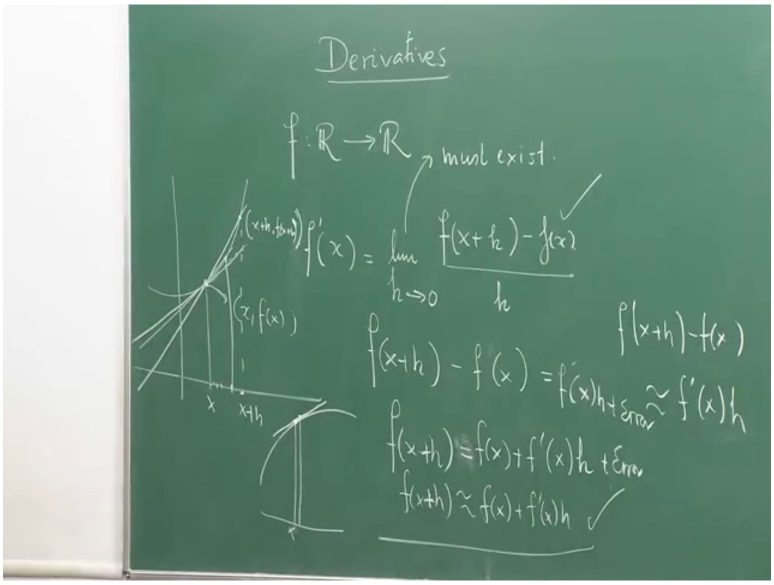
Calculus of One Real Variable
Prof. Joydeep Dutta
Department of Economic Sciences
Indian Institute of Technology, Kanpur

Lecture – 11
Derivatives

In fact, I have told you few lectures back when I had discussed the notion of the limit and we came and discussed the notion of the velocity of a particle moving say in a straight line. So, and we looked at x as a distance travelled as function of time and how do we calculate the instantaneous velocity which turned out to be same as the derivative of that function.

Now I told you that you would understand calculus far better, if you know at the very outset that every continuous function need not have a derivative. And second of all you would which we will show now and you will also understand calculus much better if you know at the very outset that this notion of derivative or integrals are names given to special limits I think I have already told you.

(Refer Slide Time: 01:19)



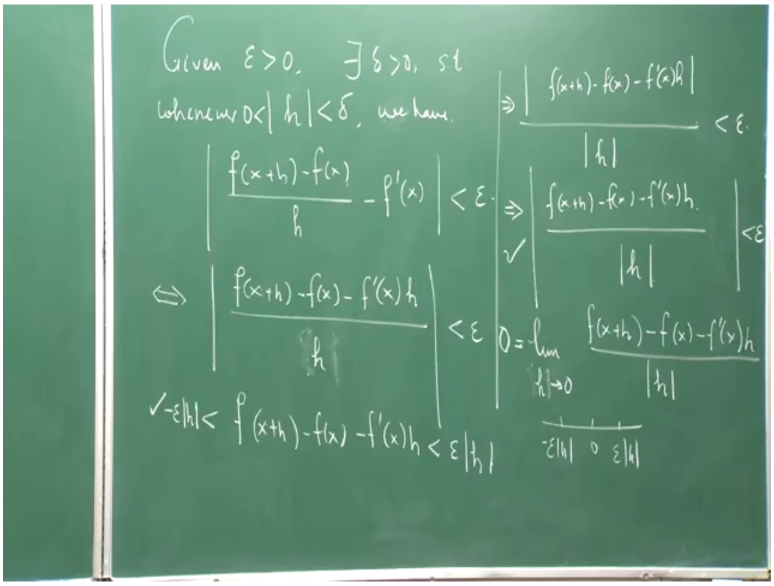
So, this is what you are thought at high school right the so, if you have a function f. So, I am just writing from R to R it could be from a b to R does not matter. So, the derivative at any point x, you see how if I make the change in x how much x has f has changed. So, basically I am calculating the slope of the graph, slope of the line joining the 2 points x f x and x plus h and f of x plus h. So, that is the this is exactly the slope. So, this is my coordinate point x f x, and this is my coordinate point x plus h f of x plus h.

So, this ratio that you see is nothing but the slope of this line. Now if you have h becoming smaller. So, this was my x plus h this was x and this was x plus h. So, this was h. So, if h becomes smaller and smaller. So, this will come here. So, here then we will be talking about a slope of. So, x plus h will come here. So, we will be talking about the slope of this line, and as h becomes smaller x plus h becomes nearer and nearer x we will finally, be talking about the slope of the tangent at the point x f x. And the slope of the tangent at the point x f x is called the derivative of the function f at x.

So, this if this limit exists then f is called the derivative. So, this limit of course, must exists. So, I if I assume the function is differentiable. So, if this limit exist. Now because we have studied the notion of the limit in terms of this epsilon in a delta stuff, then what we can do is the following.

We will write down the epsilon delta in the epsilon delta language what is the meaning of this. It means the following, given epsilon greater than 0 no matter how small, there exist a delta greater than 0 Such that whenever mod of h is strictly less than delta and strictly bigger than 0 h is not equal to 0.

(Refer Slide Time: 04:05)



Of course, then we can not take the function neighborhood around one is 0, we are taking a neighborhood around 0, but not including 0 is just that just as the definition of the limit such that by the definition of the limit whenever this happens we have that this slope of the cords approach the slope of the tangent, which we have denoted as it we call as a derivative this must be less than the epsilon, this is the meaning.

See if I look at this I can do some more simplifications. Simply means that sorry, my h this is strictly less than epsilon. So, we have just made some simple modifications, we were just made this more compact. So, what does this means? It simply means that which also simply which also means minus 0 if I can write. So, this simply means there is minus 0 means just you want to remove this is this. So, this simply means that 0 is the limit of this function as h stands to 0. So, here once I fix the x this becomes function of h, and that is the limit when. So, whenever given any epsilon and mode h is this that limit this limit of this function is 0.

So, simply it means that the limit another way of talking about or even you can write h goes to 0, does not matter whatever you want, but let me this is nothing but 0. So, this is 0 this is another way of looking at the derivative. So, derivative actually approximates, when points are when you have when you want to calculate f you know when h is very small you know that this difference from here what you get, what you get? You get f of x plus h minus f x from this part this is what, I get f of minus f dash x h is lying between epsilon mod h and minus epsilon mod h.

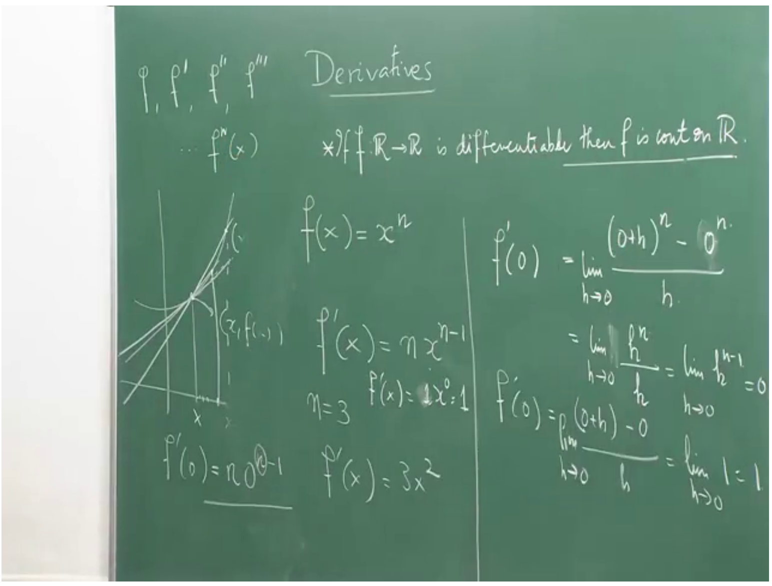
So, once the h is known, how much is my move from x, then this is then the difference between f of x plus h minus f x minus f dash x h. So, this if I can write this as some sort of an approximation. So, this difference is approximated by this, but this approximation has an error, and this error is bounded by the 2 numbers which also depends on h. So, basically we can write that this difference is plus some error. Which means I can calculate if you give me x if I know the function value at x, then f of x plus h can be calculated as f x for very small h it will almost similar to wait would not be should write f of x plus h plus error.

Now, what is happening when h is becoming smaller and smaller these 2 limits are becoming smaller and smaller, the interval minus that this interval this interval around 0 this interval is shrinking down. So, that is exactly the issue here. So, the when the error becomes smaller and smaller that is the mod of h becomes smaller and smaller x is x plus h is very near x, then f of x plus can be safely approximated by f x plus f dash h x.

So, basically function value at a given point can be approximated by the value of the y coordinate of the tangent, at that point at the point x x plus h and x plus h f of x plus h. So, this is something very important this is the key idea of about the derivative, this is what the derivative does it allows me to do a first order approximation. This is called a first order approximation, where you have at a given point. So, if you know any point a function hello here, then any point here nearby point here the function value can be computed by computing the y on the tangent because the difference is error is very small. The more you come towards the x the error goes the approximations become better and better, and in the limit they are the same, there you get back the function value.

So, this is something very important and it has to kept it keep it kept in mind. Most of the cases you evaluate your limit your functional values, when you are starting to learn derivative you evaluate your first expressions of derivatives of a given function using this limit by calculating this limit we just call the approach to the first principle. One of the most important classes of limits that we work through is a following. One of the things I want you to think about, is a first a fact which I have not mentioned But I will just state.

(Refer Slide Time: 11:59)



But I want you to try out it is proof at home, it will be interesting exercise.

So, if f from R to R is differentiable. So, it is not just differentiable at one point of x to get definition here was differentiability at given point x, but here when I say if it is a differential function means is differentiable at every point x in R if x is differentiable then f is continuous, then f is continuous or not.

So, this is something which you can try to prove yourself actually use the definition of the derivative. It is good to have some exercise done like this. So, you can be doing something interesting. Now the second point is how to calculate for example, one of the most simplest functions that you ask to find the derivative with x to the power n. So, here when we find the derivative the people immediately without blindly thinking would use the formula. So, how do you calculate this from the first principle? And what should be the value of n for which this is valid.

Now, it is very important to understand, suppose I take n equal to 2 or n equal to 3. So, by your by the definition f dash x is. So, x to the x is 3 3 x square. Now suppose you might say then it is fine you just calculate and, but when you when you look at this formula right. And when you put x equal to 0, then what you will have f dash of 0 is n 0 to the power n minus 1 right. So, if n is not bigger than 1, then you will be in trouble with at this point if n is bigger than 1 then of course, you can write ah minus 0 say if suppose it is 3 then it is 3 minus 1 2. So, 0 0 square is 0, but say if n is half. So, what will happen if n is half? It will become 0 to the power minus half means one n by 0 when a 1 by 0, 0 to the power half and immediately you have nothing you your you have you are dividing by 0. So, you have to be very cautious about this n. So, all your formulas were good.

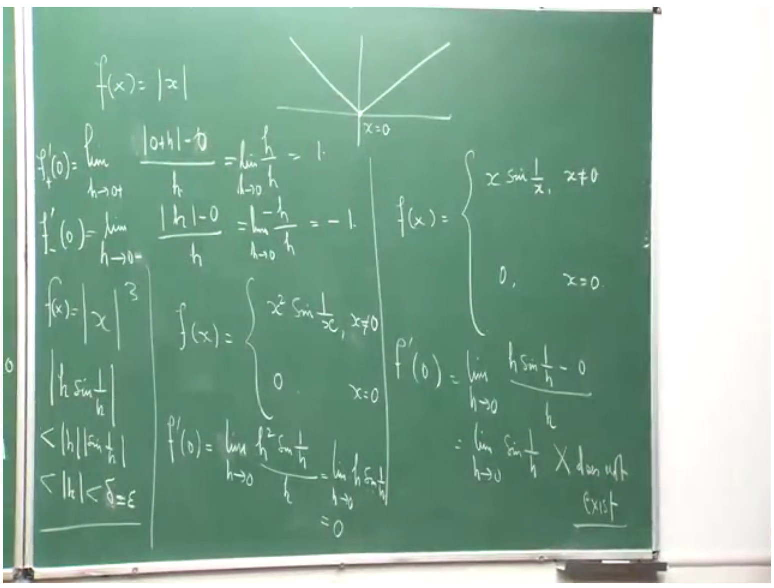
So, whenever x is for example, let me just try it out for example, if I f of I am calculating for this function f dash at 0, for this x to the power n. Then I will have 0 plus h to the power n minus, because here if I put 0, an n is see if any if n is not greater than 1. So, you have it like this. So, basically you have you have the limit as h tends to 0. So, here you immediately have this is nothing but limit, where h is not 0 and that will give you 0 if n is bigger than 1. If n is not bigger than 1 then you will have a problem with this definition. So, suppose n is equal to 1, first let us see if n is minus 2 1 by x to the power 2 x is not defined it 0. So, you can say how do I calculate the derivative either fine, but x is minus 1 then 1 by x it is not defined at 0. Now suppose x is equal to 1. So, f x equal to x right, then if you put. So, here it will become 1 minus 1 0 see if you put any x non 0. So, if is equal to 1 then if you put f dash x and x is non 0 then it will become n x to the power 0 and n is 1. So, it will become nothing but one, but what about 0? You just cannot put 0 in here. You cannot put 0 in here because (Refer Time: 17:29) 0 to the 0 to the power 0 is undefined.

So, for the case when f x equal to x comes, you really have to compute the derivative from the first principle at 0. So, in that case you will have f dash 0, 0 plus h minus 0 by h limit h tends to 0 which is h by h is 1. So, it is limit h tends to 0 or 1, it is 1. So, you can not just put that 0 into this formula and say the answer is 1. So, there are certain points where you have to be very careful even with such basic functions your logic has to be absolutely clear.

Now we will provide an example of a function which does not have a derivative, but yet a continues function. Please try this at home please this is a very good exercise. So, if you do this exercise you would have understanding of the structure of the how do we write this, how do how we have played with this things here, what is epsilon delta structure and you will have a very clear idea. Actually the expressions that we wrote here is very much required to prove this fact.

Now, this is the standard function which is studied everywhere f x is equal to mod of x, and this mod of x is not differentiable at 0.

(Refer Slide Time: 19:10)



Because So, you take the right limit which I write like this h tends to 0 plus. So, it is called the right derivative. So, f of So, mod of 0 plus h minus h by h. So, if h is positive mode of sorry, this is 0 mod of h is h. So, it is h by h is 1 and hence the limit would be 1, but if I take this as the symbol of the left limit then. So, this is mod of h since h is a negative, now is going to 0 from the negative side. So, then this will become minus h by h. So, the limit of this as h tends to 0. So, this is nothing but minus 1.

So, the left limit and right limit are not the same as f approach h, h approaches 0 for this functions but hence, but if for example, if you take mod x cube this has a limited 0, and that is 0 or mod x to the power 5. Mod x to the power 4 does not matter the limit exist and is 0 and only in this case the limit does not exist, but remember this function is continuous this is the very standard absolute value function. Absolute value function is what it measures the distance from 0 of any point. When you are at 0 the distance from 0 from 0 is 0 as you move away from 0 the distance keeps on increasing and that is exactly what the function represents, the distance of a point on the number line from the origin right.

Now, there was there was some very typical type of function things we had studied. Does every continuous function need have a derivative? This is an example another example is the following, where the function was discontinuous what we have used the technique of removal discontinuity and obtained our obtained a continuous function for example, if I have f x equal to x sign 1 by x when x is not equal to 0. And is equal to 0 when x is equal to 0.

Now, this is a continuous functionality at this moment, you know how the function behaves. I have already drawn pictures of this, but the funny part is that this function is not differentiable. At x equal to 0 when x is not equal to 0 you can easily calculate, the derivative for those who know how to calculate derivatives. But if you even from the first principle. If you do not have, but if you come to 0 then there is a problem then again if you go by the first principle then let me right down. So, I have to find this limit 0 plus 0 h sin 1 by h at a functional value at 0 is 0 divided by h and h h cancels, because h is actually not equal to 0 limit of h tends to 0 sin 1 by h. Because as h is tending to 0 say it is tending to 0 from the right side, it does not matter. What would happen? It is tending to 0 from right side means sin 1 by h becomes bigger and bigger and bigger. But as it goes bigger and bigger the if you look at remember the graph of the sin function it is sin values oscillate between 0 and 1, it does not stabilize to any limit.

Similarly, if you take the other side if h is negative right. If h is sin of say minus h basically with h positive number, then what would happen? This value would become smaller and smaller on the negative side, but bigger and bigger on the negative side means smaller and smaller, but the sin function will continue to remain oscillate between 0 and 1. So, you cannot even compute a right limit the right limit does not exist nor the left limit exist. So, the limit this limit this one this limit does not exist. So, here as an example a function continuous again which is slightly where we have made it continuous artificially by using this technique or removal it is continuity, and the derivative is not existing.

However we change the function a bit, you see why you put x square squaring is always some sort of a smoothingness it is it is a regularization sort of thing takes not equal to 0 and is 0, when x equal to 0. Now what is f dash z 0, f dash 0 is nothing but limit h tends to 0 h square sin 1 by h minus 0 by h. So, that so, here h which will cancel we will become limit h sin 1 by h as h tends to 0. Now sin 1 by his bounded function. So, sin 1 by h is a bounded function. It cannot take any value without outsides minus 1 plus 1. So, it always remain within that bound minus 1 plus 1. So, I can so, because of that as h tends to 0, this limit this value would go to 0. So, this value please remember sin 1 by h is a bounded function. So, basically if you have 2 functions f and g, and if one of them is bounded right then, then the other one and the other goes to 0 then the whole product of those 2 will go to 0.

So, this is some simple idea this idea is that you see as h becomes smaller and smaller, these value keeps on oscillating between you might say, what about these? Oh these values are oscillating. So, I do not know may be this whole thing is oscillating, but no as h becomes smaller and smaller and smaller and smaller does not matter what values you are taking. The values would ultimately it will become so small, and because with if these value is only between minus 1 and plus 1 these value product value will start becoming smaller and smaller and smaller.

So, because here I have a fix range. So, suppose I am taking any number between one and minus 1, but my value becomes. So, near 0 say from the positive side only let us check that it becomes. So, near 0. So, does not matter whatever be the value I take the value will finally, become smaller and smaller and smaller say another way of looking at it is if you want to find this limit right. So, basically I have to show that given whenever mod h is less than this. So, what is this?

So, this can be this is less than mod h into sin of mod h. So, this is less than mod h. So, whenever mod h is less than delta. So, if I choose my epsilon delta to be equal, then this is less than epsilon delta and my delta is my if I choose my if delta to be equal to my epsilon then this will immediately show me that h sin h has a limit 0. So, you can do it in either way, you can have this conceptual thinking that what can have an even if that numbers oscillating if I and keep a, and if I am forcing another thing on it which is pushing it to 0, then what would happen? But you can straight come out from this rigorous way of looking at it. But I would be more curious for you to know that once, but you of course, you can compute the derivative of this. The question is the derivative continuous at 0, that is the question and can you answer that.

So, think about this we will we have still many more things to talk about the derivative. We will talk about it is nature it is how do you differentiate of course, there are of higher derivatives. So, you can have a function f, you can take it is first derivative. And then you can again differentiate the derivative derivatives of the function of x. You can differentiate it again you can differentiate the second derivative. So, you can keep on doing it. So, something is called n f derivative f n x. So, you have differentiated it n times if it is possible to differentiate actually.

So, for example, if you have d d x of f x and you want to again differentiate it you get the second derivative of this. So, that is written as. So, similarly you can get the third derivative and 4th derivative fifth derivative (Refer Time: 29:38) derivative. So, once you know about the derivative the rest is just an extension, which we do not want to discuss which is very simple may be you would have some problems. So, with this we have this basic idea of the derivatives specially the initial approximation which is important and knowing fully well that all continuous function the in whatever derivative and also looking at the nature of this very special type of very pathological functions.

So, we have generated some idea of doing the derivative. Of course, I am not going to give you examples which are they given in standard high school classes of finding from first principles. But here I for example, I am trying to show you that you can when you are looking for very simple things you should be very, very careful.

So, this is done once this is the basic idea of the derivative is in place specifically when I have written in terms of epsilon delta. So, the whole idea is that the derivative is a first order approximation. It tells you that if you move a very little from the point x then the functional value of f of x plus h is nothing but f x plus f dash x h plus some very, very, very small error, which for many practical purposes you can ignore (Refer Time: 30:59) we will simply ignore that error. And they will take f x plus f dash x h a something which is practically applicable ok.

Thank you and see you in our next lecture.