Calculus of One Real Variable
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Lecture – 15
Monotone and Inverse Functions

We continue our discussions of the application of mean value theorem. We will see how powerful that ideas, we had we had made the statement yesterday that mean value theorem is powerful blah, blah, blah. We can give physical explanations everything, but even in calculus lot of good ideas come out because you keep on applying the mean value theorem. One such idea I will now speak on is this idea of monotone functions. So, monotonic functions.

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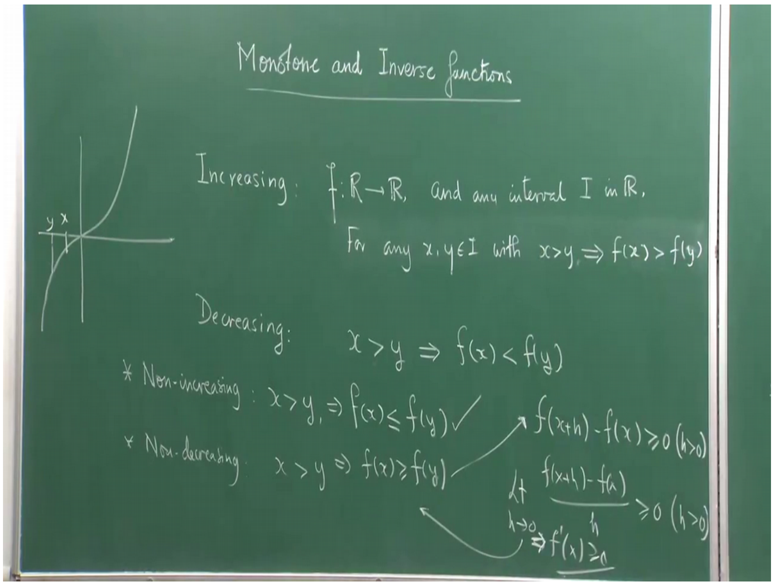

So, we will talk about 2 classes of functions increasing and decreasing, increasing decreasing. So, take any function f say from R to R. And any interval I in R, if that for any x y in I with x greater than y. If it implies that f y is strictly bigger than f x then the function is called increasing. And the decreasing thing is the same where whenever x is strictly bigger than y the function value should decrease. So when x is strictly bigger than y; f x is strictly bigger than f y this is increasing; sorry, I just made a mistake I just wrote the decreasing definition.

So, increasing means when x is bigger than y f of x should also strictly bigger than y if x is bigger than y it should imply that f of x should be strictly lesser than f y. So, that is the meaning of decreasing. So, there are certain other definitions which we have like non increasing non decreasing non decreasing, non increasing means and non decreasing. So, non increasing means it either it does it does not increase if it at all everything happens it as a remains same or it decreases. So, if x is strictly bigger than y then f of it should imply that f of x is less than or equal to f y. So, it is not increasing, it could remain equal, but it would not be increase. And here if x is strictly bigger than y, then f x is greater than or equal to f y f x need not be always strictly greater even if I have these 2 are strictly greater.

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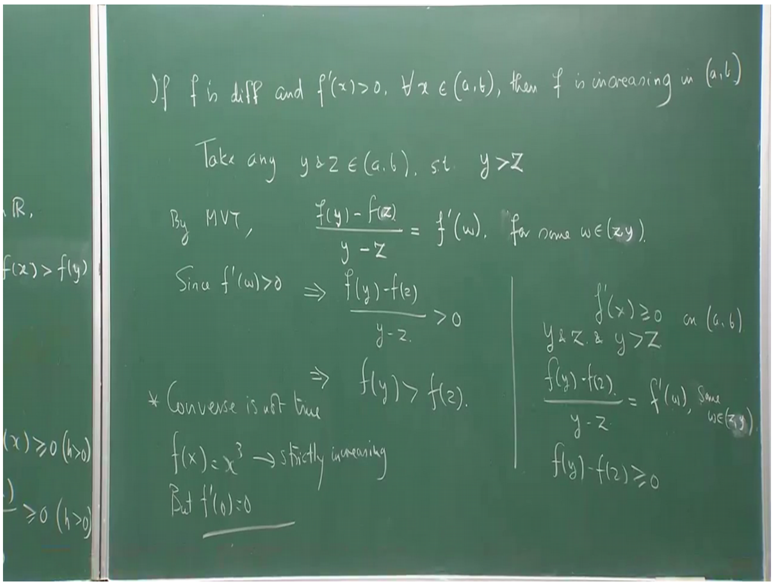

So, one of the most simple and elegant applications of the main value theorem is the following is that if f is differentiable. And f dash x is strictly greater than 0 for all x say in an interval a, b, then f is increasing. So, this term after these terms have been going in the literature some people started calling this as strictly increasing and strictly decreasing, but we keep on calling them increasing and decreasing, increasing in a, b. So, example if you draw the graph of increasing function, y is equal to x cube. So, you take any x and any y it does not matter the function value is increasing the function value is increasing. So, whenever you have x and y whenever x is bigger than y if x is bigger than f y.

So, how do I prove this fact? The fact is the following take any y and z arbitrary y and z in a, b. Then by the mean value theorem you have that f y minus f z. So, take y and z in a, b. So, among them either y is bigger than z, or y equal to z or y is less than z. So, we will choose; obviously, we can not choose y equal to z does then we have the same function value remains same. So, choose it in such a way that f y is strictly y is strictly bigger than z. So now, then my definition f y minus f z by y minus z. This must be equal to f dash w for some w in the open interval y z that is a m v t, but then this open interval y z is a subset of the open interval a, b as a result of which f dash w is strictly bigger than 0.

So, since f dash w is strictly bigger than 0, it simply implies that f y minus f z by y minus z is strictly bigger than 0. Now y so, you multiply both the side y minus z then y minus z is strictly bigger than 0. So, the this inequality has not change. So, means that f y is strictly bigger than f z and it shows that it is increasing. I would leave it to you to write down a similar result for the decreasing case when you have f x is strictly less than 0. So, if f is differentiable and f dash x is strictly less than 0 in an interval a, b then f is decreasing. Now what I want to say is that this is the sufficient condition it tells me that if I know this nature of the derivative I can tell the nature of the function. As I told you whenever you have some information of the about the derivative and you need an information of the function just go apply this beautiful lagrange mean value theorem, and that would lead to the answers see how beautiful and simple things are.

And now, you might ask the reverse question is this condition necessary, whenever I have a strictly increasing function is f dash x is strictly bigger than 0. Because the answer is of course, no converse is not true. Example is write on board x cube function. So, f x is equal to x cube this is strictly increasing, but f dash 0 is 0 is a very prototype and standard example in calculus that this is. So, the reverse is not true. But So, you cannot have an necessary condition here, but have sufficient condition, but you will take for example, this kind of stuff. Suppose your differential function which is non increasing or say non decreasing then you have necessary conditions. For example, if you take 2 points x and x plus h where h is positive. So, f of x plus h minus f x is greater than equal to 0. If it is say known decreasing because h is greater than 0. So, x plus h is strictly bigger than h x plus h is strictly bigger than x. So, which means f of x plus h is either equal or strictly greater than f x that is greater than or not equal to f x, then because h is positive you simply divide by h. Then you take the limit as h tends to 0 that will simply tell you that f dash x is greater than or equal to 0.

So, for this case the necessary condition can be implied, but this does not at all mean f dash is greater than or equal to 0. So, we might now get the idea that f dash greater than equal to 0. Might not always imply this condition, means this one. See we know that that in that case it is not sufficient, but in sorry now strictly case strict case it is sufficient, but not necessary. But here is it necessary and sufficient that is what we want to check this necessary is obvious because here that I have taken the limit h going to 0 plus, but because of that derivative exists that is a derivative and hence we get it. But what about the reverse in this case? That is the question I am want to ask. Again the mean value theorem would be helping. Suppose on an interval I know that f dash x is greater than equal to 0, say on interval a, b. Now you take any point again y and z such that y strictly bigger than z. Then again apply the mean value theorem, f dash w for some w.

So, here you see f double dash w for some w sorry, y y z for some w z e z y. Here it should be also z y because z is lesser than z is lesser than y. So, z should be on the left and y should be on the right. Sorry I am writing fast. So, and I write fast I sometimes slip. So, that human mathematics, by the way is human endives something that we need to remember. So now, what would happen that this greater than equal to 0. So, this is true for any some w and z y and this z y inside this interval this z y is inside the intervals a, b. So, then f y minus f z is greater than equal to 0.

Which proves that it is non decreasing. So, this condition is both necessary and sufficient. And that is the reason this non decreasing and this sort of functions have been defined that you can now pose through the derivative, both a necessary and sufficient condition both of backward and forward conditions. Now we had mentioned about this issue of the second derivative when we do optimization, we do second derivative, you use the second derivative if the function is twice differentiable to determine whether your given point critical point f dash a is equal to 0. That a is either a maxima or minima or whatever nothing.

So, that has to be determined by the sign of the second derivative. That that information we have written and many of you are already done this calculus in high school would definitely know that fact. But we are now going to use again what we have just learnt to prove this simple fact. And let us now try to prove whatever we have learnt about increasing and decreasing function, let us now try to prove that the sign of the second derivative is crucial in determining the actual nature of a critical point. Do you remember critical point is a point at where the derivative vanishes? But a critical point need not be a point of maxima or minima for example, the point x is equal to 0 in function f x is equal to x cube, which you already know that the that is not a point of maxima and minima But is again I remind you.

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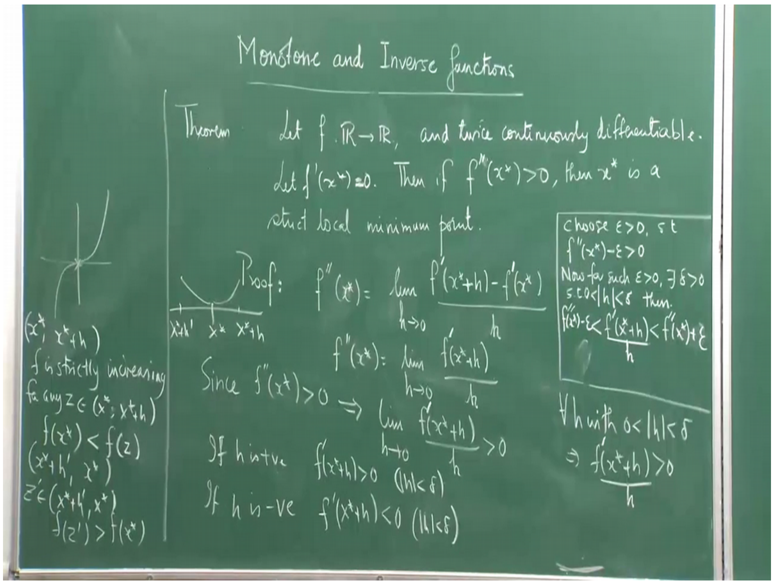

but is the point where the derivative vanishes. So, my results which should I now really write down as a theorem, while this is important as you may start from R to R do not bother about this interval and all these stuff. And because we have been really bothered much for un constant optimization in the sense that we are minimizing all the whole over.

So, the function is to be naturally needs to be defined on the whole over. So, function is from R to R and differentiable. Let f dash x star is equal to 0. So, x star is a critical point. Then if f is maybe I should write f is what toward and twice continuously differential and my second derivative exists and it is continuous. The second derivative is also a function of x. Then if f double dash x star is strictly less greater than 0 then x star is a strict local minimum x star is a strict local minimum point.

Now, how do I prove it? F double dash a how do I define this? May be twice continuously differential might not be required just continuous twice differentiable is required, but in standard problems they there use of this word twice continuously differential would be helpful, when it when it will talk about taylors theorem and use taylors theorem to actually prove this result we do this prove this result again, but would still we would we will prove this result again and in a more different way. But still does not matter if you keep this we just keep it continuous second differentiable, we do not want to take it continuous second differentiable that would second derivative exists at x star sorry is enough. So, that is limit of h tends to 0 f of x star plus h minus f of x star by h.

Now, if for f of x star is equal to 0. So, and f double dash x is greater than 0. So, this is the nothing but lien of h tends to 0, f of x star plus h by h now because this is because f double dash x star is strictly bigger than 0. I am writing since it would imply that limit h tends to 0 f of x star plus h by h is strictly bigger than 0. So now, here is a some reasoning that we will make at you to be very careful about. Let now what is that limit? That limit is f double dash x limit of this is f double dash x f double dash x is positive. So, what does it mean?

Given choose now choose an epsilon choose epsilon greater than 0, such that f double dash x star minus epsilon is strictly greater than 0, this is possible because f double dash x is not equal to 0. E f double dash x star is strictly greater than 0. So, if I take away a very small quantity still remain positive that is that is. So, between 0 and any positive number there is infinite amount of positive there are countably infinite positive numbers and that is the key of make making this statement. So, I can choose this. So now, for such epsilon there exists delta greater than 0 such that So, here there was a minus f x star.

So, x star plus h minus f x star. So, such that whenever mode h this one is within this domain then what we have, we have f of x star plus h is strictly less than f of x star plus epsilon sorry f of x star f of x star plus epsilon. And also strictly greater than f of x star minus epsilon. But f of x star minus epsilon if you look at I have already chosen the epsilon such that this is strictly bigger than 0. So, for such an epsilon I can get an delta for this is true. So, there is a delta such that whenever mode h is strictly less than delta and; obviously, h is not 0 will have. So, for all h with this it implies that f of x star plus h is strictly bigger than 0.

So, it says for h is sufficiently small h whose the real number hs absolute value is sufficiently small, for which this is true. So, what does it say? Sorry I have made a big mistake big mistake, here it should be f dash because we are taking the second derivative. So, this is f dash. So, it is f dash. So, here it is f dash here it is f double dash. So, please understand.

So, I am taking the second derivative. So, it is a derivative of the derivative. So, I made a I made rather I would if I type it out we call now what is typographical error. So, it is a chalk error rather that I forgotten this sign. So, so what is what does it show that in this small interval f dash plus x star h is strictly bigger than 0. So, basically if So, the So, f dash of x star plus h by h sorry, I made a mistake. So, by h this is strictly bigger than 0. So, if h is now positive if h is positive what I have.

So, if h is positive I will have f of x star plus h which strictly big greater than 0 whenever h is less than delta. But if h is negative I will multiply both sides by h because I want to cancel this h because I just want the information is f dash x star plus h. So, from x star to x star plus h means a up to x star plus delta f dash of x star plus h is strictly bigger than 0. But if h is negative that is you are moving towards the left of x star x star x star plus h when h is positive x star plus h dash same way f dash is a negative number. So, if h is negative which is this case then h dash because I will multiply by negative number. So, this sign will change. So, f dash x star plus h is negative.

So, again I will apply the theorem that we have already learnt here. So, because whenever h is positive and h is less than delta. And whenever h is negative and is greater than minus delta, you have this whenever h is positive and less than delta this happens whenever h is; obviously, here I should have h less than delta here I should have or rather when I can just write mode h delta h and h not equal to 0; obviously, I am not writing repeatedly because h is; obviously, assume to be not equal to 0. So, when h is positive I have this. So, between x star and x star plus h the function is increasing and between x star and x star minus h the function is strictly decreasing strictly please understand the term. So, from x between x star in this interval f is strictly increasing. And so, for any z in x star plus x star plus h what would happen? Any z in x star plus x star plus h I will have f of x star strictly less than f of f of x star 2 is strictly less than f of z.

Similarly, when I take x star plus h dash to x star and for any z, z dash element of x star plus h dash to x star what will I have? I will have f of z dash that is this strictly increasing here it is strictly decreasing. So, the value of f would decrease function value decreases f of z dash is bigger than f of x star. So, what is when what is f of x star? F of x star because z; obviously, I am taking a z dash which is not equal to x star because I have taken it in the open interval. So, for any z dash or z any in the interval say x x plus h and x minus h in this interval x star plus h and x star minus h whatever z I take it does not matter if it is not equal to x star the functional value is strictly bigger and hence it is strict local minimizor and these are very, very important results and lot of things you.

See we are continuously using this epsilon delta argument sorry for some miss typos other, but please follow the argument once again here we taking the second derivative derivatives of the derivatives are going in this fashion in there they are using the epsilon delta fashion. And that is the exactly where we learn about the several things which cannot be told in this class, but will appear in the notes, but I will just like to tell you that notes will not appear immediately. It will take time to correct the notes and put things. Many things cannot be told in the class that there are time restriction, but hence we will do something will write something in the notes which has not in the class, and which will not be told in the class.

So, will end this discussion will take 5 or 6 minutes more to talk about what is called inverse functions. When does the function of inverse and what does what is the meaning of inverse? This is something very important. So, we will just work it out.

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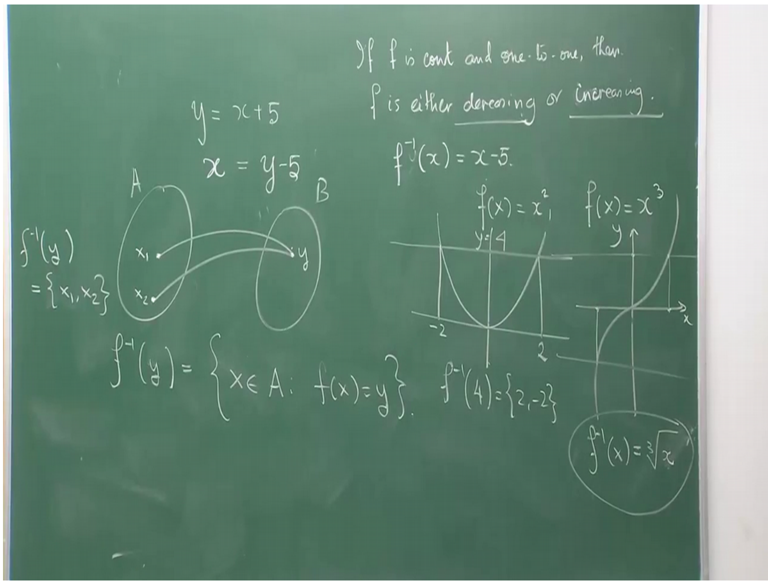

So, giving a very brief idea what a inverse function possibly is. So, basically you know what is inverse function is for example, if I have y is equal to x plus 5, then inverse function is now to write here y is the function of x an inverse function means I write x as a function of y. So, here So, if somebody asks me this statistical mathematics to tell me what is the inverse function corresponding to this function? And say f inverse of x is x minus 5 actually this is y, but this is because this is a dummy variables just this is the traditional way of writing it. So, here is your domain and here is your range. So, this is a very important concept. So, you see there can be 2 points which are going to the same 2 xs in a x1 and x 2 which can go to the same y. So, what is f inverse y f inverse y actually means the inverse image of y. So, it means the collection of all x in say a such that f of x is equal to y, that is the meaning of the inverse function.

So, in this particular case for example, here f inverse y is equal to x1 x 2. So, corresponding to this y I have 2 elements which are corresponding. So, which means here the inverse function really does not satisfy the definition of a function for a definition of function. Every y should be mapped only one x exactly one x. 2 different y s can be mapped to by if you think that if function has to be inverse you can. So, can 2 different xs can be mapped to can a single x 2 different y s be mapped to a single x no it cannot be, but then it will break that this is a function from this side. To maintain that it is a function from both side that is I can write y as a function of x and x also the function of y things can be done, then 1 x should correspond to one y only. So, that is called a one to one correspondence bijective mapping about which you have some idea in high school.

So, this is so, this is the very basic idea of an inverse mapping. You have to very clearly understand for example, I take 2 simple functions f x is equal to x square. And f x is equal to x cube, look at f x is equal to x square does it. So, if the inverse image has more than for given y if there is more than one element, then that the this inverse is not a function it is called a said valued map. But for every y in b if the inverse image has only one image element corresponding to it x, then we say the function has the inverse for example, here take the number of 4, y equal to 4 y equal to 4 is achieved by 2 numbers. 2 and minus 2 the square of minus 2 is 4 square of 2 is 4. So, in this case if f inverse of 4 is a set 2 minus 2.

So, this is, so this function where x square does not have a inverse, but f x equal to x cube does have a inverse. So, how do I look at it here x and here y. So, x is x is viewed as a function of y. I can just look at that like this and think of x now viewed as a function of y here y is first viewed as a function of x. And then if I look at it x is bigger function of y. And if you observe take any y there is only one x corresponding to y take any y there is only one x corresponding to y. So, there is a bijection between the xs and ys. So, which means this function has an inverse and f inverse x as y equal to x cube x is cube root of y, but usually we this is the standard way of writing; obviously, you can write f inverse y is cube root of y, that is that is also acceptable.

So, this is a inverse this is the, but from this function you might have some doubt on in the sense that there must be some relation between a continuous function and increasing function. So, here we said this result and we end our talk today you will find more information in the notes is that every if a continuous function is one to one, that there is a one to one correspondence between the xs and ys, then it is an either a decreasing function or an increasing function. If f is continuous and 1 to 1; so, it means it has a inverse then f is i; f is either decreasing or increasing.

So, this is something very important and with this, we end our write it near c. With this we end todays lecture, as in next class we are going to talk about a very passionating way of using the derivative to actually solve equations of the form f x equal to 0. Because this solution of equations of the form of f x is equal to 0 is required in many places for example, if you want to solve f dash x equal to 0. It might not always be easy to solve f dash x is equal to 0. So, you have to adopt some other techniques which are called numerical techniques and in that the derivative plays a clearly a major role and that was first exposed by none other than the great isac newton. Possibly with the greatest scientist of all times. So, we will talk about newton method or sometimes call the newton raphson method for solving f x is equal to 0, using the notion of a derivatives; obviously, we are assuming the derivative x is.

Thank you very much.