Calculus of One Real Variable

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Lecture – 01
Introduction to Numbers

So, welcome to my course on Calculus of One Variables. Those who are taken my course on basic calculus they have learned both one variables and two variables, but too much thing was compressed into little time. So, I decided to make the things much more detailed much more relaxed. Today we are going to start speaking about numbers.

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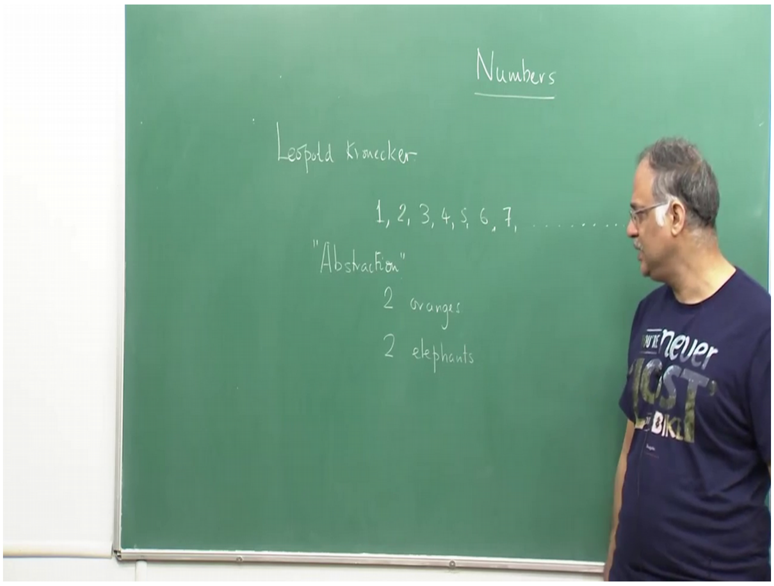

You might think that why you need to speak about numbers. Numbers also fundamental we use it every day to buy our groceries, to pay for a tickets, to look at the cricket score everything is done through numbers. So, human society thrice because of the existence of numbers in fact, a very famous German mathematician whose name is Leopold Kronecker. He once made a very famous statement the statement is that god gave us the natural numbers that is 1 2 3 4 5 6 7 8 9 and then everything is mans hand evolved once you have that you can do everything.

But do not ever think that numbers is something which you did not really, not bothered about it so common. Of course, you know about this basic numbers, so obvious that I do not need to explain to you why and how work this. Now the importance of the numbers is a following. These numbers introduce into our study what we call abstraction. So, what does abstraction mean? Abstraction means using a single symbol to represent different things. For example, if I talk about say 2 oranges and I say about 2 elephants. So, this same 2 is representing that quantity 2 the same symbol. So, it does not matter whether it is orange or whether it is an whether it is elephants. So, this is something we have to keep in mind that it brings in the notion or abstraction; however, let us look at from modern point of view. From modern point of view the mathematical language is the language of set theory.

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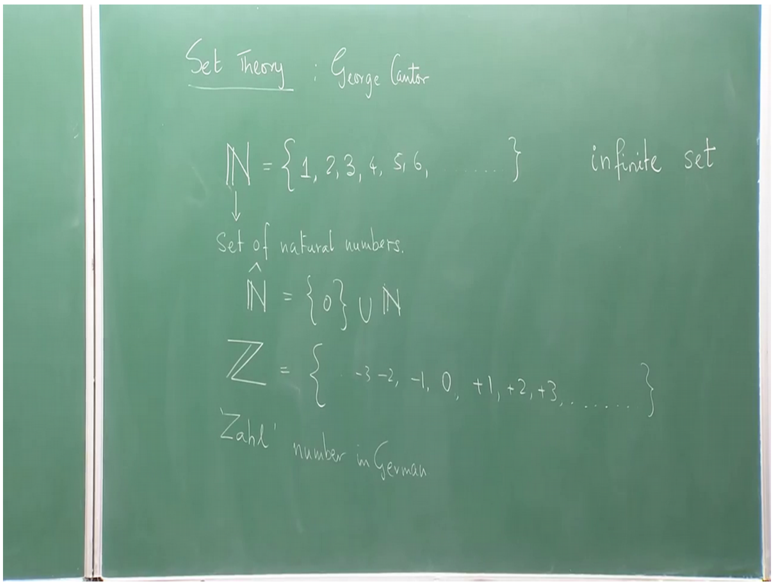

Set theory you must have learnt at school those these are some things which contains objects do not try to define sets as well defined collection of objects and all those things because that will get into logical problems. So, you know and I even know what a set means. So, you always had some say number of one set is set of your friends or set of your classmates set of all say cards in India or set of all potholes in the cities city roads.

So, set theory or the theory of sets was first introduced and studied somebody called George cantor and in his language the set of natural numbers is put into a set N symbol as N. So, this set consist of all the natural numbers that we have. So, this is called the set of natural numbers, this called the set of natural numbers. Now, if you observe very carefully this 1 2 3 4 5 6 7 8 all this keeps on going it does not end, and this is the first time you are possibly you see in your very basic thing of life 1 2 3 4 5 6 7 8 9 10 dot dot

dot dot dot you are coming face to face with the infinite. Thus this is I think your first example that you get as you taught in mathematics is of example of an infinite set the set whose elements are infinite not finite. So, this is not the kind of set that you have studied really in school and George cantor really bothered about sets of this kind.

His teacher Leopold Kronecker who Leopold Kronecker supervise a thesis of George cantor, but unfortunately his studies of taking having a headlong crash with infinite with the infinite was not really appreciated by his teacher. For his teacher infinity was just a notional thing where you take limits and all sorts of things which we will soon learned, but to him to George cantor infinity was a real thing and we will see something in the next classes also.

So, this is the set of natural number some people try to define what is called set of whole numbers by introducing 0 in the set of natural numbers. So, I will use this symbols of union intersection all those things assuming that you know this basic facts of set theory from your school days. So, is this enough for us, the fact is that it is not enough for us see given an object here I can we can define something called addition of these 2 numbers. So, if you have 2 apples in 1 box 3 apples in another box I can bring them together make them 5 apples, but there is a issue of subtraction.

So, if I have 3 apples in a box can I take away 5 apples from there if I ask very stupid possibly a question like that then is there any answer to this that gives us idea of negative numbers. In fact Blaise Pascal one of the very big mathematician, so the 17 century he; 17th I guess yeah 17th century he made a statement that how can you think of taking away something from nothing so, but negative numbers are really your bank balances go in negative many of many of us have made the bank balances. So, in that case negative numbers are the real.

So, this introduces us to the set of all integers. So, the set of al integers contains 0 contains 1 plus 1 plus 2 plus 3 which is nothing, but 1 2 3 just to separate them from minus 1 minus 2 minus 3 this is also an infinite set. This Z this symbol comes from the word Zahl, Zahl actually means number in German. I am not mistaken possibly Richard dudgeon the German professor had first introduced this symbol. Now at your school level you are also taught about fractions that if I have single cake and I have to divide them between 4 people then I have divide into 4 parts and everybody gets one-fourth of the whole right if I want to divide them equally.

So, fractions are also part of our daily life which we cannot ignore. When we use pure integers to denote fractions we introduce something called a rational number.

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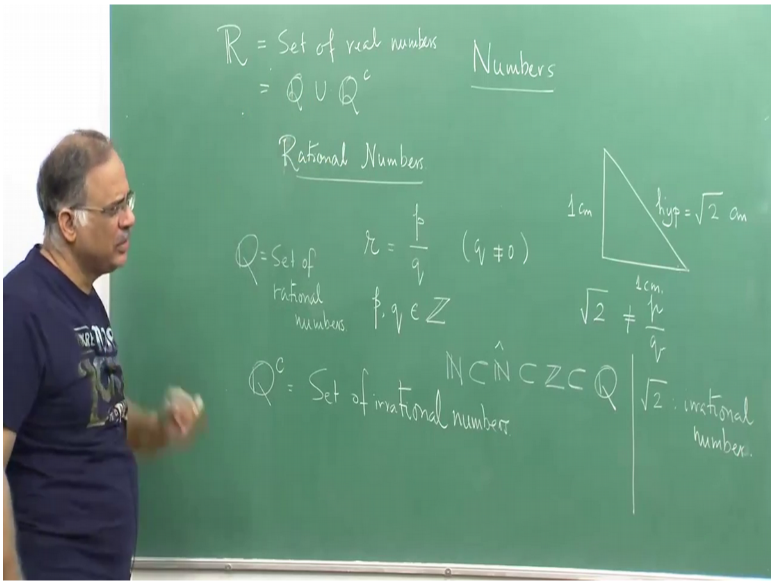

Very loosely we can tell that we are talking about fraction, but fraction actually mean something more bigger it is not just what we are trying to say. So, here we are going to talk about rational numbers. So, we essentially trying to talk about the fractions that we study at school. So, what are rational numbers? Rational numbers are any number say r which is expressed as the ratio of 2 integers p and q where q is not equal to 0 of course, you know very well that division with 0 is not permissible and here p and q both are in the set of integers Z.

So, now, we have got our natural numbers or negative numbers we have got 0 we have got fractions the question is do we need anything more other than this rational numbers. See this thing that there can be something other than rational numbers was first came from geometry it did not come from basic algebraic manipulations. If you take a right angle triangle and this is the most famous description away whose both sides are say 1 centimeter then applying the Pythagoras theorem the hypotenuse would have a length root 2 centimeter and it was one of the Pythagoreans you know the great Pythagoras had a clan a secret clan who had access to knowledge about the world and they wanted to keep it secret they believe that everything was natural numbers and fractions, they did not believe there was something else.

So, when this came out and one of his students call (Refer Time: 11:36) who actually proved that root 2 this number root 2 cannot be expressed in the form p by q. So, we will show all this things in the third lecture what we are just going to mention that this root 2 are shown by 1 of the Pythagoreans to be not rational and that Pythagoreans were all hast (Refer Time: 12:10) was thrown in the sea and killed because he made such a statement that N there is something other than rational numbers.

So, what we have obtained this for using the language of set theory is the following. That we have the natural number set which when added with 0 means the whole number set then when negative numbers coming you have the integer set and every integer can be written as that integer divided by 1. So, all the integers come in the set Q the set Q is called a set of rational numbers which I have not introduced earlier. So, there are certain numbers which are not expressible as p by q for example root 2, root 2 is thus called an irrational number. In fact, we will show in the next lecture that irrational numbers are more in abundance than rational number. So, here, so any number which is not rational that is that cannot be expressed in the form p by q is called irrational.

Now, is there anything beyond irrational of course, you can imagine that there cannot be anything beyond irrational. So, we have the most obvious things this fractions and standard numbers negative all that you can do with numbers. So, you would be perfectly happy just of rational numbers, but unfortunately geometry which is also very real thing gives us something very different story because here you have taken sides whose lengths are rational numbers 1 one centimeter. So, irrational number is a reality with which mathematicians have to put up and they play of pretty major role in mathematics. So, then if I denote where c denotes the complement of a set this is a complement of the set of all rational numbers is a set of irrational numbers. This is what we have.

And now we introduce the super number set which is call the set of real number these are the numbers which we really need, we real in the sense that this number seem to exist actually in reality, in the sense that we can actually find them even irrationals root 2. So, r the set of real numbers is nothing, but the union of q with q complement of course, you know q intersection q complement is empty there can may not be a number which is both rational and irrational. So, hence union of these 2 forms what is call the rational number set.

Now, we will start talking of properties of rational numbers and in that we will start first talking about the natural number set its properties that that is quite important actually you should have some knowledge about the set of natural numbers. So, we are now going to going to talk about the properties of natural numbers. So, we are started dealing with something call the infinite set from the very beginning we have started dealing with the infinite and the infinite is very much or the reality in calculus we cannot really get out of that. So, these natural number sets are classified into 2 different sets call odd sets set of odd numbers and even numbers which you know form very basic school mathematics which I just repeat for your convenience which need not have bother also with.

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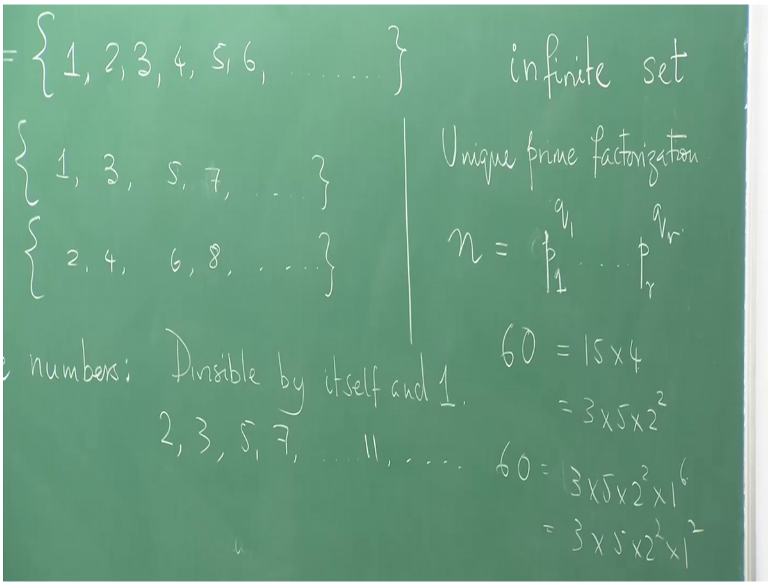

Now, apart from these 2 classification there is a interesting class of numbers. One of the see there are 4 arithmetical operations that you carry on these numbers one is addition subtraction, multiplication, multiplication tables of course, have to do with natural numbers and you also carry out division.

So, when you carry out division which is inverse of multiplication you come to some interesting classes of numbers. So, they are call the prime numbers and they are very very important in the understanding about in if you want to understand about natural numbers. So, what are prime numbers? These also taught at school level any number that is divisible by itself and one is called a prime number. So, prime numbers means divisible by itself and 1. So, you can say I can start with 1, 1 is divisible by itself and one, but 1 is not taken to be a prime number we will soon tell you why it is not taken to be a prime number. So, our prime number starts with 2, then 2 is the only even prime number and then it goes to 3 5 7 and so and so 11.

And important fact about natural numbers is something called unique prime factorization which I am writing on the side, there important fact is called unique prime factorization. So, what does it tell me it tells me that if you give me any number N you can try out I can always express it as a product of the powers of prime factors right, means a factor is a number which divides (Refer Time: 19:15) numbers among the factors there is some number which have prime numbers and you can always express it in this way. For example, so m can be written as say if you take p 1 to a prime number p 1 to the power q 1 say p r to the power q r.

You can form of for example, 60, 60 is 15 into 4, 15 is 3 into 5 and 4 is 2 square that is it. So, all the numbers are prime, so and this representation is unique now suppose I have considered 1 to be a prime number then 60 could have this representation. Unique also means this powers are also unique every number has that. So, once you do a take one of the prime number this is what can happen you do not have a unique prime factorization that is reason why 1 is not taken by convention to a prime number. So, this is something important and has to be kept in mind.

This is a very very powerful result actually, this idea would be used to show that every every know natural number has a prime factor would be used to show that there infinite primes. So, that is a very key result about numbers which has to be understood and one of the important properties of natural numbers is that it allows something called the method of induction. So, what is the method of induction? Method of induction is a following that if you have a mathematical statement involving the symbol n, so you have to decide whether the statement given statement is true or false.

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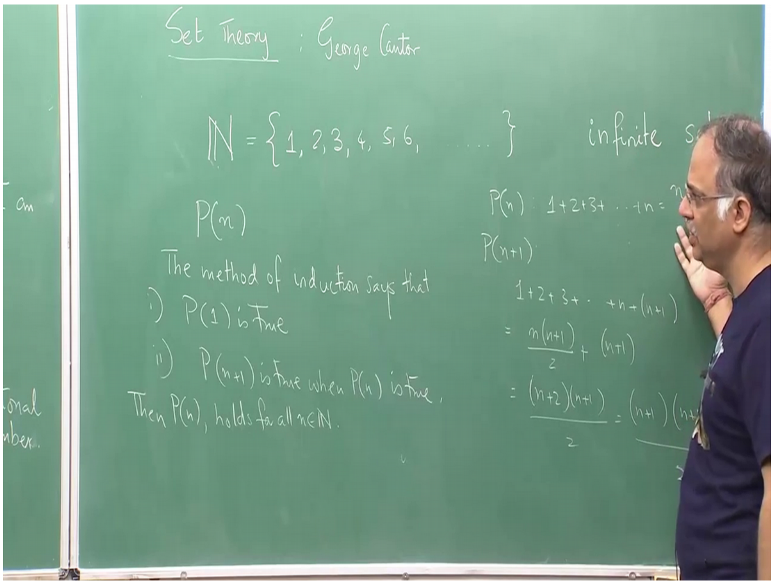

So, the method of induction says fist check is p 1 is true and then if P n plus 1 is true when P n is true then P n holds for all n, holds for all natural numbers. This is a method of induction.

For example, if you take this very simple statement. So, my P n is the following this is what I want to check is this true so; obviously, p 1 is 1 here and you can put n equal to 1 and see the left side is also 1. So, P 1 of course, satisfies this. So, now, you take assume that P n is true. So, what is P n plus 1, in this case the statement P n plus 1 is of course, 1 plus 2 plus 3 plus N plus N plus 1 this simply tells me simple fact because I know that this is true I can use this that up to N it is true. So, this simply gives me N plus 2 into N plus 1 by 2 which is same as N plus 1 into N plus 1 plus 1 by 2 and hence this holds for N plus 1 also showing that this result actually holds for all N.

So, combining these 2 features or induction as well as the fact that you have a unique prime factorization that hence we will now show that we have an infinity of prime. So, this result is a very ancient result which also present in Euclid's element a famous 1 most famous books in mathematics are all times. Now we will not do Euclid's proof we will do a very modern proof taken from a book call the proofs from the book actually a the great mathematician Paul Erdos used to think that god whom we called also supreme fascist that you have he has a book in which all the best and most elegant proofs of mathematical results are kept. So, we have to figure out that whether we can do better than the book or not. So, it does not mean. So, mathematical research does not always consist of the fact that it you would do always something new you can have look look at old things from very new angle and get new new or insights. So, here is our first theorem. So, because we are doing mathematics we have to come into the structure of the subject and our first theorem is that primes are there are there are an infinity of primes.

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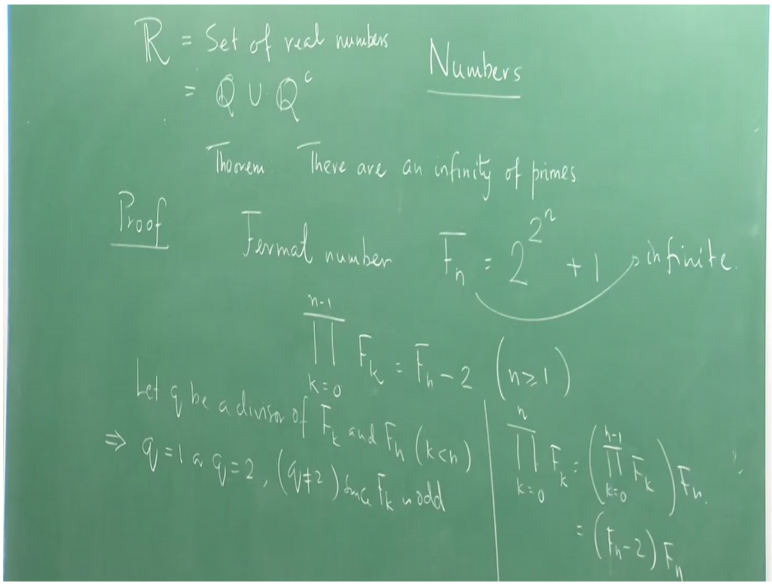

To do this proof we will take the help of something call Fermat's number or Fermat numbers a Fermat number is a number. So, we are doing a proof. So, Fermat number is a number of this form for any N Fermat number is 2, 2 to the power N plus 1. You can ask that who conjure of this strange things that you would proof like this of course, it need guess and test it has come through many many guessing and testing.

The original proof of Euclid might look much most straight forward compare to this, but this is say something interesting and, so I just want to show you this, that what happens at if you realize that this number is always an odd number this 2 to the power 2 to the power N any power of 2 is an even number. So, this is an odd number now I will take 2 Fermat numbers and show that they are relatively prime that is their greatest common divisor or HCF if you talk about a school language is 1 right. So, there are no other prime numbers which are a common dividers to this [FL] and every prime number is bigger than 1. So, there cannot be no 2, no prime number which is a common divisor to 2 different Fermat's number. So, that is why they are called relatively prime or co prime.

So, if I have a number is that 2 2 numbers. So, given of a prime number if it is a divisor of say a then it cannot be a divisor of b if a and b are co prime. Now you can understand as I keep on changing N because the set of natural numbers is infinite, I have infinite number of Fermat's number. So, this Fermat's number is infinite, infinite numbers there be infinite number of Fermat's numbers, while I will show that give me any pair of Fermat's number they are co prime. So, they are cannot be a given common prime which is a factor to both of them. So, for each of the Fermat's number there should be different primes. See Fermat's number infinite so are the number of primes that is the whole idea .

In order to do this we really use this very important recursive relation we will show it by induction actually and n of course, is greater than equal to 1. Now assume that so F k, here k is of course, strictly less than N this k is 1 0 to N minus 1. So, let q be a divisor of F k and F n. So, you are talking about common divisors now if it is. So, if it divides all the F k's and F n. So, this is the common divisor right. So, we have a, so for some k q has a divisor when a q is a divisor of F k and also of F n. So, what would happen? Here I have a product of numbers. So, if I divide this side it is completely divisible because q is dividing, but then this side should also be completely divisible then the in equality make sense which means F because F n is also divisible by q which means 2 also has to be divisible by q which means q this would imply let q is equal to 1 or q is equal to 2 right.

Now q cannot be 2 because if q is 2 if q, but an q divides F k then F k must be even 2 only divides even number 2 does not divide odd number. So, q cannot be 2 this is a conclusion since F k is odd which means what q is equal to 1. So, only common divisor possible between any F k and F n here is 1. So, for any pair you take k less than N for any pair F k F n that you chose or any F k 1, k 2 where my for any pair F k F n that you chose does not matter now you can do the same thing if you chose any k less than N you can do the same sort of induction same sort of recursiveness same structure with something lesser. So, whatever k you chose from here the only common factor is 1. So, then which means we have proved that for any 2 pairs of Fermat numbers they are relatively prime and hence it shows as we have discussed that prime numbers are infinite you see this is very beautiful and simple proof.

And now by induction we can show that this is actually true. So, here instead of N minus 1 I have to add 1 here. So, I come to this simple fact (Refer Time: 31:28) k equal to 0 to n minus 1. So, once you do this I would request you as this all given in the notes I am not going to do the proof that put in F n what is what is the structure of F n you know that this F n is given as this 1. So, use that and finally, prove that this is nothing, but F of n plus 1 minus 2 that is what you can prove. So, now, we are coming more into rational number things. How do we represent rational numbers? There is an interesting way of representing rational numbers call decimal representation. So, decimal representation means I am looking at multiples of 10 right, or division through multiples of tens. So, that when we give representation through multiples of 10 that gives raise to decimal representation of numbers.

Now, any rational number or any fraction if you want to be this more lose fraction is actually more broader term than rational number because I will tell you later why once we introduce functions you see if you take a rational number say p by q.

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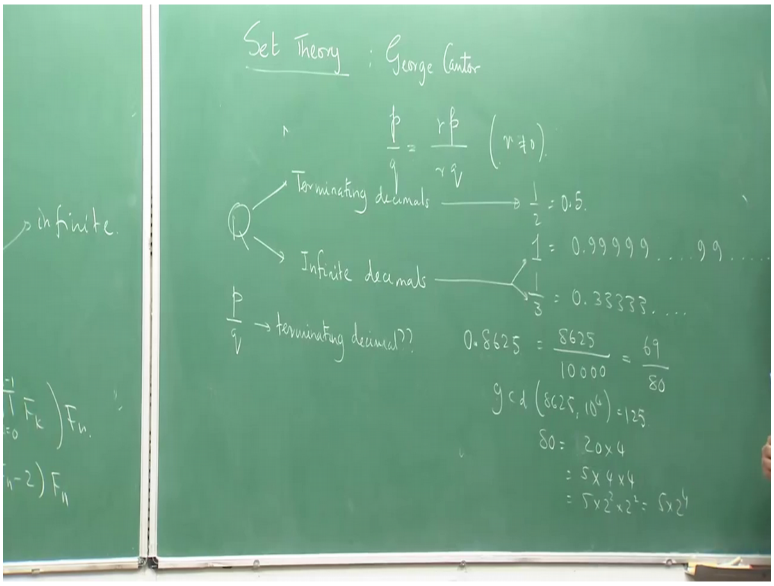

Then if you multiply them by any non 0 number or then p by q is same as r p by r q this is something you have to understand. Now decimal representation means that you can represent there is something called a decimal point and which is a I am sorry there is something called decimal point and on the right hand side of the decimal point there will be some chain of numbers. So, decimal representation is important because it shows if the number is between 0 and 1 the left hand side of the decimal point where you have the whole numbers actually that part would be 0. So, you have already done some decimal points called trying to learn division in school, but we will talk about something. So, in more detail. So, rational numbers are 2 types. So, you take the set Q there 2 types one which is represented by terminating decimals another is representing by infinite decimal that is their own terminate and decimal points are as you keep on dividing of bigger number smaller number by a bigger number you are there on the right hand side you will keep on getting your division will continue your division will continue.

So, these decimal things arise through the division process right. For example, if you say half is 0.5, but luckily your number 1 can be expressed as a non terminating decimal you might be wondering how come this would be equal to this. Actually it means as you keep on increasing the 9 the distance between 1 and that number would go to 0. Here also is hidden the stamp of the infinite which we will come in more detail later on. For example, one-third would be 0.33333.

So, this is the terminating decimal this is the example of a termination decimal these are examples of non terminating decimals. Take this for example, this nice explanation given by Niven that when does a of course, you would liked our terminating decimal or the non terminating decimal terminating decimal looks much more nicer. So, you ask the question when would p by q give me a terminating decimal that is a question, when will it give me when will it give me a terminating decimal.

So, consider this example this is nothing, but 8625 that is you have learnt in school divided by 10000 as a 4 digit. So, of one after one you put 4 zeros and this is same as 69 by 80 you observe that the greatest common divisor between 8625 and 10000 is 125 a greatest common divisor or HCF whatever you want to call GCD between 8625 and 10 to the power 4 is 125, so if we divide we have basically divided both of both the sides by the greatest common divisor and if you look at 80.

So, this is the terminating decimal actually now we have (Refer Time: 37:37) into a rational number form. Now this look at this 80 and look at its prime factorization its prime factorization gives me what is, what is this it is you can write it as 20 to 4, so 5 into 4 into 4 5 into 2 square into 2 square, so its 5 into 2 to the power 4 6 into 5 that is it. So, this is this is this prime factorization. So, what is happening that when you have a terminating decimal it looks like as if the lower, when you put the thing in the see what is happening that the lower thing the lower because you are ultimately dividing by the

GCD to get the p by q in the lowest form you cannot do any more your cross cuttings that you are learnt in school any more divisions.

So, in this case what happens what is 80 what is this - this is nothing, but a factor of or rather this number is a multiple of this number because you have divided by the GCD. So, GCD into 80 is this which is multiple of this number. So, you have observe that because 10 is equal to 2 into 5 the only prime factors of this number is 2 into 5 and so only prime factor of this number will also be 2 and 5.

So, at decimal can be a terminating decimal if you have a rational number and if Q is only prime factors are 2 and 5 we will always there are terminating decimal and that is the correct and this can be as we have proved which we will not statement which you can look into it later on. Now, but among these, so this is our terminating decimal what is what about non terminating decimal. So, we will discuss about non terminating decimals and finish. Do they have some pattern?

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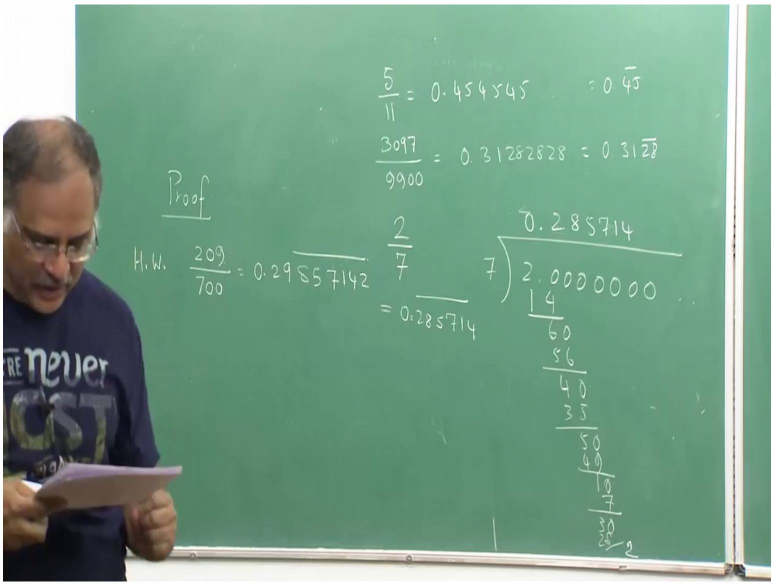

Non terminating decimals have some interesting pattern see if you look at this number say 5 by 11. So, I am giving this explanations from very beautiful small book called numbers rational and irrational by Ivan Niven very well written very beautifully explained book. So, if you look you see 4 5 4 5 4 5 get s repeated. So, what is happening. So, for example, you take 3097 by 9900 you have 0.31282828. So, initially 2 things are not repeated then again it is repeated. So, the shortened of writing this is 0.45 bar on the shortened of writing this is 0.3128 bar the shortened. So, it means that these things are repeated periodically. So, every non terminating decimal has some periodic structure built in do it that one particular patterns, after some time one particular pattern will keep repeating itself. So, why this happens? This just because of division, is simple division.

For example you take the number 2 by 7 let us see what sort of a 7 is a prime number. So, it cannot have 2 and 5 that is prime factors right the only factor are 7 and 1. So, this means it cannot be a non it cannot be a terminating decimal it must be a non terminating decimal. Let us see if I divide 2 by 7 what would happen. So, here is a 7 2 can be written as 2.0000000. So, first is of course, 0 then you put a point any of 20.

So, you to 7 2s 14, 6 it at 60 and I am just writing it down I am writing down answer which you can verify because this is very simple to do I need not do this thing repeatedly you can do this. So, what is happening you have started with do these divisions and finally, you come back to 2; 20. So, again you start over from this 2 and once you start from this 20, see if it is 7 what could be the reminders, reminders will less than 7 it has to be 6 5 4 3 2 1. So, once you have record if any one of the things have record here then you go back the division process again goes back. So, this number again gets repeated.

So, 2 by 7 is nothing, but 0.285714. So, it is not that the very first number would always to get repeated may be the second or third number reminder suddenly gets, it will get repeated. So, from there the pattern would start repeating again. So, you can check this out by dividing for example, I will give you an exercise just to check it is you can take as a homework and I click quite a good amount of fun to understand what actually happens we really understand that 2 need not start repeating here. So, some other reminder can start repeating and so the factor would just go from that point.

In this case p by q this can be represented you just can do the division and just see. So, it is a second reminder which starts repeating here and this is what happens. The very interesting fact is that that is given in the notes I will not do the proof here because of time is that if you give me any repeated decimal I can always represented by a rational number. So, any decimal which is a periodic representation like this can always be represented as a decimal this will be in the notes. So, you can see it from the web.

So, in the notes we have also spoken about real numbers and what are the rules of handling combining real numbers because those rules or rules through which you do addition subtraction multiplication for example, you will see a rule called distribution of multiplication over addition this is exactly the way when you multiply 2 digit number with the 2 digit number or 3 digit number with a 2 digit number that is exactly what is happening that rule is actually followed (Refer Time: 45:04) you learned at school; obviously, you do not know that there is a very very basic rule that has been followed. So, we have now some broad idea about numbers how to handle them and what are the properties you see this prime factorization is such a crucial idea and that is that is why it was introduced. And with this I stop for today and tomorrow we are going into the world of countability, uncountabilities of finite and the infinite.

Thank you very much.