Probability and Stochastics for finance-II

Prof. Joydeep Dutta
Department of Humanities and Social Sciences
Indian Institute of Technology, Kanpur

Lecture - 09
Mean-Variance Portfolio Optimization (Part-IV)

So, welcome back to the lecture once. We are almost end our second week, almost the end the last part one lecture for the second week; and you must be getting pretty bored and upset whether I am always writing mean variance portfolio optimization. Please understand that there are two important aspects of the financial profession or finance profession. One important aspect is to manage risk, which could be your own investment or which could be others investment.

(Refer Slide Time: 00:50)

So, basically we have two parts; management of risk plus pricing financial instruments. We are essentially now studying the management of risk. I am sure that many of among you at least some of you who are listening to this lecture, who would one day become risk managers in reputed financial firms, where you have to essentially manage other peoples risk, your customers risk as we invest in the market, we have to develop a portfolio for him or her. So, one important aspect of risk management is a mean variance portfolio optimization. It is not that it is the only way by which this means variance stuff is done; you know it can also be done by using other measures of risk. For example, the value at risk or conditional value at risk and so and so forth, but the mean variance approach due to Markowitz remains very, very fundamental one. We have discussed a lot about how to handle that, what is it, and how to compute even an under certain natural assumptions how to compute the weights, in fact, unique weights. So, you can uniquely define your portfolio that is what we had discussed in the last class.

But what I intent to do now here as I as promised you to show how reward and risk interacts means how does the return and the risks interact between them. So, here I largely discussed from chapter two of this book called Introduction to Mathematical Finance from by Steven Romans. So, I will just discuss this from this book where it has been beautifully described and this risk has been I mean the risk versus reward or risk versus rate of return or has been described in terms of only two assets that is you are investing in two of the asserts. The market containing two assets is the very very simplified modal, but gives you someone understanding of how thing might just work.

So, here I will have two assets asset A 1 and asset A 2. So, these are my assets from which this is a two market two asset market stocks of some companies. And I have an expected return mu 1 here - return. And the risk is just the variance. Similarly, mu 2 here and the risk, so here when portfolio consisting of these two things and that portfolio would essentially consist of, so portfolio instantly consisting of these two things and the expected return of the portfolio is t plus mu 1 1 minus t plus mu 2, where t is the amount of weight. I have given to the first asset 1 minus t is the amount weight associated with the second asset.

So, there variance is this one; rho 1, 2 is a correlation coefficient between 1 and 2. So, correlation coefficient actually means covariance 1, 2 divided by sigma 1 square that is the meaning of correlation coefficient. And this correlation coefficient of course, you can understand has to be between 1 and minus 1. Now, you have this scenario. So, this is what you have. Now, assume for sake of gravity that is 0 bigger than sigma 1, I written as a positive risk, but it calls sigma 2 possibilities. So, this is assumption. So, we will work under that particular assumption. So, you know how to calculate the mean of the portfolio and the variance of the portfolio which we have already done earlier. So, here I just I am writing it down, but you will immediately understand.

Suppose, first case one, in case one, let us assume that this is uncorrelated that these two events are uncorrelated that is this two assets have no correlation between them. So, I assume that rho 1, 2. Once I do that my sigma square become t square sigma 1 square 1 minus t square sigma 2 square plus 2 t into 1 minus t sigma 1 sigma 2. So, once you have done that you can rearrange these and put it in this form. So, this is what I have after simplification. So, you know that sigma 1 is strictly bigger than 0, and sigma 2 is either equal or strictly bigger than sigma 1. So, sigma 1 square plus sigma 2 square that comes here is non-negative and thus it is truly a quadratic function rather than sin function. So, this is something you have to keep in mind.

So, basically now what I have represented, I represented the variance or the risk in terms of weights. And these are the sigma 1 and sigma 2 are fixed numbers know earlier about this assets. Now, before I move on to do something, let me see for what value of t, t is playing the role of the weight here w 1 and w 2 is 1 minus g. So, let me see for what value of t thus this sigma square achieves a minimum that is exactly what we want to know. Then by symbol differentiation notes standard one t m sigma 2 square, so the minimum value of t, which minimize the sigma square is this. So, this is the weight I have to provide to each of and you observed that this is strictly bigger than 0. This is the weight I have to actually send to the first element and 1 minus t m as a weight I have send to the second element, so that is that is something crucial. And the variants here, so the variance here is as follows.

Now, we will come to two cases. So, this is the value of the weight that I have to attach the first element and this is 1 minus seem the second element, which is strictly bigger than 0. And for that, the allowed risk is this minimum value of the risk is this. So, see how it is linked with whatever variance we have for the two assets problem. Now, this is the minimum weight that you required 1 minus t m is also positive. So, this is what we have with the counter part of the weight. So, this w 1, this is w 2.

Now, what we do once we know this we will talk about what happens if I take rho 1 to be perfectly that is 1 and 2 are perfectly correlated. So, rho 1, 2 live it in 1 and minus 1; and it is perfectly correlated rho 1, 2 is 0. When it is not correlated then sorry when it is perfectly correlated rho 1, 2 is 1; and when it is not perfectly what to say correlated then we can say rho 1, 2 is minus 1 or you can have rho 1, 2 is to be 0. So, what it shows here, it shows that if rho 1, 2 is 0 what we get, we get this nonnegative weights and then risk which depends on sigma 1 square sigma 2 square, sigma 1 square sigma two square which is constant. So, what it shows here is the following. Even if you have no correlations, you can find weights which sum up to 1, but sigma m square is our certain amount.

(Refer Slide Time: 14:13)

So, we have just written down quite hurriedly if there is no correlation, then what is the weight which would minimize the variance, and then we have written now what would be the variance under that minimum weight. And we said that ok, the minimum variance would be something quite significant variance does not go down even if the things are uncorrelated or that total variance can be made lesser than the minimum of the two variances and you do not have any short (Refer Time: 14:37) to do.

Now, how am I reaching and how am I getting these t m values. So, here I am considering two cases. So, 1 was 0, the when there was no correlation. So, 1 was 1 is in the correlation is perfect, so that is call perfect correlation when rho 1, 2 is 1 and it is completely negative correlation that rho 1, 2 is minus 1. So, if something increases something decreases, so that is sort of thing. So, which are completely opposite. So, what I do here is here I put rho 1, 2 is 1 in the expression of sigma square then I make a simplification. Then I take the derivative equal to 0, and then I say whatever you get whatever t solves that makes the derivative 0 is the answer. Of course, I want to a sigma

2 not equal to sigma 1, because if I have sigma 2 equal to sigma 1 then sigma 1 square is equal to sigma 2 square, there is nothing to say.

Now, here you may ask me, there how do you know that this is truly the minimizer, which could ask me even in the case in the rho and rho 1, 2 a 0? The answer comes from the fact that this is actually a convex function. In fact, any variance is a convex function. So, when you have a convex function, for example, we have defined a convex function we will again define a convex function, a convex function is of this form. So, again let me recall that for example, if you take an example like this x is half, x A x; once A is positive semi definite which I am writing in short what is psd then f is convex. Now for here, so the variance that we have is always convex, the variance written as w sigma w. Now, for a convex function, we said it is very important to know that when you want to minimize a convex function, you can over a convex said every local minimize global there is nothing call a local minimize as such.

Now, the interesting part is this, this is differentiating convex function. So, you are we trying to say that every critical point means any point which satisfies the derivative is equal to 0 is also a minimize or global minimizer of our convex function the answer is yes, because if you look at the diagram of a convex function of graph is something like this. So, if you take a tangent any point time if the function differentiable you will have unique tangent at every point and the tangent is lying below the graph, this fact in higher dimension is expressed by the fact that the function is convex if and only if the following results holds true. So, f is convex if and only if this thing happens.

Now if x start is point were grad f x is 0 now grad f of x star is 0 and from here n f is known to be convex this function is convex, and f y minus f of x star is grad f of x star y minus x star for all y. So, if I fix y x this is happen with all y, because these true for all x y. So, what does it mean because this is 0, so this thing become 0, so that will immediately tell me that f y is greater than or equal to f x star for all y showing that x star is a minimizer. So, any critical point that is a point which the gradient is 0 is a minimizer. So, here we have equation in one variable. So, we take the derivative. So, once we know that the derivative is 0, you know that this is minimizer.

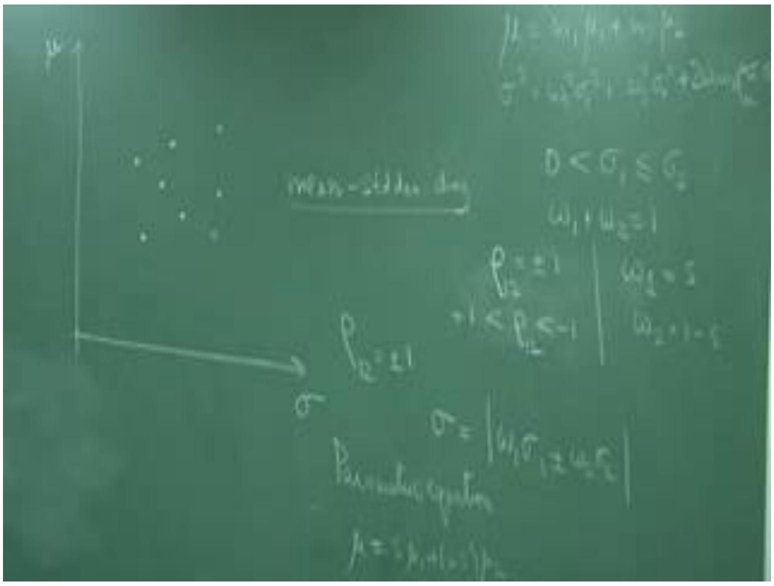
But in this particular case when there is a perfect correlation this is something pretty amazing, amazing part is the following that if you have t m one of the weights which is strictly bigger than 0. The other way would be negative, which means if you want to invest in the first one - A 1, if you want to invest in the first instrument you actually cannot invest in the second instrument, but rather short selling it you have to get it from someone and sell it in the market. So, here we have to do short selling.

Let us see what is the case of sigma now value of sigma square. Sigma square is sigma 1 minus sigma 2 sigma 2 sorry this is sigma 2 sigma 2 minus sigma 1 plus sigma 1. So, this will cancel out by minus 1. So, sorry sigma 2 this will cancel out to be minus minus sigma 2 and so this is 0. So, if you have a perfect correlation then you have to short sell, but your risk would be 0. So, let us come to the case when you do not have perfect correlation there are negative re-correlated there essentially give an opposite fashion then your rho 1, 2 equal to minus 1. And now so rho 1, 2 is equal to minus 1, and then you again do the thing, you write the sigma square which is now of this form, take the derivative and put is equal to 0 and take t m. So, here t m is positive 1 minus t m is positive. So, no short selling is required.

So, now, I want to find the sigma square, where is the sigma square when I got the optimal weights. The sigma square is sigma 1 plus sigma 2. And here I have sigma 2 y sigma 1 plus sigma 2 minus sigma 2 which is 0. So, again the variance becomes 0. So, when you have positive correlation even if you have short sell you have zero risk. When you have negative correlation were you do not have short selling you can simply have invest and there is no risk you will get your expected return possibly. So, this is idealistic situation which cannot happen.

So, your rho has to be somewhere, 0 is acceptable rho has to be strictly some were between minus 1 and plus 1. So, this is the very nice way of is very difficult to get situation where you have even if you have two thing where rho to be called 2 minus 1. So, it (Refer Time: 22:25) to conclude at least from this example that rho usually between two objects rho 1, 2 say two assets should always be between minus 1 and plus 2. Now, can this idea tell us something more? In the finance world, for example, people do not really go out in this way relating sigma in terms of the weights, and then trying to analyze that they do it is some call called a variance mean diagram on the standard division mean diagram and that is exactly what we are now going to study.

(Refer Slide Time: 23:11)



Now, what we are going to discuss is a mean variance diagram. This is very funny that the mean for expected mean r bar which we will write as a mu for because we will assume only things - two asset portfolio and sigma. And people want to discuss interaction with sigma and r bar that is this called the mean. So, you can have various portfolios with various means and variances. This is called the mean standard deviation diagram. So, now what is interesting to me for the moment is the following.

Again let us have two assets and let us have mu 1 and mu 2 be their expected returns and portfolio expected return as this. And the variance is omega 1 square sigma 1 square w 2 square sigma 2 square plus 2 rho 1, 2 w 1 w 2 sigma 1 sigma 2. So, or rather I should write it in this way. So, I should write it as plus 2 rho 1 w 2 rho into sigma 2 sigma 2 rho 1, 2, this whole thing is nothing but the covariance within 1 and 2. Now, what I am going to do again I will consider this is also simplicity. In fact, when I draw the diagram I will possibly take mu 2 bigger than mu 1 that is that will make it easy for us to understand.

So, here of course, I should have the weights. So, a normalized weight w 1 plus w 2 is equal to 1, so that is the standard thing. So, now, one side put w 1 plus w 2 is equal to 1 I can further rewrite you know. So, we will divide the whole thing into couple of things. So, we will first looking in the case rho is plus minus 1 and then of course, we look into the case rho is between minus 1 and plus 1, that is not rho, rho 1, 2.

Now, if now this is my standard thing, now we will try to put w 1 as s, w 2 as 1 minus s, for simplicity we will do that. First let us so we can write everything in terms of s. So, basically then mu and sigma square would they have be instead of writing a function relation in mu and sigma square, we will make a parametric representation of mu sigma square. So, then we can draw the locus of point which passes to the point mu sigma square connected by the parametric relationship, so this is basic coordinate geometric.

So, let us consider the first case where rho 1, 2 is equal to plus minus 1 if you do that then you can immediately write. Now sigma, sigma square the here that do not write in terms of sigma square what sigma, so instead of r bar let me just put mu sigma mu diagram means standard deviation diagram. So, sigma can now be written as so square, so you get the mod of this and then essentially you are talking about ok plus and minus depending on whether rho 1, 2 is plus 1 or minus 1. So, the parametric equations, this is like for example, if you take a parabola y square is equal to four x then you have x is equal to a t and y is equal to a t square that that is the type of parameterization that you do basically.

So, when you have you know for example, if you write if you have a circle, so you can write x is equal to r cos theta and y is equal to r sin theta and of course, x square plus y square is equal to r square. So, this is the parametric representation in terms of the circles. So, we have instead of theta I am just takes x and y we have been looking at everything in terms of theta. So, you write the parametric equation as sigma sorry maybe I should start with mu, mu is nothing but the same thing, but you write s mu 1 plus 1 minus s mu 2, then you write sigma in this particular case as sorry s plus 1 minus s sigma 2.

So, here again we on that perfectly correlated and that correlated space. Now, instead of this, how would I write this diagram, how will I draw this diagram. So, this is our next step. So, how will I draw this diagram? So, drawing this diagram is very, very important. So, instead of taking a step today, in this class or this particular class to draw the diagram, I leave it to you to ponder over it how to draw the diagram. So, in tomorrow’s class, we will start with discussing about drawing the diagram. So, once we know how to draw the diagram things would be much simpler.

Once we do that we will talk about what would happen in the case of rho is between minus 1 and plus 1 and that would give you fairly good idea about how the reward and risk is actually interacting.

Thank you, I will see you in the next class.