; I do not know why x hat.

So, here again you guess and test. So, this what the art of actually computing; that you start with an x naught and if you see you are not converging after a long time, then you understand something else is happened change your x naught. Another way now we mathematical software, you can graph the function and then try to guess where you want to start. Sometimes, from the graph you might get an idea about the solution; sometimes not, but some fair solutions you can guess see where in which part of the graph is very near 0; then you can start something nearby.

So, this is thing that one has to do and one really goes for an implementation, but mathematically; if you assume that there is a solution, then we say that there is a solution x there is a delta greater than 0 and if you take the delta neighborhood around x hat and then for any x naught in that x hat will very fast go and hit the solution. It will not hit the solution means will reach the solution.

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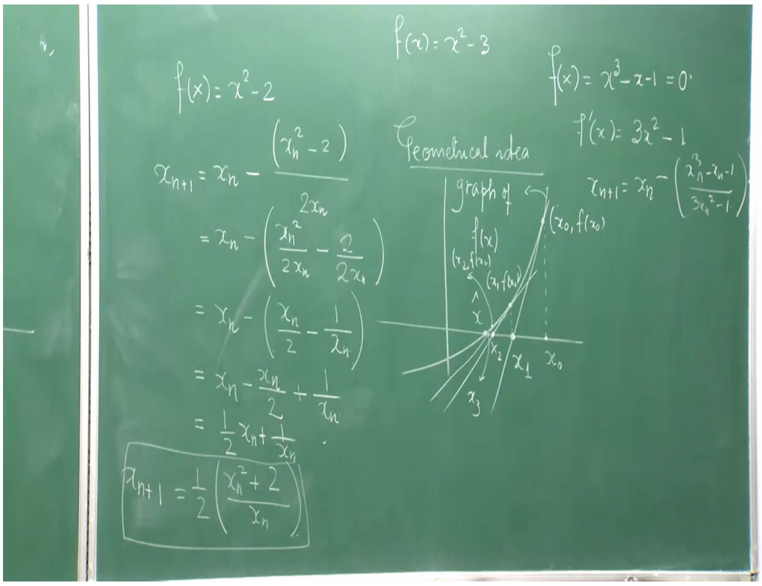

For example; if I want to say solve now this is the equation; f x is equal to x square minus 2. So, what I am trying to find what is x? x is the square root of 2; so, how do I compute this? By this methodology here. So, my technique would be x n plus 1; s o, my Newton iteration scheme is x n plus 1 minus x n; x n minus f of x n; which is x n square minus 2 and divided by 2 x n. So, this is a x n minus; x n square by 2 x n; minus 2 by 2 x n.

So, I can write this as x n minus x; x n square cancels with x n. So, x n by 2 minus 1 by x n; so, what I have here? So, look at it very carefully this is what I have. So, it is x n minus x n by 2, minus 1 by x n; of course, you can say half of x n minus 1 by x n or you can say now x n plus 1. So, not minus 1 by x n it will be plus 1; so, what do I get from here? I can write this as half of x n square plus 2 by x n. So, this is the scheme; so now, you do not have to again calculate the derivative anything.

So you just have to know what is x n; start with some x naught and just put this formula you will get x n plus 1. So, you can do it for many many functions; this I am sure t s would give some homework in this for example, you may take for example, there is a problem from Thomases calculus; x cube minus x, this is a very standard problem given in many classes; I also did this when I was a under graduate student.

So, how will you handle this situation? So, you will start with some called one point and then start moving in and here you have to find f dash x is equal to 3 x square minus 1. So, your x n plus 1 in this case would become x n minus x cube n minus x n minus 1 by 3 x n square minus 1. Of course, I would also ask you to try to find f x equal to x square minus 3; so, that is the square root of 3.

So, here you have a brief idea of how actual numerical computation goes. So, this is your first step into numerical computation and I am pretty; (Refer Time: 33:27) I will just try to give you a very basic idea how things are there. I hope I have done my job, I hope that you have got a pretty decent idea of how to handle it, but when you compute because here we cannot go on computing and showing the things.

If you compute, please understand that this is the way; you should actually stop your algorithm; not any other way, not by looking; guys come on [FL] its looks like discreetly stabilized; no that is not the way. You really have to check whether your solution is really near 0; the function value is really near 0. Because you cannot reach the 0; we are happy with this sort of approximation, this sort of that the function value is very very smaller; absolute function value that will do.

Thank you very much. We will speak about optimization problems in our next class and hope that you are enjoying the course.