Calculus of One Real Variable
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Lecture – 12
Rules of Differentiation

So, once you have now defined the derivative the game is now to know how to play with it. Many of you who have already done calculus in a high school I am sure knows how to play with it. So, in order to play with it some rules have to be formulated.

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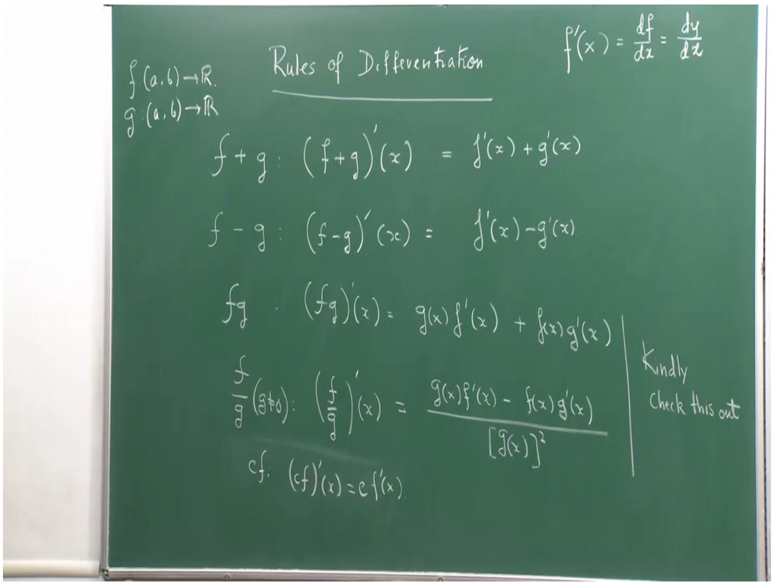

So, this is what are called rules for rules of differentiation. It tells you if I combine 2 functions in different ways then how am I going to find the derivative of that combined function. So, how many ways you can combine 2 functions. First of course, is if I add the 2 functions f and g. And then second if I subtract the 2 functions f and g. Then of course, comes multiplication, and then comes division where of course, I need to know that g cannot be 0.

See if I know that the function f and the function g has a derivative, how do I compute the derivative of f and g. So, derivative is sometimes also written like this. Or if you write y is equal to f x, some people also prefer to write this. It depends on what is your choice it is no hard and fast rule you can invent your own if you want. Now so, what is the what how do I compute the derivative of this function. Knowing the fact that limit of f plus is limit of f plus limit of g just. So, if you write down the derivative, definition of the derivative of this and separate them out, you can easily prove that. Proving this is not a very difficult situation for another thing I just wanted to add it in give you is. If I have f of So, I multiply c with x where c is constant. Then how would I calculate the derivative.

Now, in this case you know what is want is the meaning of f minus g, at that any point x the functional value f minus g is evaluated as f x minus g x. So, if I compute the derivative of this would give me just give me this. So, this is pretty well known a slightly complicated one is this product and you know that if I the product function f g when evaluated at x is nothing but f of x into g of x. And so, when I am trying to find the derivative of this function then I have some little we have to make a little more manipulation, and I think this is this is a good exercise for anyone to try at home who wants to sharpen his confidence in handling things in calculus.

So, this is g x into f dash x plus f x into g dash x. So, here is a interesting looking formula. This is essentially nothing but manipulation of that limit. So, how do you manipulate the limit and what you do? So, this is exactly what you will get I would very strongly suggest that those of you are watching the program take a pen and pad, you can stop the program for some time work it out and see whether I come to this conclusion or not.

This is slightly tricky one of course, g cannot be g x cannot be 0 for any x. So, when I want to evaluate this is a slightly tricky one, but again it is a same thing what you do with because this is nothing but f multiplied with 1 by g. Where g is a non negative function. So, this what you have. So, these are suggested things you need I would suggest that this is slightly more complicated then this 2. So, you have to do some manipulation. So, kindly check out. This is very simple of course, this says that if you have this function that is you multiply c with x and operate f. Of course, you can ask what is the domain and range of f n g all this things can be asked. You can take the domain out domain could be say the interval a b to R and g could would also interval a b to R or interval open interval c d to R where a b and c d has some intersection. Or you can just say both of them are from R to R does not matter.

So, this domain and range it does not matter if the functions is differentiable at a given x, and then that is exactly. So, essentially this is a point based result. That we are talking about the differentiability at this particular point. So, that is how you will calculate the derivative for this. So, if I know that derivative of this and this at this point x then I know the derivative of f plus g at the point x. And so And so forth the rest of rest of the writings.

So, this seem so, if I want to find out the derivative of this. This is nothing by, but sorry it should be c of f. So, c of f means basically c of f at x is nothing but c of f x. So, when you take this function and try to find the derivative at x, this gives you c of f dash x. That is the constant comes out of the differentiation. So, these are standard one.

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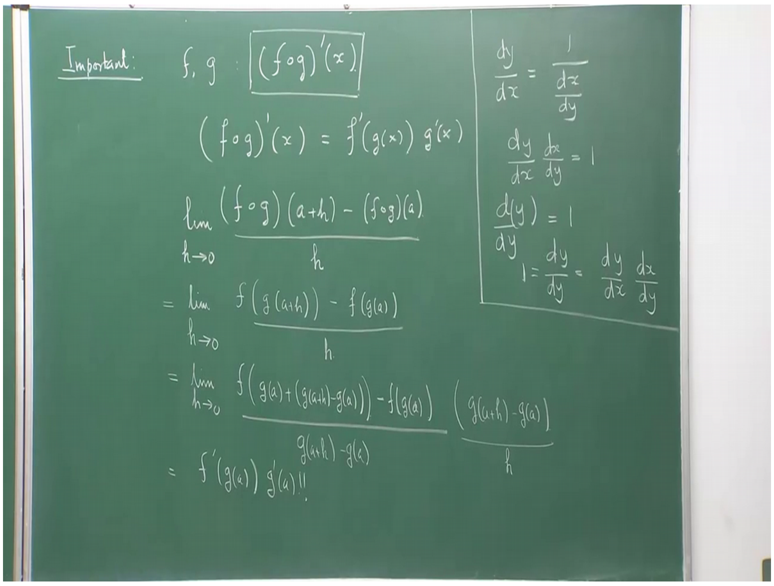

So, you see we have covered almost all the known operations, but there is another important operation. And this is important, and this is one of the key rules that you need to bother about. Here we are talking about 2 functions f and g. And we are talking about the composition of the 2 functions. And we want to compute the derivative of this composition, how do we do this?

Many of you would immediate jump. And so, this is the chain rule yes this is the chain rule the, but the ease with which you will prove this, the same is with the same ease you would not be proving this. This thing and you need to be slightly careful when you talk about this problem. So, if you want to compute say the derivative at x or say at a point a then what you do is you compute the derivative of f dash at g x, and in then you compute g dash x. So, how does this formula arrive? You can So, I can just do the say I can just write down the limit and I can do something and I can for example, I can do this. So, I am computing it at some asa. So, basically I should have right. So, of course, we are already talking about differential functions, when a please do not please do not ask questions in the forum which are very obvious when you are talking about rules of differentiation. I have to really focus only on differential functions; I cannot be talking on functions which are not differentiable.

So, that is the definition of the derivative. So, you think I have to get something like this. So, somehow I have to bring that g dash x. In this is the formula somebody tells you and you decide they are I really have to bring in this and that would be enough right. [FL] what you indent to do? Now how you will progress to get this g dash x. So, let us see that. So, you think somehow I have to get this g of a plus h minus g of h that sort of things should come in. So, I can write this as, and that is exactly what you will be thought in most places.

I will do some manipulations here I will add a g a and subtract a g a. What I do now? I multiply both top and below by see so, that is done because of course, for the definition of composition f must be defined on the range of g. And hence by (Refer Time: 11:49) this is acting as my h, g a plus h minus g a is acting as my h or k basically the increment. So, then this term basically tells you. So, what I did was I divided and multiplied by g a plus h minus g a.

So, this term simply tells you that fine. So, this will be. So now, I operate the limit here limit of f f into f g is limit f into limit g if both the limit exists. So now, what I will simply do is take the limit of this and then the limit of this. And this will simply give me f of f dash of g a and that would give me g dash of a (Refer Time: 12:37) I have done it, but I would ask you to as I explained some goodies which a chain rule gives us, I would ask you to think whether this proof is correct or not whether this proof is mathematically rigorous or not. And that is why and you once you can figure out, what I am trying to say you will really start appreciating the nature of mathematical proof, and why we really need in mathematics to demonstrate things rigorously.

See one thing which I have seen students often do. It does not matter where are you the student see it is usually assumed d y d x is 1 by d x d y as if it is just like a fraction right, but it is not a fraction. Essentially whether this is this an operator it is a it is a. So, how do I, how do I? So, it simply means that d y d x into d x d y is equal to 1. So, how do I justify this. This justification actually comes from the chain rule. This justification actually comes from the chain rule, because the function y if I take this is 1 right.

So, once I can write x also as a function of y that is x this function or rather y y y is a function of x that you of course know. But if you write a you if you can write x also as a function of y means the function is invertible, only for that case you can definitely write you can talk about this term d x d y you know y is the function of x and if it is invertible then x can be represented as the function of y, and then in that case you can write this thing as that composition of a function y is a composition of the function in to it is inverse. If you compose a function with it is inverse if the inverse x is it gives you back the function.

So, then what happens is this can be written as first and then x as a function of y, and this is you it is standard that that is one. So, this is what happens. So, it is a chain rule which is a key fact which proves such things. And of course, it is not you just do not write things like this it is not always true is only true, when a function is inverted you can actually apply the chain rule right. So, you have to be very, very careful that I can I can I should be able to express x as a function of y, then only you can do d x d y right. You cannot just write something like this from the air, but most of the things in calculus that we do are essentially of that nicer form. And so, you can do things, but not always every function does not really have an inverse.

So now where do you think is the problem with this proof? The problem with the proof lies in the fact that no one told me that g a plus h minus g a is not equal to 0. If I move a little from g a the function could be slight could hold would remain constant over a little zone or maybe it is just constant after sometime. And g of a plus g of a plus h could be just 0, how do you accommodate that situation? There is no way I can tell anything if that happens here. And that tells us that this approach are actually proving the chain rule which is quite standard in many ordinary calculus texts is actually wrong.

So, you really have to know the correct steps of proving this rule and that is exactly what we are going to know do. So, our discussion today would end with actual there are true proof of this chain, chain rule formula. And this proof I am doing from the book of spewak as I told you that I suggest that this book be actually kept in the library of every good school, and students should have an access to it this is a school or high school or even college or whatever does not matter. That this book now at least in the colleges in India. So, that the kids could have a better access to proper knowledge basically.

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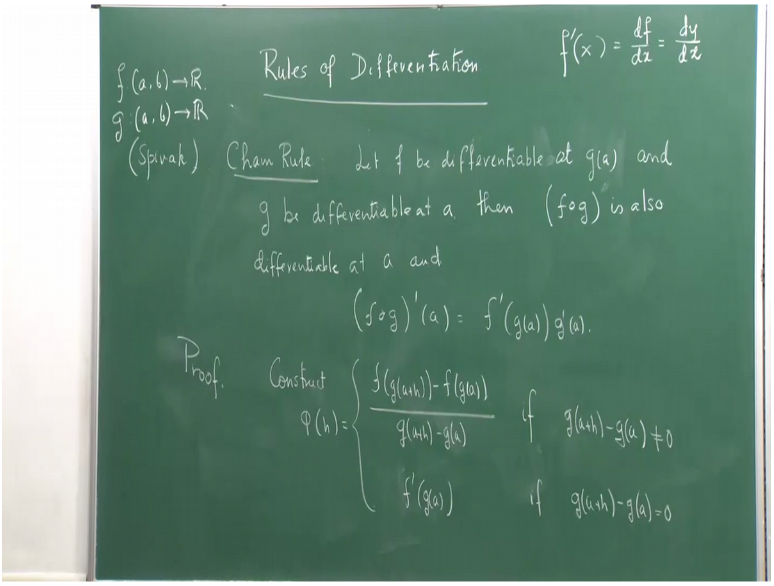

So, here is the chain rule and I right down the theorem. Let f be differentiable at g a and g be differentiable at a. So, I will write the author whose proof I am now using whose proof style rather, thus a quiet apart of this course I will follow this book, but I will also show you some more interesting books which you can read. G be differentiable at a then f compose g is also differentiable at a. See we are not bother much about the domain and range and of f and g this is obvious (Refer Time: 19:46) this is just you can do it a over any open interval or the whole R it does not matter. And this is what you will have that is exactly the same thing that I am going to repeat here.

So, f dash evaluated at g a into g dash a. Now how do I start doing the proof of this? The proof begins by constructing a function called phi h, which has similar structure like this right. Because you see the this thing is nothing but g of a plus h minus f g a by this. So, I will construct phi h which is of this form, that is f of if and it is f dash of g a if. So, that is how I have defined this function.

So, the next step that we will use is a trying to prove the continuity of the function 5. So, the function 5 you can easily prove that it is the continuous at the point 0, when h goes to 0 because that is what we have to check. At the function this function 5 is continuous at 0 right. So, that you can figure out because if h goes to 0. It does not matter if h and this is not equal to 0 and if h goes to 0 this is nothing but f dash g of a and if this is 0 does not matter if h goes to 0 and if this continues to remain 0 then also the limit is nothing but f dash g of a. So, this is nothing but f of that is nothing but phi of 0.

So, whenever h is 0 this is 0. So, you will always have this. So, phi is continuous at 0. So now, let us do it step by step and you will see how the whole thing evolves, and that will be very nice and beautiful working with the chain rules is something which you can easily figure out in any calculus book. My idea here again as I told you in this course is not really to show you doing examples and examples is to give you the conceptual issues in calculus. When how do you really think about a problem. How do you really think in calculus. So, that is the key idea. So now So, whatever we have just said that phi is a continuous at 0, that was just a idea brief idea that we have given, but we will just prove that phi is actually continuous at 0, And that we will do very step by step.

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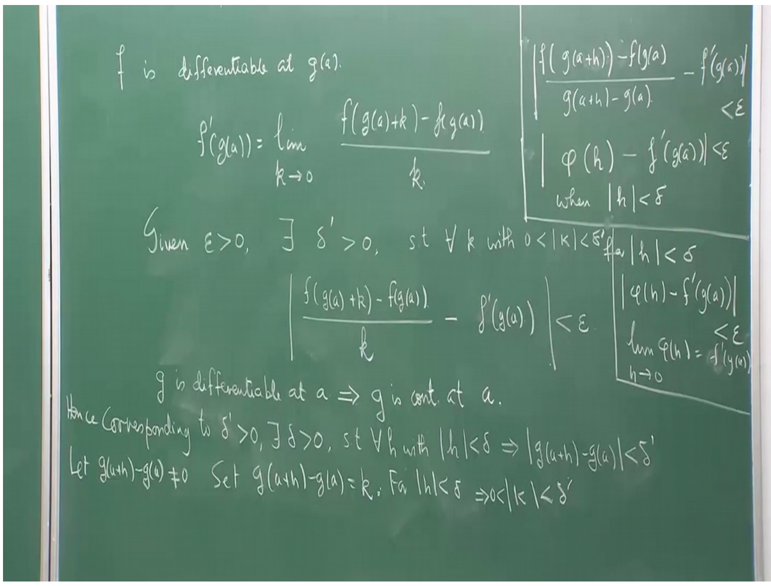

So, f is differentiable at g a or whatever f is differentiable at g a.

So, what does it mean f is differentiable at g a. It means the following that f dash of g a is equal to limit k tends to 0, f of g a plus k minus f of g a by k. So, given epsilon greater than 0. So, given epsilon greater than 0. There exist say delta dash greater than 0, such that for all k with what would I have I would have f of g a plus k minus f of g a. You see this is a rigorous argument, here there would be no flaws in the logic. I am not going to get into the foundation all these aspects of mathematics, but there is no flaw in the logic as per as the standard mathematical logic goes there is no flaw in all logics.

So, these demonstration is absolutely alright you know mathematics as evolved over the ages. And I want to tell you what was a very logical thing in the early part of the twentieth century or late part of the 19 century is not possibly could be considered logical many of proof of great mathematicians like oilers whose results are all correct and the proofs are completely wrong by our current standards. So, the whole notion of proof evolves as the subjective evolves itself. So, here so, till now what we are doing at this point of our evolution of mathematics what we are now doing is expected to be rigorous. So, everything there is a there is a little pinch of salt, but that is alright. So, minus f dash of g a this must be less than epsilon. So, this is the standard definition you already know, I have already spoken about the derivative and given this definition. So, this is what you already know [FL].

Now, g is differentiable at a right. So, which implies that g must be continuous at a [FL], because every differential function is continuous something which I told that all of you try at home those who are observing this. Those who are looking at this video should try this that at home because without trying stuff you would not be getting anywhere and that is the most important thing.

So, g is differentiable at a this is the crux, this argument that here you are using an important fact that whenever g is differentiable at a g is continuous a this is the crux of this whole proof, that phi is continuous at 0 if you want to make it rigorous. This is the crux of the proof if you just take use a differentiability definition of g you would not get anywhere that is the crux. So, implies g is continuous at a. So, if I now for g I considered this delta dash to be my epsilon.

So, corresponding to delta dash corresponding to that delta dash corresponding, hence corresponding to delta dash greater than 0 there exit delta greater than 0, such that for all h with mod h strictly less than delta. It implies that g of a plus h minus g of a is strictly less than delta dash. Now this result this fact given epsilon there is a delta dash for which this holds is true for any k which satisfies this [FL]. Now assume that g of a plus h now assume let g of a plus h minus g a is not equal to 0.

Student: sir there can be equality there 0 less than equals to mod k, because here also again the same thing can happen like g of a plus.

No these are the definition of the limit. You see this is the definition of the limit. Limit it always take a punctured neighborhood here it, if writing the definition of the limit the functional value at when k tends to 0 is not this of this. This differential quotient this is not the functional value whereas, k tends to 0 this has at k equal to 0 this has a no definition that is undefined at k equal to 0 this differential quotient. So, that cannot be continuous. So, this is correct this is a standard definition of the limit. So, here I have I have using the continuity. So, because it will has to be defined at h actually at a.

So, here h can be 0 because as you know it has to be defined at a. Now suppose this is not equal to 0, but I have shown that for all mod h less than delta this is less than delta dash. Now set g of a plus h minus g a to be k. Now for all mod h so. So now, for mod h less than delta. So, what happens it implies that mod k which is bigger than 0, now because this is not 0 is less than delta dash. Because of this statement because now we assume that g of a plus h minus g of k. So, it is satisfies it. So, g of a plus h minus g of a is one of this ks.

So, which implies that if I replace that k. So, f of g a plus k that is. So, if I write here k as g of a plus minus h minus g a. So, I will simply have here g of a plus h minus f of g a. So, so the let me I am doing this part of the proof here delta dash. So, we will have this divided by instead of k I will now have g of a plus h minus g a minus f dash g a strictly lees than epsilon.

So, for all mod h less than delta, this is this. So, what is this? This is nothing but that is nothing but 5 h minus, because whenever g of a plus h minus g a is not equal to 0 phi h is defined like that is phi h f dash of g a is strictly less than epsilon whenever mod h is less than delta. So, this is true when, so, which means whenever h is tending to 0 then this is also can be made smaller and smaller.

Now, if this is equal to 0 g of a plus h minus g a is equal to I have already defined f dash f of phi of h as f dash of g a. So, that difference would a would be anyway 0 and would be less than any epsilon that you choose because epsilon is positive. So, it means that for mod h when for mod h less than delta phi h minus f dash of g a is strictly less than epsilon right. It does not does not matter. So, this is true. So, so what I have proved that limit phi of h as h tends to 0 is f dash of g a.

This is the crucial step by which you overcome the fact that g of a plus h minus g a could be 0. So now, we will finish this spot the rest of the proof which is not difficult I am just removing this part. And we will finish this rest of the part and hence the our discussion on rules for differentiation this is the most important rule one of the most important rules of calculus and when many people actually do not understand it very clearly how to handle it.

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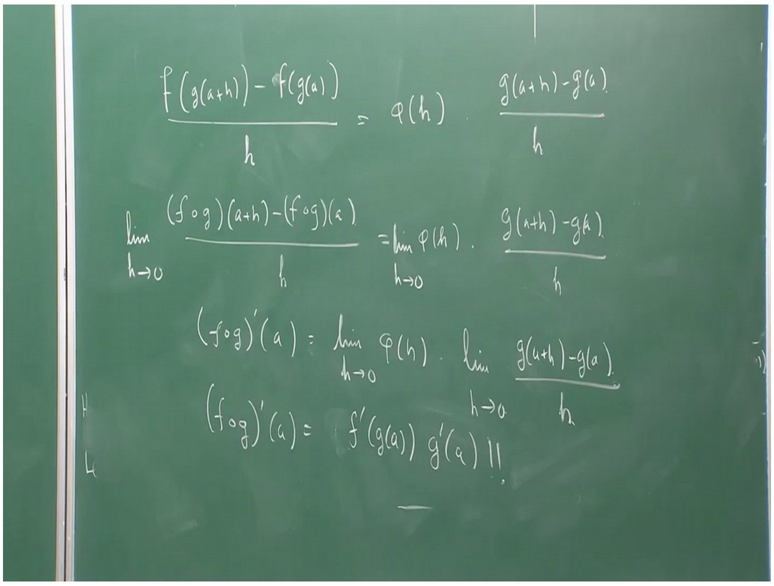

So now, what I can write. So now, let me write f of g a plus h minus f of g a by h. This thing is can be written as phi h into means this thing into g of a plus h minus g of a by h right. So, because this is nothing but. So, this. So, what I have. So, what I have now written I have written f of g of a plus h minus f of g of a divided by h is equal to phi h into g of a plus h minus g a by h. Now here if it is g of a plus h minus g a is 0 does not matter. Because if g of a plus h is equal to g a then this is 0 and then this is 0 0 0 both sides it does not matter it is holds.

Now, you see why this phi h tending to 0 was crucial. Now if I apply the limit as h tends to 0 I will apply it on both the sides, limit as h tends to 0. So, this is nothing what is by definition? By definition is this now because the limit of these 2 quantities exist. So, the limit of this side also must exist and hence f composite is differentiable at a and this is nothing but limit of h tends to 0 phi of h into limit of h tends to 0 g of a plus h minus g a by h. And you have already proved that limit of phi h is equal to 0, and see this is the beauty of mathematics everything can be proved in a very rigorous fashion.

So, this is you already know that this is nothing but f dash of g of and you see the power of the epsilon delta reasoning just because of this proof is rigorously done because of the power of that reasoning. And this is nothing but g dash a and (Refer Time: 36:44) we have proved this. So, we have proved this game. So, with this we end this discussion I hope you enjoyed this proof because I think it is a very nice very beautifully done thing. And after this we will talk about one of the things, which is very close to my heart about maxima and minima. And which is having very important application of this side of this portion of derivative, which we will start in our next lecture.

Thank you very much.