**Calculus of One Real Variable**
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Lecture - 10
Supremum and Infimum

So, we are going to discuss today about something called supremum and infimum. In a standard calculus course that you might have done in your high school you might not have heard these words, they are not really told in a high school. But these are a very slight generalization of the idea of maximum and minimum. One might say that real calculus course possibly does not need it, but it does need it because without these ideas is very difficult to prove for example, the notion of this intermediate value theorem. We will show of the intermediate value theorem as an application of these ideas. Now, what do I mean by supremum? Supremum means something super, so something related to the maximum; and infimum means something lower inferior, so it is something related to the minimum.

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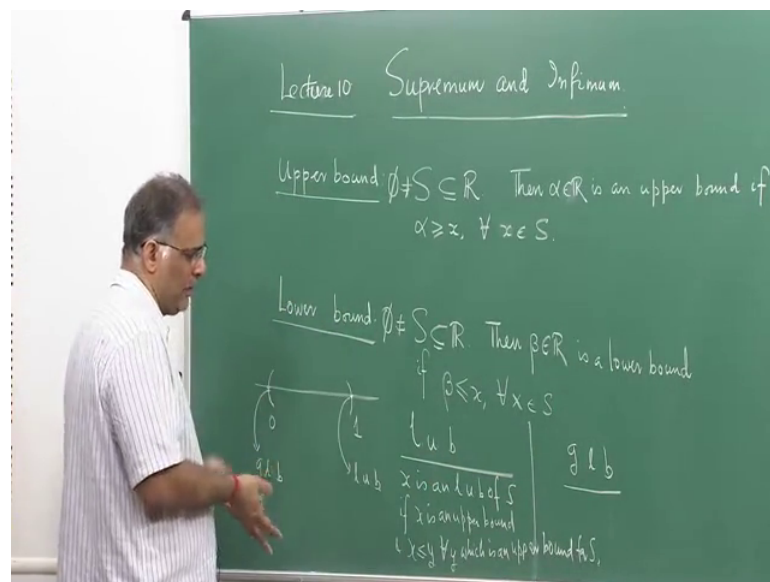
So, let us now define two things. One is an upper bound; one is called a lower bound. So, of course, you might have upper bound or what lower bound or what. So, upper bound and lower bound of a given subset of the real line. So, S is a subset of R, then alpha is an upper bound, if alpha is greater than equal to x for all this is the sign for all this inverted

A for all x element of S. So, any number which is bigger than alpha is also an upper bound. So, there could be infinite such upper bounds as you know that the real number line is uncountable. So, of course, if you want to be more precise then you should write alpha element of the real line. So, real number is an upper bound to S, if the source. Similarly, so we take again S subset of R then beta element of R is a lower bound if beta itself is less than or equal to x for all element of S.

Now, observe one thing that any number less than beta is also lower bound. So, for example, if you look at this set, the open interval 0, 1, so 2 is a upper bound, 1.5 is a upper bound, in fact, 1 is also an upper bound. In this case, for example, minus 1 is a lower bound, minus 2 is a lower bound, minus half is a lower bound and 0 is a lower bound by these definition. But among all these upper bounds you see one appears to be the one which is the mean least has a leased value. And among the lower up lower bounds 0 is the one which seems to has the greatest value. So, this looks. So, obvious that you really do not go and try to prove that there exist a lower bound or there exist an upper bound when a least upper bound or an greatest lower bound, you take them as an axiom.

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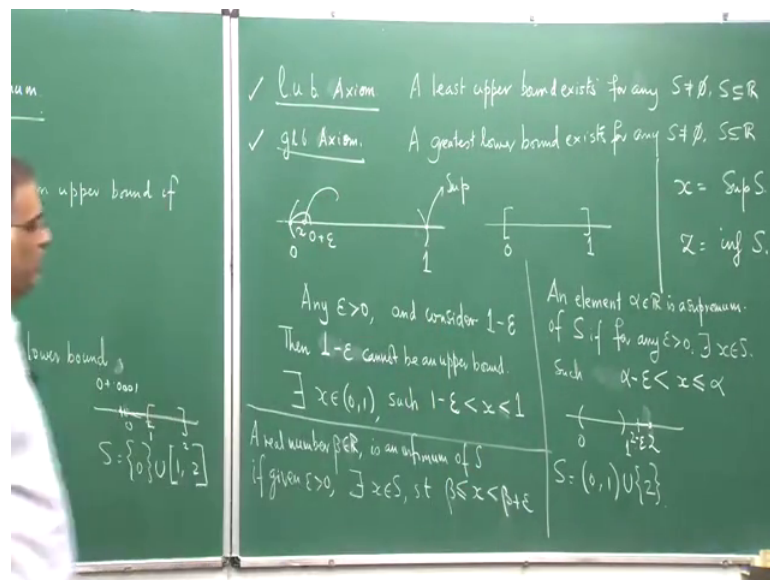
So, first axiom is called the least upper bound axiom or l u b axiom, a least upper bound exists. So, axiom is something which is so obvious a truth that you do not really define it. So, what is a least upper bounds. So, similar there is something called a greatest lower

bound. So, here I call this as the 1 as the l u b; and 0 as a greatest lower bound. The another be more nice looking name for l u b supremum and nice looking name for g l b is infimum. L u b axiom say a least upper bound exist. Similarly, you have the g l b axiom which also says a greatest lower bound exist for S for actually for any subset of R basically maybe I should write for any S not equal to phi, and S subset of R.

Now, obviously, when I when I was discussing this upper bound and lower bound and because I have mentioned this x element of S, I am in my hint side actually I am assuming this fact that this set must be nonempty. Because otherwise you can say that if you take the empty set, this called a vacuous reasoning in mathematical logic that because there is no element I can say that every element in the real line is a greatest is the upper bound; every line is a lower element is the great lower bound, hence there is no element there. So, every number can thought of convention make a make a convention that every non real number is bigger than any element in phi, because phi has really no number. So, then there is no upper bound supremum or infimum. So, this is what happens.

And of course, I am not going to define these this simply means that x is an l u b of S, if x is an upper bound; and x is less than equal to y for all y which is an upper bound for S. So, I will expect that the students can write down the definition of g l b which will simply say that x is a lower bound and x is bigger than all the ys which are also lower bounds of S. So, because this so obvious we take this for granted that these exist, but what does it more mathematically means structurally what does it mean it means the following. Suppose, you have taken this line 0 and 1, and you are claiming that 1 is a supremum.

So, sometimes you mention this you write in this way x is equal to supremum of the set S or say z equal to the infimum of the set S this is the standard notation that you use in mathematics. So, if you take the set 0, 1, what is happening what does this lowest upper bound means that is any number which is lower than that given number x it does not mean how much lower even if it is very slightly less it cannot be a low upper bound. Means if I decrease the value of 1, which I claiming here is a greatest lower bound or the supremum is the supremum of the interval 0, 1. Now, I am claiming that one is the least upper bound means any number which is lower than one cannot be an upper bound.

So, if I take any epsilon greater than 0, no matter how small and consider 1 minus epsilon then 1 is an upper bound. So, if what is a upper bound, if I consider 1 minus epsilon means 1 minus epsilon then 1 minus epsilon cannot be an upper bound, it cannot be an upper bound. See, if it cannot be an upper bound, what does it mean, it means that there exist which I write in short that there exist x from 0, 1 such that x is strictly bigger than 1 minus epsilon. So, 1 minus epsilon cannot be an upper bound; and is less than equal to 1, oh sorry, wait.

So, for all x element of 0, 1 such that this would happen. So, x there has to be some x which has to be bigger than 1 minus epsilon then only you can break the fact then only one becomes the greatest lower bound or the supremum. Similarly, for zero, see if you input increase the value of 0 to 0 plus epsilon there must be some x which is strictly lesser than zero plus epsilon which breaks the fact that this new number 0 plus epsilon or epsilon cannot be a lower bound. So, this can be generalized, this idea.

For example, if you look at this if you take now the closed interval 0,1 then 1 and 0 is again the supremum and infimum, but the difference between the open interval 0,1 and the closed interval 0,1 is that 1 and 0 belongs the supremum and infimum belongs to the set. But here the supremum and infimum does not belong to the set. When I am talking about maximum and minimum essentially I am trying to talk about something which is belonging to the set that is the idea that is if you go back and look in to the discussion of maximum and minimum that is what it means. That if you look at the image space of a function then f of a, b then we are trying to find the maximum and minimum basically a supremum and infimum which is in that set. So, this is the subtle difference between infimum and minimum supremum and maximum.

So, what is the general way of writing an element alpha element of R is a supremum of S if for any epsilon greater than 0, there exist x element of S such that sorry 0 is strict oh such that alpha minus epsilon is so you reduce the alpha. It cannot be a upper bound, hence it will be strictly less than x and this is less than equal to alpha. So, you might ask me why I put less than here, here you have not put less than, because here what happens that when no x can be equal to one in this set, but here in general you do not know. So, if this is the general definition because for example, if you take a set like this so and 2. So, your set is 0, 1 union the singleton set 2, so disconnected set.

Now, what is the supremum of this set 2 is an upper bound 3 is an upper bound, but 1.5 is not an upper bound here, because 2 is bigger than 1.5. But then if I decrease 2 by say a little very small amount, it becomes 2 minus epsilon, then what is that element x, which is bigger than 2 minus epsilon strictly, but less than or equal to this, in this particular cases value of this these connected situations then equality would come.

So, in this particular case, 2 minus epsilon, 2 is my chosen x. So, this is something you have to be very careful about, and you have to remember that. Similarly, if you look at now infimum, so a real number beta greater than sorry beta element of R is an infimum of S. If given epsilon greater than 0, no matter how small, there exist x element of S, such that beta is less than equal to x which is strictly less than beta plus sorry beta plus epsilon.

So, again you could have a similar situation like this where you have say thus set S is 0 union 1, 2. So, if 0 and here is the inter closed interval 1, 2. See, you know the 2 is a supremum, in this case 0 was the infimum, but you know 1 is not the infimum 0 is the infimum; 0 is the greater slower bound because one point say half is not the lower bound because zero is less than that. So, in this course sort of cases in this particular choice if you takes 0 plus 0.00001 as epsilon. So, you come just very near zero some point here. Then what is the element x which will satisfy this, in this case you have to choose x 2 be zero.

So, these are the cases for which we have equality here, but all you always have to strict inequality here because you have to break the fact that b plus epsilon cannot be an upper bound, cannot be an lower bound. In this case alpha minus epsilon cannot be an upper bound, this is something which you have to keep in mind and this is exceptionally important. So, here is this little difference between supremum and infimum. So, supremum and infimum may not be an element of the set, but maximum and minimum has to be in the element of the set. So, in this case, I can say one is the maximum of all the values all the numbers here in 0, 1; and 0 is the least value or the minimum value, so that is the subtle difference, but the subtle difference actually helps you.

And we will now after having a brief idea we will now go and prove the intermediate value theorem in this simplest form using this idea of supremum and infimum. So, the proof given in Spewak, Spewak as I told you is a very good book on calculus written by

Michael Spewak. So, here you will see we will use this idea of supremum and infimum in a very intelligent way and this is the same idea that we had given you already in when we discussed the intermediate value theorem I gave you the same idea. So, that same idea that partitioning up and getting things done that same idea is very tactfully given here, but it is very easy to remember.

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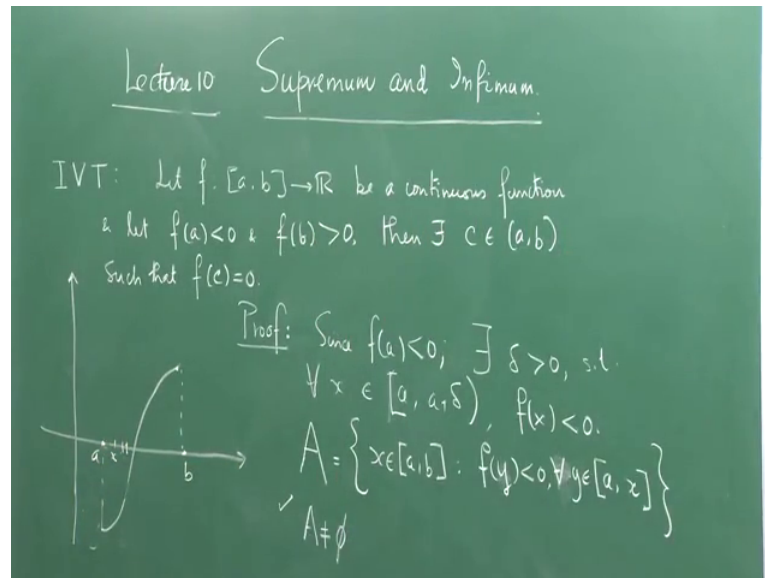
So, we are going to prove the IVT or the intermediate value theorem in its most basic form, intermediate value theorem. Let f, so you already know this a, b to R be a continuous function. And let even if the product of f a into f b should be negative and just let us for the sake of it. And let f a is negative and f b is positive then there exist an c strictly lying between a and b, so that f c is 0. Then there exist c in the open interval a b or even if you write in the closed interval does not matter because they c cannot either a or b.

Such that, so here is your function a drawing always helps. So, this is my a and b is not that they have to always draw it on the positive part, I can take any part, but just for the sake of it. So, this is your f of a and this is your f of b. So, f of a is negative, f of b is positive. Now, how would I do the proof what are the steps? In order to do the proof first I have to observe that because since f a is strictly, since f is continuous and f a is strictly greater than 0, there is a neighborhood around a, here I cannot have full neighborhoods, but only half neighborhoods. So, there is a neighborhood around a, such that f of a would

be strictly less than f of x would be strictly less than 0. So, I can find a delta such that in between the interval a to a plus delta, there for all x, f of x would remain to be strictly less than negative.

So, given which means since f a is strictly negative, there exist delta greater than zero such that for all x element of a in fact, you can say a also does not matter, f of x is strictly negative. So, there maybe not one delta there can be more than any deltas because you can as if you keep on shrinking the delta values you will have more deltas. So, you can just keep on trying and seek to which extent I can keep on moving that is basically the idea. So, construct a set A, what does it contain, it contains all x in a, b such that or I should write it in a much more simpler way it consists of all x, x in a, b maybe I such that f of x is negative on a x, let me construct this set.

So, if I take any delta, if I take any number x which is smaller than a plus delta then over that a x it is still negative. So, find all those x's such that f x is negative on the set. So, this is one set that we have constructed. Now, obviously, x is nonempty because f of a is negative, a to a is; obviously, one interval a single run point, because single run point is always interval a degenerate interval, but still a interval. So, A is not equal to phi for all x in a, x; for y element of x for [FL].

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So, here is a little correction sorry I just. So, what I am trying to do is find those x's those endpoints. So, here is one x here all the a x is negative you all the y values are you take between this interval is negative. Then you take this the function value will remains negative, then you take this function value remains negative. Now, here I have set which is nonempty now what am I supposed to do with this. So, this is what I am going to do next. So, we will hear the idea of supremum and infimum, we will see we will become really effective. Now, once we have a not equal to phi, so basically we are generating these x's here. So, we want to say that a has a supremum, how will we say that.
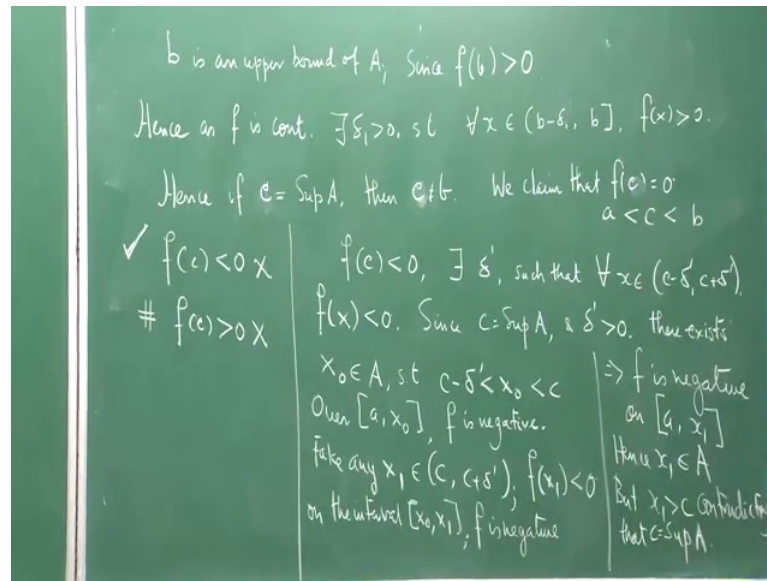
Because you see b is an upper bound of A, b is an upper bound of A. Since f b is strictly bigger than 0, because of for all values there the function value is strictly less than 0. So, f b is strictly bigger than 0. Please understand this point that b has to be an upper bound, because b cannot be included in this set. So, here what we are doing we are increasing this xs, the xs are gradually increasing not decreasing. See if you have for 1 x then obviously, you decrease the x the function value will still remain negative on that interval.

So, basically you are increasing these xs, the x value is here are increasing in the interval it is moving away from a. So, b over the whole interval a, b, if you take b at the point b, if you take a b the whole function does not remain negative because the function value a b is positive. So, b must be an upper bound right, b has to be bigger than all those xs. So, b is an upper bound, but hence since f is continuous, f is continuous in short I am writing cont, there exist delta 1 greater than 0 such that for all x element of b minus delta 1 b f of x must be strictly less than 0, yes, strictly greater than 0. So, this is what is happening, this is the definition, but it is comes from the fact that we have already studied in the last on when we studied continuous functions.

Now, what is this issue what does this do? So, what does it tell me? So, any x here also must be an upper bound. So, we are basically telling that v is not only upper bound, there are elements which are less than b which are upper bound. So, the least upper bound

must be less than b. So, hence if alpha is equal to supremum of A, then alpha is equal is not equal to b. Now, we do not know f of alpha could be strictly bigger than 0, f of alpha could be strictly less than 0. So, now we claim that f alpha is equal to 0.

Actually what would happen here you are increasing from x, increasing the x is still the function remains negative on this that interval a x; and only at the crossing point if that that thing will stop. And here you decreasing the values of x you are decreasing the x values you are making the interval x be bigger and bigger, but the values of x are decreasing on which the function is positive and where the function crosses it has to hit that supremum. So, the supremum is a point where the function would actually supremum of the set A is the point alpha at the function will actually take the value 0, alpha or c maybe. If c is equal to supremum then we are going to prove that c is f c is equal to 0. Thus we have written c here.

So, let us put c equal to so that. So, we claim that f c is equal to 0. So, how do we do it? So, we have to say f c strictly less than 0, and f c strictly greater than 0, these two cases are not possible, it cannot take this values. So, we have to take away this things. So, I will argue for this case. So, you can similarly argue for this case this is for the student to argue out the viewers. And this is for me to tell you how the argument will go.

So, f c is strictly less than 0, for c is obviously, bigger than A, because it is supremum; and c is obviously, less than b naturally. So, naturally you know that c this c is essentially lying between a and b that is; obviously, clear from the constructions that we have done in. So, if f c is strictly greater than less than 0, so there exist delta dash such that for all x element of c minus delta to c plus delta f of x sorry c minus delta dash to c plus delta dash f of x would be strictly less than 0.

Now, since a c is equal to supremum of a and delta dash greater than 0, there exist x naught element of A, such that now you say why it is not equal to c x naught. See here the interval is continuous it is not A, this set this set that you will have it will form of xs which are continuous; it will start from a, and you make a very small move the x will move continuously and it keep on forming those intervals. So, here the set is A is the set A is a connected set when they here there is it is not one set union some other set, here it is just one full set right, there is no gap like the once we are discussed in the previous examples. So, hence you may always when you do not have any gaps, you can always

choose in this conditions this way, but if you have a gap then there is a issue, but here you can always choose in this condition this way. So, there is a x naught such that this happens right.

Now of course, over a x naught what happens a x naught, f is negative, a is negative over a x naught, because x naught is in A. And over a x naught you know that this is negative. Now, take any x 1 element of c to c plus delta dash sorry this is delta dash c to c plus delta dash. So, what would happen f of x 1 must be strictly less than 0. Now, in this whole interval because f c is strictly less than 0 in this whole interval f of function value is less than 0. So, on the interval x 0 to x 1, f is negative, f is negative. And you know the interval a to x naught, it is any way negative.

So, what does it mean, it implies that f is negative on a to x 1 on this. Hence x 1 is an element of A, but x 1 is strictly bigger than c; contradicting the fact that c is the supremum, contradicting that c is equal to supremum of A. In a similar way, you can argue for the other case two that f c cannot be strictly bigger than 0; f c is strictly bigger than zero what would happen. So, you have to argue from the other side from the b side right. So, I leave that argument to you.

So, we have essentially shown that these things cannot happen and which means that f c must be equal to 0. So, using this idea of supremum and infimum, of course, we have actually framed this fact, but of course, this is the argument to reject this is much more simpler, I would allow ask you to think about this argument because it is very good to think about some mathematical proof, because that will give you deeper insights and more courage to do the problems that you will get and by doing so you will learn more about the nature very nature of mathematics.

So, I will end this thing here, but of course in the note that I am gradually preparing you will have the proof of this done also. I am not talking about the other theorem that I proved there in my bounded maximum thing, and whether that can be that I can use supremum and infimum, I can also use that. But there is another approach to that you think sequence as which we will speak when we learn about sequence I have put sequences at the end when it becomes much more useful.

So, with these we end this we see an application of the idea of a supremum and infimum. So, there are some more things for example, you can use this idea to prove that the set n

is unbounded, the natural number set is unbounded, you can use the idea of supremum and infimum. So, that let this can be your when a coffee time thinking, take a cup of tea or coffee and think about it, n is unbounded. So, as a mathematician your idea should be as a math student, you should always use one of the major tool in the hand of mathematician is called proof by contradiction. Assumed that n is bounded say bounded above or bounded below whatever bounded above, bounded below you can say there is 1. So, 1 is less than everything there is a lower bound it is not bounded. So, it is not bound you unbounded up. So, unbounded means it is one of the sides is unbounded. So, it is unbounded above there is no supremum. So, prove that, assume that there is a supremum and show some contradiction, so that can be your coffee time job.

So, with this we end the thing and the next class we are going to talk about derivatives not in the usual way you are thought in high school, but we in a slightly has a slightly different flavor though we have doing things. So, you might be wondering that I have written here two axioms called l u b axioms, I am just giving this is before I finish this lecture I will just want to remind you a two things which I have written the l u b axiom and the g l b axiom.

Where I have said that for a any given set non empty set the least upper bound exist or a given non empty set the least greatest lower bound exist. But please understand that there is a additional meaning to it actually it says to be more precise I should say if a given set if S is subset of R proper subset of R and non empty has an upper bound, then it has a least of upper bound. If it has a lower bound then it has a greatest lower bound. So, without having lower bound and with having upper bound a given set cannot really have a upper bound I mean a least upper bound and greatest lower bound.

So, the l u b axiom should to be more precise I was at that time give just giving on a connection. So, without the notation of upper bound existing, there cannot be any notion of least upper bound, so that you have to keep in mind. So, I am just writing it more in a more very precise way. So, if S proper subset of R and S is not equal to phi and further if S has an upper bound then it has a least upper bound.

See the idea is that if you have a set like this say 0 to plus infinity many of this one sided write sets. So, these are non empty sets, but these does not have a upper bounds, so there is no question of infinite and real number which can act as an l u b. Similarly, if you take

a set like minus infinity to 0, then it does not have a infimum because there is no lower bound here you can go as less as possible. So, only when a set has an upper bound, it has a supremum n has the set has a lower bound, it has a infimum.

So, without the existence an upper bound, a set does not cannot have an real number which acts as a upper bound or in the other case if it does not have a lower bound, there is no real number which act can act as a infimum. So, these are the two things you have to very keep in mind. So, I you might just get confused if we just read that g l b the set has a g l b. So, it was very are lightly written the idea is that that once there is an upper bound, there is a least upper bound once there is a lower bound, there is a greatest lower bound.

So, thank you. So, we will see each other in the next lecture. Thank you.